BULLETIN OF THE INTERNATIONAL MATHEMATICAL VIRTUAL INSTITUTE
ISSN (p) 2303-4874, ISSN (o) 2303-4955
www.imvibl.org /JOURNALS / BULLETIN
Vol. 6(2016), 25-36

Former
BULLETIN OF THE SOCIETY OF MATHEMATICIANS BANJA LUKA
ISSN 0354-5792 (o), ISSN 1986-521X (p)

Existence Results for Multiple Positive Solutions of Riemann–Liouville Fractional Order Three-Point Boundary Value Problems

K. R. Prasad[1], **B. M. B. Krushna**[2] **and L. T. Wesen**[1,3]

Abstract. In this paper, we investigate sufficient conditions for the existence of at least three positive solutions to the Riemann–Liouville fractional order three-point boundary value problems by means of fixed point theorem. We also establish the existence of at least $2m-1$ positive solutions to the boundary value problems for an arbitrary positive integer m.

1. Introduction

Fractional order differential equations have attracted considerable interest because of their ability to model complex phenomena. Fractional calculus is the field of mathematical analysis which unifies the theories of integration and differentiation of any arbitrary real order [**4, 6, 8, 9**]. In describing the properties of various real materials, the derivatives and integrals of non-integer order are very much suitable. They arise in many engineering and scientific disciplines like mathematical modeling of systems and processes in various fields such as physics, mechanics, control systems, flow in porous media, electromagnetics and viscoelasticity.

Boundary value problems associated with linear as well as nonlinear ordinary or fractional order differential equations have achieved a great deal of interest and play a pivotal role in many areas of applied mathematics like engineering design and manufacturing. Major established industries such as automobile, chemical, electronics and communications, biotechnology and nanotechnology rely on boundary value problems to simulate complex phenomena at various scales for designing and manufacturing of high technological products and in these applied settings, positive solutions are meaningful. The existence of positive solutions to fractional

2010 *Mathematics Subject Classification.* 26A33, 34B15, 34B18.
Key words and phrases. Fractional derivative, boundary value problem, three-point, positive solution.

boundary value problems have been studied by many researchers using different techniques [**2, 5, 3, 1, 10, 11, 13, 12**].

This paper is concerned with the existence of multiple positive solutions to the fractional order differential equations

$$D^{\alpha}_{0^+}y(t) + f\big(t, y(t)\big) = 0,\ \ t \in (0, 1), \tag{1.1}$$

satisfying three-point boundary conditions

$$\begin{cases} y^{(k)}(0) = 0,\ \ k = 0, 1, 2, \cdots, n-2, \\ \qquad \zeta D^{\beta}_{0^+}y(1) - \vartheta D^{\beta}_{0^+}y(\eta) = 0, \end{cases} \tag{1.2}$$

where $\alpha \in (n-1, n]$, $n \geqslant 2$, $\eta \in (0, 1)$, $\beta \in (1, \alpha)$, ζ, ϑ are positive constants and $D^{\alpha}_{0^+}, D^{\beta}_{0^+}$ are the standard Riemann–Liouville fractional order derivatives.

The rest of the paper is organized as follows. In Section 2, we construct the Green's function for the associated linear fractional order boundary value problem and estimate the bounds for the Green's function. In Section 3, we investigate sufficient conditions for the existence of at least three positive solutions to the fractional order boundary value problem (1.1)-(1.2) by using Leggett–Williams fixed point theorem. We also establish the existence of at least $2m-1$ positive solutions to the fractional order boundary value problem (1.1)-(1.2) for an arbitrary positive integer m. In Section 4, as an application, we demonstrate our results with an example.

2. Green's Function and Bounds

In this section, we construct the Green's function for the associated linear fractional order boundary value problem and estimate the bounds for the Green's function, which are needed to establish the main results.

LEMMA 2.1. *Let* $\Delta = \Gamma(\alpha)\mathcal{N} \neq 0$. *If* $h(t) \in C[0, 1]$, *then the fractional order differential equations*

$$D^{\alpha}_{0^+}y(t) + h(t) = 0,\ \ t \in (0, 1), \tag{2.1}$$

satisfying the boundary conditions (1.2), *has a unique solution*

$$y(t) = \int_0^1 G(t, s)h(s)ds,$$

where $G(t, s)$ *is the Green's function for the problem* (2.1), (1.2) *and is given by*

$$G(t, s) = \begin{cases} \underset{t\in[0,\eta]}{G(t,s)} = \begin{cases} G_{11}(t, s), & 0 \leqslant t \leqslant s \leqslant \eta < 1, \\ G_{12}(t, s), & 0 \leqslant s \leqslant \min\{t, \eta\} < 1, \end{cases} \\ \\ \underset{t\in[\eta,1]}{G(t,s)} = \begin{cases} G_{13}(t, s), & 0 \leqslant \max\{t, \eta\} \leqslant s \leqslant 1, \\ G_{14}(t, s), & 0 < \eta \leqslant s \leqslant t \leqslant 1, \end{cases} \end{cases} \tag{2.2}$$

$$G_{11}(t,s) = \frac{1}{\Delta}\Big[\vartheta t^{\alpha-1}(1-s)^{\alpha-\beta-1} - \zeta t^{\alpha-1}(\eta-s)^{\alpha-\beta-1}\Big],$$
$$G_{12}(t,s) = \frac{1}{\Delta}\Big[\vartheta t^{\alpha-1}(1-s)^{\alpha-\beta-1} - \mathcal{N}(t-s)^{\alpha-1} - \zeta t^{\alpha-1}(\eta-s)^{\alpha-\beta-1}\Big],$$
$$G_{13}(t,s) = \frac{1}{\Delta}\Big[\vartheta t^{\alpha-1}(1-s)^{\alpha-\beta-1}\Big],$$
$$G_{14}(t,s) = \frac{1}{\Delta}\Big[\vartheta t^{\alpha-1}(1-s)^{\alpha-\beta-1} - \mathcal{N}(t-s)^{\alpha-1}\Big],$$
$$\textit{and } \mathcal{N} = \vartheta - \zeta\eta^{\alpha-\beta-1}.$$

Proof. Let $y(t) \in C^n[0,1]$ be the solution of fractional order boundary value problem given by (2.1) and (1.2). An equivalent integral equation for (2.1) is given by

$$y(t) = -\int_0^t \frac{(t-s)^{\alpha-1}}{\Gamma(\alpha)} h(s)ds + c_1 t^{\alpha-1} + c_2 t^{\alpha-2} + \cdots + c_n t^{\alpha-n}.$$

Utilizing the conditions (1.2), we obtain $c_n = c_{n-1} = \cdots = c_2 = 0$ and

$$c_1 = \frac{1}{\Delta}\left[\vartheta \int_0^1 (1-s)^{\alpha-\beta-1}h(s)ds - \zeta \int_0^\eta (\eta-s)^{\alpha-\beta-1}h(s)ds\right].$$

Hence the unique solution of the problem given by (2.1) and (1.2) is

$$\begin{aligned} y(t) = & \frac{t^{\alpha-1}}{\Delta}\Big[\vartheta \int_0^1 (1-s)^{\alpha-\beta-1}h(s)ds - \zeta \int_0^\eta (\eta-s)^{\alpha-\beta-1}h(s)ds\Big] \\ & - \frac{\mathcal{N}}{\Delta}\int_0^t (t-s)^{\alpha-1}h(s)ds \\ = & \int_0^1 G(t,s)h(s)ds. \end{aligned}$$

□

Lemma 2.2. *Let $\mathcal{N} > 0$. Then the Green's function $G(t,s)$ given in* (2.2) *is nonnegative, for all* $(t,s) \in [0,1]\times[0,1]$.

Proof. Consider the Green's function $G(t,s)$ given by (2.2).
Let $0 \leqslant t \leqslant s \leqslant \eta \leqslant 1$. Then

$$\begin{aligned} G_{11}(t,s) &= \frac{1}{\Delta}\Big[\vartheta t^{\alpha-1}(1-s)^{\alpha-\beta-1} - \zeta t^{\alpha-1}(\eta-s)^{\alpha-\beta-1}\Big] \\ &\geqslant \frac{1}{\Delta}\Big[\vartheta t^{\alpha-1}(1-s)^{\alpha-\beta-1} - \zeta t^{\alpha-1}(\eta-\eta s)^{\alpha-\beta-1}\Big] \\ &= \frac{t^{\alpha-1}}{\Delta}\Big[\mathcal{N}\Big(1+\beta s + O(s^2)\Big)\Big](1-s)^{\alpha-1} \geqslant 0. \end{aligned}$$

Let $0 \leqslant s \leqslant \min\{t, \eta\} \leqslant 1$. Then

$$\begin{aligned} G_{12}(t,s) &= \frac{1}{\Delta}\Big[\vartheta t^{\alpha-1}(1-s)^{\alpha-\beta-1} - \mathcal{N}(t-s)^{\alpha-1} - \zeta t^{\alpha-1}(\eta-s)^{\alpha-\beta-1}\Big] \\ &\geqslant \frac{1}{\Delta}\Big[\vartheta t^{\alpha-1}(1-s)^{\alpha-\beta-1} - \mathcal{N}(t-ts)^{\alpha-1} - \zeta t^{\alpha-1}(\eta-\eta s)^{\alpha-\beta-1}\Big] \\ &= \frac{t^{\alpha-1}}{\Delta}\Big[\mathcal{N}\Big((1-s)^{-\beta}-1\Big)\Big](1-s)^{\alpha-1} \\ &= \frac{t^{\alpha-1}}{\Delta}\Big[\beta s\mathcal{N} + O(s^2)\Big](1-s)^{\alpha-1} \geqslant 0. \end{aligned}$$

Let $0 \leqslant \max\{t, \eta\} \leqslant s \leqslant 1$. Then

$$G_{13}(t,s) = \frac{1}{\Delta}\Big[\vartheta t^{\alpha-1}(1-s)^{\alpha-\beta-1}\Big] \geqslant 0.$$

Let $0 \leqslant \xi \leqslant s \leqslant t \leqslant 1$. Then

$$\begin{aligned} G_{14}(t,s) &= \frac{1}{\Delta}\Big[\vartheta t^{\alpha-1}(1-s)^{\alpha-\beta-1} - \mathcal{N}(t-s)^{\alpha-1}\Big] \\ &\geqslant \frac{1}{\Delta}\Big[\vartheta t^{\alpha-1}(1-s)^{\alpha-\beta-1} - \mathcal{N}(t-ts)^{\alpha-1}\Big] \\ &= \frac{t^{\alpha-1}}{\Delta}\Big[\vartheta\beta s + \zeta\eta^{\alpha-\beta-1} + O(s^2)\Big](1-s)^{\alpha-1} \geqslant 0. \end{aligned}$$

□

LEMMA 2.3. *Let $\mathcal{N} > 0$ and $\tau \in (0,1)$. Then the Green's function $G(t,s)$ given in* (2.2) *satisfies the inequalities*

$$\begin{aligned} &(P1)\ G(t,s) \leqslant G(1,s),\ \ for\ all\ \ (t,s) \in [0,1]\times[0,1], \\ &(P2)\ G(t,s) \geqslant \tau^{\alpha-1}G(1,s),\ \ for\ all\ \ (t,s) \in [\tau,1]\times[0,1]. \end{aligned}$$

PROOF. Consider the Green's function $G(t,s)$ given by (2.2).
Let $0 \leqslant t \leqslant s \leqslant \eta \leqslant 1$. Then, we have

$$\begin{aligned} \frac{\partial G_{11}(t,s)}{\partial t} &= \frac{(\alpha-1)}{\Delta}\Big[\vartheta t^{\alpha-2}(1-s)^{\alpha-\beta-1} - \zeta t^{\alpha-2}(\eta-s)^{\alpha-\beta-1}\Big] \\ &\geqslant \frac{(\alpha-1)}{\Delta}\Big[\vartheta t^{\alpha-2}(1-s)^{\alpha-\beta-1} - \zeta t^{\alpha-2}(\eta-\eta s)^{\alpha-\beta-1}\Big] \\ &= \frac{(\alpha-1)t^{\alpha-1}}{\Delta}\Big[\mathcal{N}\Big(1+\beta s+O(s^2)\Big)\Big](1-s)^{\alpha-1} \geqslant 0. \end{aligned}$$

Therefore $G_{11}(t,s)$ is increasing in t, which implies $G_{11}(t,s) \leqslant G_{11}(1,s)$.
Let $0 \leqslant s \leqslant \min\{t,\eta\} \leqslant 1$. Then, we have

$$\begin{aligned}
&\frac{\partial G_{12}(t,s)}{\partial t}\\
&= \frac{(\alpha-1)}{\Delta}\Big[\vartheta t^{\alpha-2}(1-s)^{\alpha-\beta-1} - \mathcal{N}(t-s)^{\alpha-2} - \zeta t^{\alpha-2}(\eta-s)^{\alpha-\beta-1}\Big]\\
&\geqslant \frac{(\alpha-1)}{\Delta}\Big[\vartheta t^{\alpha-2}(1-s)^{\alpha-\beta-1} - \mathcal{N}(t-ts)^{\alpha-2} - \zeta t^{\alpha-2}(\eta-\eta s)^{\alpha-\beta-1}\Big]\\
&= \frac{(\alpha-1)t^{\alpha-2}}{\Delta}\Big[\mathcal{N}\Big((1-s)^{-(\beta-1)}-1\Big)\Big](1-s)^{\alpha-2}\\
&= \frac{(\alpha-1)t^{\alpha-2}}{\Delta}\Big[(\beta-1)s\mathcal{N} + O(s^2)\Big](1-s)^{\alpha-2} \geqslant 0.
\end{aligned}$$

Therefore $G_{12}(t,s)$ is increasing in t, which implies $G_{12}(t,s) \leqslant G_{12}(1,s)$.
Let $0 \leqslant \max\{t,\eta\} \leqslant s \leqslant 1$. Then, we have

$$\frac{\partial G_{13}(t,s)}{\partial t} = \frac{(\alpha-1)}{\Delta}\Big[\vartheta t^{\alpha-2}(1-s)^{\alpha-\beta-1}\Big] \geqslant 0.$$

Therefore $G_{13}(t,s)$ is increasing in t, which implies $G_{13}(t,s) \leqslant G_{13}(1,s)$.
Let $0 \leqslant \eta \leqslant s \leqslant t \leqslant 1$. Then, we have

$$\begin{aligned}
\frac{\partial G_{14}(t,s)}{\partial t} &= \frac{(\alpha-1)}{\Delta}\Big[\vartheta t^{\alpha-2}(1-s)^{\alpha-\beta-1} - \mathcal{N}(t-s)^{\alpha-2}\Big]\\
&\geqslant \frac{(\alpha-1)}{\Delta}\Big[\vartheta t^{\alpha-2}(1-s)^{\alpha-\beta-1} - \mathcal{N}(t-ts)^{\alpha-2}\Big]\\
&= \frac{(\alpha-1)t^{\alpha-2}}{\Delta}\Big[\vartheta(\beta-1)s + O(s^2) + \zeta\eta^{\alpha-\beta-1}\Big](1-s)^{\alpha-2} \geqslant 0.
\end{aligned}$$

Therefore $G_{14}(t,s)$ is increasing in t, which implies $G_{14}(t,s) \leqslant G_{14}(1,s)$.
Let $0 \leqslant t \leqslant s \leqslant \eta \leqslant 1$ and $t \in [\tau,1]$. Then

$$\begin{aligned}
G_{11}(t,s) &= \frac{1}{\Delta}\Big[\vartheta t^{\alpha-1}(1-s)^{\alpha-\beta-1} - \zeta t^{\alpha-1}(\eta-s)^{\alpha-\beta-1}\Big]\\
&= \frac{t^{\alpha-1}}{\Delta}\Big[\vartheta(1-s)^{\alpha-\beta-1} - \zeta(\eta-s)^{\alpha-\beta-1}\Big]\\
&= t^{\alpha-1}G_{11}(1,s) \geqslant \tau^{\alpha-1}G_{11}(1,s).
\end{aligned}$$

Let $0 \leqslant s \leqslant \min\{t,\eta\} \leqslant 1$ and $t \in [\tau,1]$. Then

$$\begin{aligned}
G_{12}(t,s) &= \frac{1}{\Delta}\Big[\vartheta t^{\alpha-1}(1-s)^{\alpha-\beta-1} - \mathcal{N}(t-s)^{\alpha-1} - \zeta t^{\alpha-1}(\eta-s)^{\alpha-\beta-1}\Big]\\
&\geqslant \frac{1}{\Delta}\Big[\vartheta t^{\alpha-1}(1-s)^{\alpha-\beta-1} - \mathcal{N}(t-ts)^{\alpha-1} - \zeta t^{\alpha-1}(\eta-s)^{\alpha-\beta-1}\Big]\\
&= \frac{t^{\alpha-1}}{\Delta}\Big[\vartheta(1-s)^{\alpha-\beta-1} - \mathcal{N}(1-s)^{\alpha-1} - \zeta(\eta-s)^{\alpha-\beta-1}\Big]\\
&= t^{\alpha-1}G_{12}(1,s) \geqslant \tau^{\alpha-1}G_{12}(1,s).
\end{aligned}$$

Let $0 \leqslant \max\{t, \eta\} \leqslant s \leqslant 1$ and $t \in [\tau, 1]$. Then

$$\begin{aligned} G_{13}(t,s) &= \frac{1}{\Delta}\Big[\vartheta t^{\alpha-1}(1-s)^{\alpha-\beta-1}\Big] \\ &= t^{\alpha-1}G_{13}(1,s) \geqslant \tau^{\alpha-1}G_{13}(1,s). \end{aligned}$$

Let $0 \leqslant \eta \leqslant s \leqslant t \leqslant 1$ and $t \in [\tau, 1]$. Then

$$\begin{aligned} G_{14}(t,s) &= \frac{1}{\Delta}\Big[\vartheta t^{\alpha-1}(1-s)^{\alpha-\beta-1} - \mathcal{N}(t-s)^{\alpha-1}\Big] \\ &\geqslant \frac{t^{\alpha-1}}{\Delta}\Big[\vartheta(1-s)^{\alpha-\beta-1} - \mathcal{N}(1-s)^{\alpha-1}\Big] \\ &= t^{\alpha-1}G_{14}(1,s) \geqslant \tau^{\alpha-1}G_{14}(1,s), \end{aligned}$$

where $\tau \in (0,1)$ satisfies $\displaystyle\int_\tau^1 G(1,s)ds > 0$. □

3. Existence of Multiple Positive Solutions in a Cone

In this section, we establish the existence of at least three positive solutions to the fractional order boundary value problems (1.1)-(1.2) by using Leggett–Williams fixed point theorem. We also establish the existence of at least $2m-1$ positive solutions to the fractional order boundary value problem (1.1)-(1.2) for an arbitrary positive integer m.

Let a' and b' be be two real numbers such that $0 < a' < b'$ and S be a nonnegative continuous concave functional on a cone P. We define the following convex sets

$$P_{a'} = \Big\{y \in P : \|y\| < a'\Big\} \text{ and } P\big(S, a', b'\big) = \Big\{y \in P : a' \leqslant S(y), \|y\| < b'\Big\}.$$

For $y \in P$, we have

$$S\big(y(t)\big) = \min_{t\in[\tau,1]}\Big\{y(t)\Big\}. \tag{3.1}$$

To establish the existence of multiple positive solutions to the fractional order boundary value problem (1.1)-(1.2) by employing the following Leggett–Williams fixed point theorem.

Theorem 3.1. [**7**] *Let $T : \overline{P}_c \to \overline{P}_c$ be completely continuous and S be a nonnegative continuous concave functional on P such that $S(y) \leqslant \|y\|$ for all $y \in \overline{P}_c$. Suppose that there exist a, b, c, and d with $0 < d < a < b \leqslant c$ such that*

(i) $\big\{y \in P(S,a,b) : S(y) > a\big\} \neq \emptyset$ *and* $S(Ty) > a$ *for* $y \in P(S,a,b)$,

(ii) $\|Ty\| < d$ *for* $\|y\| \leqslant d$,

(iii) $S(Ty) > a$ *for* $y \in P(S,a,c)$ *with* $\|Ty\| > b$.

Then T has at least three fixed points y_1, y_2, y_3 in $\overline{P}_c$ satisfying

$$\|y_1\| < d, a < S(y_2), \|y_3\| > d,\ S(y_3) < a.$$

Consider the Banach space $E = \{y : y \in C[0,1]\}$ equipped with the norm

$$\|y\| = \max_{t\in[0,1]} |y(t)|.$$

Define a cone $P \subset E$ by

$$P = \Big\{ y \in E : y(t) \geqslant 0,\ t \in [0,1] \text{ and } \min_{t\in[\tau,1]} y(t) \geqslant \Phi\|y\| \Big\},$$

where $\Phi = \tau^{\alpha-1}$.

Let $T : P \to E$ be the operator defined by

$$Ty(t) = \int_0^1 G(t,s) f\big(s, y(s)\big) ds,\ t \in [0,1]. \tag{3.2}$$

Lemma 3.1. *The operator T defined by* (3.2) *is a self map on P.*

Proof. Let $y \in P$. Clearly, $Ty(t) \geqslant 0$ for $t \in [0,1]$. Also for $y \in P$,

$$\|Ty\| \leqslant \int_0^1 G(1,s) f\big(s, y(s)\big) ds$$

and

$$\begin{aligned} \min_{t\in[\tau,1]} Ty(t) &= \min_{t\in[\tau,1]} \int_0^1 G(t,s) f\big(s, y(s)\big) ds \\ &\geqslant \tau^{\alpha-1} \int_0^1 G(1,s) f\big(s, y(s)\big) ds \\ &\geqslant \tau^{\alpha-1}\|Ty\| = \Phi\|Ty\|. \end{aligned}$$

Hence, $Ty \in P$ and so $T : P \to P$. Standard arguments involving the Arzela–Ascoli theorem shows that T is completely continuous. □

Let

$$\mathcal{R} = \max_{t\in[0,1]} \left\{ \int_0^1 G(t,s) ds \right\} \text{ and } \mathcal{S} = \min_{t\in[\tau,1]} \left\{ \int_\tau^1 G(t,s) ds \right\}.$$

Theorem 3.2. *Assume that there exist real numbers d_0, d_1 and c with $0 < d_0 < d_1 < \frac{d_1}{\Phi} < c$ such that such that the following hold, such that f satisfies the following conditions:*

$$(A1)\ f\big(t, y(t)\big) < \frac{d_0}{\mathcal{S}},\ for\ t \in [0,1]\ and\ y \in [0, d_0],$$

$$(A2)\ f\big(t, y(t)\big) > \frac{d_1}{\mathcal{R}},\ for\ t \in [\tau,1]\ and\ y \in \left[d_1, \frac{d_1}{\Phi}\right],$$

$$(A3)\ f\big(t, y(t)\big) < \frac{c}{\mathcal{S}},\ for\ t \in [0,1]\ and\ y \in [0, c].$$

Then the fractional order boundary value problem (1.1)-(1.2) *has at least three positive solutions.*

Proof. We seek three fixed points $y_1, y_2, y_3 \in P$ of T defined by (3.2). It is easy to check that S is a nonnegative continuous concave functional on P with $S(y) \leqslant \|y\|$ for $y \in P$ and from Lemma 3.1, the operator T is completely continuous and fixed points of T are solutions of the fractional order boundary value problem (1.1)-(1.2). First we prove that if there exist a positive number r such that $f\big(t, y(t)\big) < \dfrac{r}{\mathcal{S}}$, for $t \in [0,1]$ and $y \in [0,r]$, then $T : \overline{P}_r \to \overline{P}_r$. For $y \in P_r$ and $t \in [0,1]$, we have

$$\begin{aligned}\|Ty\| &= \max_{t\in[0,1]} \left\{ \Big| \int_0^1 G(t,s) f\big(s, y(s)\big) ds \Big| \right\} \\ &\leqslant \frac{r}{\mathcal{S}} \min_{t\in[\tau,1]} \int_\tau^1 G(t,s) ds = r.\end{aligned}$$

Thus $\|Ty\| \leqslant r$. Hence $Ty \in P_r$. Hence, we have shown that if $(A1)$ and $(A3)$ hold then T maps $\overline{P}_{d_0}$ into P_{d_0} and $\overline{P}_c$ into P_c. Next, we show that $\Big\{ y \in P\Big(S, d_1, \frac{d_1}{\Phi}\Big) : S(y) > d_1 \Big\} \neq \emptyset$ and $S(Ty) > d_1$ for all $y \in P\Big(S, d_1, \frac{d_1}{\Phi}\Big)$. In fact, the constant function

$$\frac{d_1 + \dfrac{d_1}{\Phi}}{2} \in \left\{ y \in P\Big(S, d_1, \frac{d_1}{\Phi}\Big) : S(y) > d_1 \right\}.$$

Moreover, for $y \in P\left(S, d_1, \dfrac{d_1}{\Phi}\right)$, we have

$$\frac{d_1}{\Phi} \geqslant \|y\| \geqslant y(t) \geqslant \min_{t\in[\tau,1]} \Big\{ y(t) \Big\} = S(y) \geqslant d_1,$$

for all $t \in [\tau, 1]$. Thus, in view of $(A2)$ we see that

$$\begin{aligned}S\big(y(t)\big) &= \min_{t\in[\tau,1]} \left\{ \int_0^1 G(t,s) f\big(s, y(s)\big) ds \right\} \\ &\geqslant \min_{t\in[\tau,1]} \left\{ \int_\tau^1 G(t,s) f\big(s, y(s)\big) ds \right\} \\ &> \frac{d_1}{\mathcal{R}} \max_{t\in[0,1]} \left\{ \int_0^1 G(t,s) ds \right\} \\ &= d_1,\end{aligned}$$

as required. Finally, we show that $S(Ty) > d_1$ if $y \in P(S, d_1, c)$ and $\|Ty\| > \dfrac{d_1}{\Phi}$. For this, we suppose that $y \in P(S, d_1, c)$ and $\|Ty\| > \dfrac{d_1}{\Phi}$. Then

$$\begin{aligned} S\big(Ty(t)\big) &= \min_{t\in[\tau,1]}\left\{\int_0^1 G(t,s) f\big(s, y(s)\big) ds\right\} \\ &\geqslant \Phi \int_0^1 G(1,s) f\big(s, y(s)\big) ds \\ &\geqslant \max_{t\in[0,1]}\left\{\int_0^1 G(t,s) f\big(s, y(s)\big) ds\right\} \\ &> \frac{d_1}{\mathcal{R}} \max_{t\in[0,1]}\left\{\int_0^1 G(t,s) ds\right\} \\ &= d_1. \end{aligned}$$

Thus, all the conditions of Theorem 3.2 are satisfied. Therefore, the fractional order boundary value problem (1.1)-(1.2) has at least three positive solutions y_1, y_2, y_3 such that

$$\|y_1\| < d_0, d_1 < \min_{t\in[\tau,1]}\big\{y_2\big\}, \|y_3\| > d_0, \min_{t\in[\tau,1]}\big\{y_2\big\} < d_1.$$

□

Now we establish the existence of at least $2m-1$ positive solutions to the fractional order boundary value problem (1.1)-(1.2), by using induction on m.

THEOREM 3.3. *Let m be an arbitrary positive integer. Assume that there exist numbers $d_r(r = 1, 2, 3, \cdots, m)$ and $b_s(s = 1, 2, 3, \cdots, m-1)$ with $0 < d_1 < b_1 < \dfrac{b_1}{\Phi} < d_2 < b_2 < \dfrac{b_2}{\Phi} < \cdots < d_{m-1} < b_{m-1} < \dfrac{b_{m-1}}{\Phi} < d_m$ such that f satisfies the following conditions:*

(A4) $f\big(t, y(t)\big) < \dfrac{d_r}{\mathcal{S}}$, $t \in [0, 1]$ *and* $y \in \big[0, d_r\big]$, $r = 1, 2, 3, \cdots, m$,

(A5) $f\big(t, y(t)\big) > \dfrac{b_s}{\mathcal{R}}$, $t \in [\tau, 1]$ *and* $y \in \left[b_s, \dfrac{b_s}{\Phi}\right]$, $s = 1, 2, 3, \cdots, m-1$.

Then the fractional order boundary value problem (1.1)-(1.2) *has at least* $2m-1$ *positive solutions in* $\overline{P}_{d_m}$.

PROOF. We use induction on m. First, for $m = 1$, we know from the condition $(A4)$ that $T : \overline{P}_{d_1} \to P_{d_1}$, then it follows from the Schauder fixed point theorem that the fractional order boundary value problem (1.1)-(1.2) has at least one positive solution in $\overline{P}_{d_1}$. Next, we assume that this conclusion holds for $m = l$. In order to prove that this conclusion holds for $m = l+1$, we suppose that there exist numbers $d_r(r = 1, 2, 3, \cdots, l+1)$ and $b_s(s = 1, 2, 3, \cdots, l)$ with $0 < d_1 < b_1 < \dfrac{b_1}{\Phi} < d_2 < b_2 <$

$\frac{b_2}{\Phi} < \cdots < d_l < b_l < \frac{b_l}{\Phi} < d_{l+1}$ such that f satisfies the following conditions:

$$\begin{cases} f\big(t, y(t)\big) < \frac{d_r}{\mathcal{S}},\ t \in [0,1] \text{ and } y \in [0, d_r], \\ \qquad\qquad r = 1, 2, 3, \cdots, l+1, \end{cases} \tag{3.3}$$

$$\begin{cases} f\big(t, y(t)\big) > \frac{b_s}{\mathcal{R}},\ t \in [\tau, 1] \text{ and } y \in \Big[b_s, \frac{b_s}{\Phi}\Big], \\ \qquad\qquad s = 1, 2, 3, \cdots, l. \end{cases} \tag{3.4}$$

By assumption $m = l$, the fractional order boundary value problem (1.1)-(1.2) has at least $2l - 1$ positive solutions $y_i^*,\ i = 1, 2, 3, \cdots, 2l-1$ in $\overline{P}_{d_l}$. At the same time, it follows from Theorem 3.2, (3.3) and (3.4) that the fractional order boundary value problem (1.1)-(1.2) has at least three positive solutions y_1, y_2 and y_3 in $\overline{P}_{d_{l+1}}$ such that $\|y_1\| < d_l,\ b_l < \min_{t\in[0,1]} y_2(t), \|y_3\| > d_l,\ \min_{t\in[0,1]} y_3(t) < d_l$. Obviously, y_2 and y_3 are distinct from $y_i^*,\ i = 1, 2, 3, \cdots, 2l-1$ in $\overline{P}_{d_l}$. Therefore the fractional order boundary value problem (1.1)-(1.2) has at least $2l + 1$ positive solutions in $\overline{P}_{d_{l+1}}$ which shows that this conclusion also holds for $m = l + 1$. This completes the proof. □

4. Example

In this section, as an application, we demonstrate our results with an example.

Consider the fractional order three-point boundary value problem

$$D_{0^+}^{2.8} y(t) + f(t, y) = 0,\ t \in (0, 1), \tag{4.1}$$

$$y(0) = 0,\ y'(0) = 0,\ 7D_{0^+}^{1.6} y(1) - \frac{5}{2} D_{0^+}^{1.6} y\Big(\frac{1}{2}\Big) = 0, \tag{4.2}$$

where

$$f(t, y) = \begin{cases} \frac{\sqrt{1-t^2}}{165} + 12[y^2 - y] + \frac{1}{196}, & 0 \leqslant y \leqslant 2, \\ \frac{\sqrt{1-t^2}}{165} + 8[log_2 y + y] + \frac{1}{196}, & y > 2. \end{cases}$$

Then, the Green's function $G(t, s)$ for the associated linear fractional order boundary value problem is given by

$$\begin{aligned} G_{11}(t, s) =& \frac{1}{8.05}\Big[7t^{1.8}(1-s)^{0.2} - \frac{1}{2} t^{1.8}\Big(\frac{1}{2} - s\Big)^{0.2}\Big], \\ G_{12}(t, s) =& \frac{1}{8.05}\Big[7t^{1.8}(1-s)^{0.2} - 4.82(t-s)^{1.8} - \frac{1}{2} t^{1.8}\Big(\frac{1}{2} - s\Big)^{0.2}\Big], \\ G_{13}(t, s) =& \frac{1}{8.05}\Big[7t^{1.8}(1-s)^{0.2}\Big], \\ G_{14}(t, s) =& \frac{1}{8.05}\Big[7t^{1.8}(1-s)^{0.2} - 4.82(t-s)^{1.8}\Big]. \end{aligned}$$

Clearly, the Green's function $G(t,s)$ is positive and f is continuous and increasing on $[0,\infty)$. By direct calculations, we get $\Phi = 0.02368, \mathcal{R} = 0.085341, \mathcal{S} = 0.068697$. If we choose $d_0 = 1.5, d_1 = 2.05$ and $c = 1500$, then $0 < d_0 < d_1 < \dfrac{d_1}{\Phi} \leqslant c$ and f satisfies

$$(i)\ f(t,y) < 21.83501 = \frac{d_0}{\mathcal{S}},\ t \in [0,1] \text{ and } y \in [0,1.5]\,,$$

$$(ii)\ f(t,y) > 24.02128 = \frac{d_1}{\mathcal{R}},\ t \in [0.65,1] \text{ and } y \in \big[2.05, 86.57095\big],$$

$$(iii)\ f(t,y) < 21835.01463 = \frac{c}{\mathcal{S}},\ t \in [0,1] \text{ and } y \in [0,1500]\,.$$

Then, all the conditions of Theorem 3.2 are satisfied. Therefore, by Theorem 3.2, the fractional order boundary value problem (4.1)-(4.2) has at least three positive solutions.

References

[1] B. Ahmed and J. J. Nieto, *Existence results for a coupled system of nonlinear fractional differential equations with three-point boundary conditions,* Comput. Math. Appl., **58** (2009), 1838–1843.

[2] Z. Bai and H. Lü , *Positive solutions for boundary value problems of nonlinear fractional differential equations,* J. Math. Anal. Appl., **311** (2005), 495–505.

[3] M. Benchohra, J. Henderson, S. K. Ntoyuas and A. Ouahab, *Existence results for fractional order functional differential equations with infinite delay,* J. Math. Anal. Appl., **338** (2008), 1340–1350.

[4] K. Diethelm and N. Ford, *Analysis of fractional differential equations,* J. Math. Anal. Appl., **265** (2002), 229–248.

[5] E. R. Kauffman and E. Mboumi, *Positive solutions of a boundary value problem for a nonlinear fractional differential equation,* Elec. J. Qual. Theory Diff. Equ., **3** (2008), 1–11.

[6] A. A. Kilbas, H. M. Srivasthava and J. J. Trujillo, *Theory and Applications of Fractional Differential Equations,* North-Holland Mathematics Studies, vol. 204, Elsevier Science, Amsterdam, 2006.

[7] R. W. Leggett and L. R. Williams, *Multiple positive fixed points of nonlinear operator on order Banach spaces,* Indiana Univer. Math. J., **28** (1979), 673–688.

[8] K. S. Miller and B. Ross, *An Introduction to Fractional Calculus and Fractional Differential Equations,* John Wiley and Sons, New York, 1993.

[9] I. Podulbny, *Fractional Diffrential Equations,* Academic Press, San Diego, 1999.

[10] K. R. Prasad and B. M. B. Krushna, *Multiple positive solutions for the system of (n,p)-type fractional order boundary value problems,* Bull. Int. Math. Virtual Inst., **5** (2015), 1–12.

[11] K. R. Prasad and B. M. B. Krushna, *Eigenvalues for iterative systems of Sturm–Liouville fractional order two-point boundary value problems,* Fract. Calc. Appl. Anal., **17**(2014), 638–653, DOI: 10.2478/s13540-014-0190-4.

[12] K. R. Prasad, B. M. B. Krushna and N. Sreedhar, *Eigenvalues for iterative systems of (n,p)-type fractional order boundary value problems,* Int. J. Anal. Appl., **5** (2014), 136–146.

[13] K. R. Prasad and B. M. B. Krushna, *Lower and upper solutions for general two-point fractional order boundary value problems,* TWMS J. App. Eng. Math., **5**(2015), 80–87.

Received by editors 07.08.2015; Revised version 01.11.2015;
Available online 09.11.2015.

[1]Department of Applied Mathematics, Andhra University, Visakhapatnam, 530 003, India

E-mail address: rajendra92@rediffmail.com

[2]Department of Mathematics, MVGR College of Engineering, Vizianagaram, 535 005, India

E-mail address: muraleebalu@yahoo.com

[3]Department of Mathematics,, Jimma University, Jimma, Oromia, 378, Ethiopia

E-mail address: wesen08@gmail.com