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A UNIQUE COMMON FIXED POINT THEOREM FOR FOUR MAPS WITH RATIONAL INEQUALITY USING α - ADMISSIBLE FUNCTIONS IN ORDERED PARTIAL METRIC SPACES

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Abstract. In this paper, we obtain a unique common fixed point theorem for four self maps satisfying rational (ψ, ϕ, φ)-contractive condition using α-admissible function in ordered partial metric spaces. Also we give an example to illustrate our main theorem.

1. Introduction and Preliminaries

There are many generalizations of the concept of metric spaces in the literature. One of them is a partial metric space introduced by Matthews [**16**] as a part of study of denotational semantics of data flow networks. After that fixed and common fixed point results in partial metric spaces were studied by many other authors, for example,[**1, 3, 4, 5, 7, 9, 10, 11, 13, 15, 17, 18**].

Throughout this paper, $\mathcal{R}^+$ and $\mathcal{N}$ denote the set of all non-negative real numbers and set of all positive integers respectively.
First we recall some basic definitions and lemmas which play crucial role in the theory of partial metric spaces.

Definition 1.1. ([**16**]) A partial metric on a nonempty set X is a function $p : X \times X \to \mathcal{R}^+$ such that for all $x, y, z \in X$:

(p_1) $x = y \Leftrightarrow p(x,x) = p(x,y) = p(y,y)$,
(p_2) $p(x,x) \leqslant p(x,y), p(y,y) \leqslant p(x,y)$,
(p_3) $p(x,y) = p(y,x)$,
(p_4) $p(x,y) \leqslant p(x,z) + p(z,y) - p(z,z)$.

The pair (X, p) is called a partial metric space (PMS).

If p is a partial metric on X, then the function $p^s : X \times X \to \mathcal{R}^+$ given by $p^s(x,y) = 2p(x,y) - p(x,x) - p(y,y)$ is a metric on X. It is clear that
(i) $p(x,y) = 0 \Rightarrow x = y$,

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(ii) $x \neq y \Rightarrow p(x,y) > 0$ and
(iii) $p(x,x)$ may not be 0.

EXAMPLE 1.1. (See [**4, 16, 18**]) Consider $X = \mathcal{R}^+$ with $p(x,y) = \max\{x,y\}$. Then (X,p) is a partial metric space. It is clear that p is not a usual metric. Note that in this case $p^s(x,y) = |x-y|$.

We now state some basic topological notions (such as convergence, completeness, continuity) on partial metric spaces (see [**4, 5, 9, 16, 18**]).

DEFINITION 1.2. (i) A sequence $\{x_n\}$ in the PMS (X,p) converges to the limit x if and only if $p(x,x) = \lim_{n\to\infty} p(x,x_n)$.
(ii) A sequence $\{x_n\}$ in the PMS (X,p) is called a Cauchy sequence if $\lim_{n,m\to\infty} p(x_n,x_m)$ exists and is finite.
(iii) A PMS (X,p) is called complete if every Cauchy sequence $\{x_n\}$ in X converges with respect to τ_p, to a point $x \in X$ such that $p(x,x) = \lim_{n,m\to\infty} p(x_n,x_m)$.
(iv) A mapping $F : X \to X$ is said to be continuous at $x \in X$ if for every $\epsilon > 0$ there exists $\delta > 0$ such that $F(B_p(x,\delta)) \subseteq B_p(Fx,\epsilon)$.

It is clear that if F is continuous at $x \in X$ then $\{Fx_n\}$ converges to Fx whenever the sequence $\{x_n\} \in X$ converges to x.

The following lemma is one of the basic results in PMS ([**4, 5, 9, 16, 18**]).

LEMMA 1.1.

(i) *A sequence $\{x_n\}$ is a Cauchy sequence in the PMS (X,p) if and only if it is a Cauchy sequence in the metric space (X,p^s).*

(ii) *A PMS (X,p) is complete if and only if the metric space (X,p^s) is complete. Moreover*
$$\lim_{n,m\to\infty} p^s(x,x_n) = 0 \Leftrightarrow p(x,x) = \lim_{n\to\infty} p(x,x_n) = \lim_{n,m\to\infty} p(x_n,x_m).$$

Next, we give a simple lemma which will be used in the proof of our main result. For the proof we refer to [**18**].

LEMMA 1.2. *Assume $x_n \to z$ as $n \to \infty$ in a PMS (X,p) such that $p(z,z) = 0$. Then $\lim_{n\to\infty} p(x_n,y) = p(z,y)$ for every $y \in X$.*

DEFINITION 1.3. ([**1**]) Let (X,p) be a partial metric space and $F,g : X \to X$.Then the pair (F,g) is said to be partial compatible if the following conditions hold:
(i) $p(x,x) = 0 \Rightarrow p(gx,gx) = 0$ whenever $x \in X$,
(ii) $\lim_{n\to\infty} p(Fgx_n, gFx_n) = 0$ whenever there exists a sequence $\{x_n\}$ in X such that $Fx_n \to t$ and $gx_n \to t$ for some $t \in X$.

We observe that the above definition is not sufficient. Hence we added $p(t,t) = 0$ in (ii) and we call the pair (F,g) as partial$^{(*)}$ compatible pair.

DEFINITION 1.4. ([**8**]) Let X be a non-empty set and $f,S : X \to X$.The pair (f,S) is said to be weakly compatible if $fSu = Sfu$ whenever $fu = Su$ for $u \in X$.

Samet et al. [**2**] introduced the notion of α- admissible mappings as follows:

Definition 1.5. ([**2**]) Let X be a non empty set, $T : X \to X$ and $\alpha : X \times X \to \mathcal{R}^+$ be mappings. Then T is called α- admissible if for all $x, y \in X$, we have $\alpha(x, y) \geqslant 1$ implies $\alpha(Tx, Ty) \geqslant 1$.

Some interesting examples of such mappings are given in ([**2**]). Actually they proved the following:

Theorem 1.1 ([**2**]). *Let (X, d) be a complete metric space.Suppose that $\alpha : X \times X \to \mathcal{R}^+$ and $\phi : \mathcal{R}^+ \to \mathcal{R}^+$, where ϕ is non-decreasing and $\sum \phi^n(t) < \infty$ for each $t > 0$. Suppose that $T : X \to X$ satisfies $\alpha(x, y)d(Tx, Ty) \leqslant \phi(d(x, y))$ for all $x, y \in X$.*
Assume the following:
(i) T is α-admissible,
(ii) there exits $x_0 \in X$ such that $\alpha(x_0, Tx_0) \geqslant 1$,
(iii) either T is continuous or if $\{x_n\}$ is a sequence in X with $\alpha(x_n, x_{n+1}) \geqslant 1$ for all $n \in \mathcal{N}$ and $x_n \to x$ as $n \to \infty$, then $\alpha(x_n, x) \geqslant 1$ for all $n \in \mathcal{N}$.
Then T has a fixed point in X. Further, if for any $x, y \in X$, there exists $z \in X$ such that $\alpha(x, z) \geqslant 1$ and $\alpha(y, z) \geqslant 1$ then T has a unique fixed point in X.

Recently, Karapinar et al. [**6**] defined the notion of triangular α- admissible mappings as follows:

Definition 1.6. ([**6**]) Let X be a non empty set, $T : X \to X$ and $\alpha : X \times X \to \mathcal{R}^+$. Then T is called triangular α- admissible if

(i) $x, y \in X$, $\alpha(x, y) \geqslant 1 \Rightarrow \alpha(Tx, Ty) \geqslant 1$,
(ii) $x, y, z \in X$, $\alpha(x, z) \geqslant 1$ and $\alpha(z, y) \geqslant 1 \Rightarrow \alpha(x, y) \geqslant 1$.

Later Shahi et al. [**14**] and Abdeljawad [**19**] defined the following:

Definition 1.7. ([**14**]) Let X be a non empty set, $\alpha : X \times X \to \mathcal{R}^+$ and $f, g : X \to X$. Then f is said to be α- admissible with respect to g if $\alpha(gx, gy) \geqslant 1$ implies $\alpha(fx, fy) \geqslant 1$ for all $x, y \in X$.

Definition 1.8. ([**19**]) Let X be a non empty set, $\alpha : X \times X \to \mathcal{R}^+$ and $f, g : X \to X$. Then the pair (f, g) is said to be α-admissible if $\alpha(x, y) \geqslant 1$ implies $\alpha(fx, gy) \geqslant 1$ and $\alpha(gx, fy) \geqslant 1$ for all $x, y \in X$.

Using these definitions, we introduce the following:

Definition 1.9. Let X be a non empty set, $\alpha : X \times X \to \mathcal{R}^+$ and $f, g, S, T : X \to X$. The pair (f, g) is said to be α- admissible w.r.to the pair (S, T) if $\alpha(Sx, Ty) \geqslant 1$ implies $\alpha(fx, gy) \geqslant 1$ and $\alpha(Tx, Sy) \geqslant 1$ implies $\alpha(gx, fy) \geqslant 1$ for all $x, y \in X$.

Definition 1.10. (f, g) is called triangular α- admissible w.r.to (S, T) if

(i) (f, g) is α- admissible w.r.t. (S, T) and
(ii) $\alpha(x, y) \geqslant 1$ and $\alpha(y, z) \geqslant 1 \Rightarrow \alpha(x, z) \geqslant 1$ for all $x, y, z \in X$.

Recently, Abbas et al. [**12**] introduced the new concepts in a partially ordered set as follows

Definition 1.11. ([**12**]) Let $(X, \preceq)$ be a partially ordered set and $f, g : X \to X$. Then
(i) f is said to be a dominating map if $x \preceq fx$.
(ii) f is said to be a weak annihilator of g if $fgx \preceq x$.

Definition 1.12. Let $\psi : \mathcal{R}^+ \to \mathcal{R}^+$. The function ψ is called an altering distance function if it is non-decreasing, continuous and $\psi(t) = 0 \Leftrightarrow t = 0$.

2. Main Result

Now we prove our main result.

Theorem 2.1. *Let $(X, p, \preceq)$ be a partially ordered complete partial metric space and $\alpha : X \times X \to \mathcal{R}^+$ be a function. Let f, g, S and T be self mappings on X satisfying:*

(2.1.1) *f and g are dominating maps and f and g are weak annihilators of T and S respectively,*

(2.1.2) *$f(X) \subseteq T(X)$, $g(X) \subseteq S(X)$,*

(2.1.3) *$\alpha(Sx, Ty)\ \psi(p(fx, gy)) \leqslant \phi(M(x, y)) - \varphi(M(x, y))$ for all comparable elements $x, y \in X$, where*

$$M(x, y) = \max\left\{\frac{p(Ty, gy)[1 + p(fx, Sx)]}{1 + p(Sx, Ty)}, p(Sx, Ty)\right\}$$

and $\psi, \phi, \varphi : \mathcal{R}^+ \to \mathcal{R}^+$ are such that ψ is an altering distance function and ϕ and φ are upper and lower semi continuous respectively, $\phi(0) = 0, \varphi(0) = 0$ and satisfying the following condition:

$$\psi(t) - \phi(t) + \varphi(t) > 0 \text{ for } t > 0........(A),$$

(2.1.4) *the pair (f, g) is triangular α-admissible w.r.to the pair (S, T),*

(2.1.5) *$\alpha(Sx_1, fx_1) \geqslant 1$ and $\alpha(fx_1, Sx_1) \geqslant 1$ for some $x_1 \in X$,*

(2.1.6)(a) *S is continuous,the pair (f, S) is partial$^{(*)}$ compatible and the pair (g, T) is weakly compatible and if there exists a sequence $\{y_n\}$ in X such that $\alpha(y_n, y_{n+1}) \geqslant 1$, $\alpha(y_{n+1}, y_n) \geqslant 1$ for all $n \in \mathcal{N}$ and $y_n \to z$ for some $z \in X$, then we have $\alpha(Sy_{2n}, y_{2n-1}) \geqslant 1, \alpha(z, y_{2n-1}) \geqslant 1, \alpha(z, z) \geqslant 1$ and $\alpha(z, Tz) \geqslant 1$,*

(or)

(2.1.6)(b) *T is continuous,the pair (g, T) is partial$^{(*)}$ compatible and the pair (f, S) is weakly compatible and if there exists a sequence $\{y_n\}$ in X such that $\alpha(y_n, y_{n+1}) \geqslant 1$, $\alpha(y_{n+1}, y_n) \geqslant 1$ for all $n \in \mathcal{N}$ and $y_n \to z$ for some $z \in X$, then we have $\alpha(y_{2n}, Ty_{2n-1}) \geqslant 1, \alpha(y_{2n}, z) \geqslant 1, \alpha(z, z) \geqslant 1$ and $\alpha(Sz, z) \geqslant 1$,*

(2.1.7) *if for a non-decreasing sequence $\{x_n\}$ in X with $x_n \preceq y_n, \forall n \in \mathcal{N}$ and $y_n \to u$ implies $x_n \preceq u, \forall n \in \mathcal{N}$.*

Then f, g, S and T have a common fixed point in X .

(2.1.8) *Further if we assume that* $\alpha(u,v) \geqslant 1$ *whenever* u *and* v *are common fixed points of* f, g, S *and* T *and the set of common fixed points of* f, g, S *and* T *is well ordered then* f, g, S *and* T *have unique common fixed point in* X.

PROOF. From (2.1.5), we have $\alpha(Sx_1, fx_1) \geqslant 1$ for some $x_1 \in X$. From (2.1.2), there exist sequences $\{x_n\}$ and $\{y_n\}$ as follows:
$y_{2n+1} = fx_{2n+1} = Tx_{2n+2}$, $y_{2n+2} = gx_{2n+2} = Sx_{2n+3}$, $n = 0, 1, 2, \cdots$.

Now

$$\begin{aligned}
\alpha(Sx_1, fx_1) \geqslant 1 \ &\Rightarrow \alpha(Sx_1, Tx_2) \geqslant 1, \text{ from definition of } \{y_n\} \\
&\Rightarrow \alpha(fx_1, gx_2) \geqslant 1, \text{ from (2.1.4), } i.e\ \alpha(y_1, y_2) \geqslant 1 \\
&\Rightarrow \alpha(Tx_2, Sx_3) \geqslant 1, \text{ from definition of } \{y_n\} \\
&\Rightarrow \alpha(gx_2, fx_3) \geqslant 1, \text{ from (2.1.4), } i.e\ \alpha(y_2, y_3) \geqslant 1 \\
&\Rightarrow \alpha(Sx_3, Tx_4) \geqslant 1, \text{ from definition of } \{y_n\} \\
&\Rightarrow \alpha(fx_3, gx_4) \geqslant 1, \text{ from (2.1.4), } i.e\ \alpha(y_3, y_4) \geqslant 1.
\end{aligned}$$

Continuing in this way, we have

$$\alpha(y_n, y_{n+1}) \geqslant 1, \ \forall\, n \in \mathcal{N}. \tag{2.1}$$

Similarly, by using $\alpha(fx_1, Sx_1) \geqslant 1$, we can show that

$$\alpha(y_{n+1}, y_n) \geqslant 1, \ \forall\, n \in \mathcal{N} \tag{2.2}$$

From (2.1.4), using triangular property, we have

$$\alpha(y_m, y_n) \geqslant 1 \text{ for } m < n. \tag{2.3}$$

From(2.1.1), we have
$x_{2n+1} \preceq fx_{2n+1} = Tx_{2n+2} \preceq fTx_{2n+2} \preceq x_{2n+2}$,
$x_{2n+2} \preceq gx_{2n+2} = Sx_{2n+3} \preceq gSx_{2n+3} \preceq x_{2n+3}$.

Thus

$$x_n \preceq x_{n+1}, \forall n \in \mathcal{N} \tag{2.4}$$

Case (i): Suppose $y_{2m} = y_{2m+1}$ for some m. Assume that $y_{2m+1} \neq y_{2m+2}$. i.e. $p(y_{2m+1}, y_{2m+2}) > 0$. Now $\alpha(Sx_{2m+1}, Tx_{2m+2}) = \alpha(y_{2m}, y_{2m+1}) \geqslant 1$, from (2.1). From (2.1.3) and (2.4), we have

$$\begin{aligned}
\psi(p(y_{2m+1}, y_{2m+2})) &= \psi(p(fx_{2m+1}, gx_{2m+2})), \\
&\leqslant \alpha(Sx_{2m+1}, Tx_{2m+2})\psi(p(fx_{2m+1}, gx_{2m+2})), \\
&\leqslant \phi(M(x_{2m+1}, x_{2m+2})) - \varphi(M(x_{2m+1}, x_{2m+2}))
\end{aligned} \tag{2.5}$$

where

$$M(x_{2m+1}, x_{2m+2}) \quad = \max\left\{ \tfrac{p(y_{2m+1}, y_{2m+2})[1+p(y_{2m}, y_{2m+1})]}{1+p(y_{2m}, y_{2m+1})}, p(y_{2m}, y_{2m+1}) \right\}.$$

But from (p_2) we have
$p(y_{2m}, y_{2m+1}) = p(y_{2m+1}, y_{2m+1}) \leqslant p(y_{2m+1}, y_{2m+2})$. Hence $M(x_{2m+1}, x_{2m+2}) = p(y_{2m+1}, y_{2m+2})$. Now (2.5) becomes

$$\psi(p(y_{2m+1}, y_{2m+2})) \leqslant \phi(p(y_{2m+1}, y_{2m+2})) - \varphi(p(y_{2m+1}, y_{2m+2})).$$

It is a contradiction to (A). Hence $y_{2m+1} = y_{2m+2}$. Continuing in this way we can conclude that $y_n = y_{n+k}$ for all positive integers k. Thus $\{y_n\}$ is a Cauchy sequence in X.

Case (ii): Suppose that $y_n \neq y_{n+1}$ for all $n \in \mathcal{N}$. Now from (2.1) follows $\alpha(Sx_{2n+1}, Tx_{2n+2}) = \alpha(y_{2n}, y_{2n+1}) \geqslant 1$. As in Case (i), we have

$$\psi(p(y_{2n+1}, y_{2n+2})) \leqslant \phi(M(x_{2n+1}, x_{2n+2})) - \varphi(M(x_{2n+1}, x_{2n+2}))$$

where

$$M(x_{2n+1}, x_{2n+2}) = \max\{p(y_{2n}, y_{2n+1}), p(y_{2n+1}, y_{2n+2})\}.$$

If $M(x_{2n+1}, x_{2n+2}) = p(y_{2n+1}, y_{2n+2})$, then

$$\psi(p(y_{2n+1}, y_{2n+2})) \leqslant \phi(p(y_{2n+1}, y_{2n+2})) - \varphi(p(y_{2n+1}, y_{2n+2})).$$

It is a contradiction to (A). Hence

$$\begin{aligned} \psi(p(y_{2n+1}, y_{2n+2})) &\leqslant \phi(p(y_{2n}, y_{2n+1})) - \varphi(p(y_{2n}, y_{2n+1})) \\ &< \psi(p(y_{2n}, y_{2n+1})), \text{ from(A)}. \end{aligned} \tag{2.6}$$

Since ψ is increasing, we have $p(y_{2n+1}, y_{2n+2}) \leqslant p(y_{2n}, y_{2n+1})$. Similarly using (2.2), we can show that $p(y_{2n}, y_{2n+1}) \leqslant p(y_{2n-1}, y_{2n})$. Thus $\{p(y_n, y_{n+1})\}$ is a decreasing sequence of non-negative real numbers and hence converges to some real number $r \geqslant 0$. Hence

$$\lim_{n\to\infty} p(y_n, y_{n+1}) = r.$$

Suppose $r > 0$. Letting $n \to \infty$ in (2.6), we get $\psi(r) \leqslant \phi(r) - \varphi(r)$. It is a contradiction to (A). Hence $r = 0$. Thus

$$\lim_{n\to\infty} p(y_n, y_{n+1}) = 0 \tag{2.7}$$

From (p_2), we have

$$\lim_{n\to\infty} p(y_n, y_n) = 0 \tag{2.8}$$

By the definition of p^s, (2.7) and (2.8), we have

$$\lim_{n\to\infty} p^s(y_n, y_{n+1}) = 0 \tag{2.9}$$

Now we prove that $\{y_{2n}\}$ is a Cauchy sequence in (X, p^s).

On contrary, suppose that $\{y_{2n}\}$ is not a Cauchy sequence. Then there exist $\epsilon > 0$ and monotone increasing sequences of natural numbers $\{y_{2m_k}\}$ and $\{y_{2n_k}\}$ such that $n_k > m_k$,

$$p^s(y_{2m_k}, y_{2n_k}) \geqslant \epsilon \tag{2.10}$$

$$p^s(y_{2m_k}, y_{2n_k-2}) < \epsilon. \tag{2.11}$$

Now from (2.10) and (2.11), we obtain

$$\begin{aligned} \epsilon \leqslant p^s(y_{2m_k}, y_{2n_k}) &\leqslant p^s(y_{2m_k}, y_{2n_k-2}) + p^s(y_{2n_k-2}, y_{2n_k-1}) + p^s(y_{2n_k-1}, y_{2n_k}) \\ &< \epsilon + p^s(y_{2n_k-2}, y_{2n_k-1}) + p^s(y_{2n_k-1}, y_{2n_k}). \end{aligned}$$

Letting $k \to \infty$ and using (2.9), we get

$$\lim_{k\to\infty} p^s(y_{2m_k}, y_{2n_k}) = \epsilon. \tag{2.12}$$

Hence, from the definition of p^s and (2.8), we have

$$\lim_{k\to\infty} p(y_{2m_k}, y_{2n_k}) = \frac{\epsilon}{2}. \tag{2.13}$$

Letting $k \to \infty$ and then using (2.12) and (2.9), in

$$|p^s(y_{2m_k+1}, y_{2n_k}) - p^s(y_{2m_k}, y_{2n_k})| \leqslant p^s(y_{2m_k}, y_{2m_k+1}),$$
$$|p^s(y_{2m_k}, y_{2n_k-1}) - p^s(y_{2m_k}, y_{2n_k})| \leqslant p^s(y_{2n_k-1}, y_{2n_k})$$

we obtain upon using definition of p^s and (2.8) that

$$\lim_{k\to\infty} p(y_{2m_k+1}, y_{2n_k}) = \frac{\epsilon}{2}, \tag{2.14}$$

$$\lim_{k\to\infty} p(y_{2m_k}, y_{2n_k-1}) = \frac{\epsilon}{2}. \tag{2.15}$$

Also $\alpha(Sx_{2m_k+1}, Tx_{2n_k}) = \alpha(y_{2m_k}, y_{2n_k-1}) \geqslant 1$, from (2.3). Hence from (2.1.3) and (2.4), we have

$$\begin{aligned} \psi(p(y_{2m_k+1}, y_{2n_k})) &= \psi(p(fx_{2m_k+1}, gx_{2n_k})) \\ &\leqslant \alpha(Sx_{2m_k+1}, Tx_{2n_k})\ \psi(p(fx_{2m_k+1}, gx_{2n_k})) \\ &\leqslant \phi(M(x_{2m_k+1}, x_{2n_k})) - \varphi(M(x_{2m_k+1}, x_{2n_k})) \end{aligned} \tag{2.16}$$

where

$$\begin{aligned} M(x_{2m_k+1}, x_{2n_k}) &= \max\left\{ \tfrac{p(y_{2n_k-1}, y_{2n_k})[1+p(y_{2m_k}, y_{2m_k-1})]}{1+p(y_{2m_k}, y_{2n_k-1})}, p(y_{2m_k}, y_{2n_k-1}) \right\} \\ &\to \tfrac{\epsilon}{2} \text{ as } k \to \infty, \text{ from (2.15), (2.7)} \end{aligned}$$

Letting $k \to \infty$ in (2.16) and using (2.14) we obtain

$$\psi(\frac{\epsilon}{2}) \leqslant \phi(\frac{\epsilon}{2}) - \varphi(\frac{\epsilon}{2}).$$

It is a contradiction to (A). Hence $\{y_{2n}\}$ is a Cauchy sequence in (X, p^s).

Letting $n \to \infty$, $m \to \infty$ and using (2.9) in

$$|p^s(y_{2n+1}, y_{2m+1}) - p^s(y_{2m}, y_{2n})| \leqslant p^s(y_{2n+1}, y_{2n}) + p^s(y_{2m}, y_{2m+1}),$$

we obtain $\lim_{n\to\infty} p^s(y_{2n+1}, y_{2m+1}) = 0$. Hence $\{y_{2n+1}\}$ is a Cauchy sequence in (X, p^s). Thus $\{y_n\}$ is a Cauchy sequence in (X, p^s). Hence we have

$$\lim_{n\to\infty} p^s(y_n, y_m) = 0$$

and hence from definition of p^s and (2.8), we have

$$\lim_{n\to\infty} p(y_n, y_m) = 0 \tag{2.17}$$

Thus $\{y_n\}$ is a Cauchy sequence in (X, p). Since (X, p) is a complete partial metric space, there exists $z \in X$ such that $p(z, z) = \lim_{n\to\infty} p(y_n, y_m)$.

From (2.17),

$$p(z, z) = 0. \tag{2.18}$$

Hence

$$p(z,z) = \lim_{n\to\infty} p(fx_{2n+1}, z) = \lim_{n\to\infty} p(gx_{2n+2}, z) = \lim_{n\to\infty} p(Sx_{2n+1}, z) = \lim_{n\to\infty} p(Tx_{2n+2}, z) = 0. \tag{2.19}$$

Suppose (2.1.6)(a) holds. Since the pair (f,S) is partial$^{(*)}$ compatible,from (2.18), we have $p(Sz,Sz) = 0$.
and

$$\lim_{n\to\infty} p(fSx_{2n+1}, Sfx_{2n+1}) = 0. \tag{2.20}$$

Since S is continuous at z, we have

$$\lim_{n\to\infty} p(SSx_{2n+1}, Sz) = p(Sz,Sz) = 0 \tag{2.21}$$

and

$$\lim_{n\to\infty} p(Sfx_{2n+1}, Sz) = p(Sz,Sz) = 0. \tag{2.22}$$

Also $p(fSx_{2n+1}, Sz) \leqslant p(fSx_{2n+1}, Sfx_{2n+1}) + p(Sfx_{2n+1}, Sz)$. Now by using (2.20) and (2.22), we have $\lim_{n\to\infty} p(fSx_{2n+1}, Sz) \leqslant 0$. Hence

$$\lim_{n\to\infty} p(fSx_{2n+1}, Sz) = 0. \tag{2.23}$$

Now $p(fSx_{2n+1}, SSx_{2n+1}) \leqslant p(fSx_{2n+1}, Sz) + p(Sz, SSx_{2n+1})$.

$$\lim_{n\to\infty} p(fSx_{2n+1}, SSx_{2n+1}) \leqslant 0 \; from \; (2.23) \; and \; (2.21).$$

Hence

$$\lim_{n\to\infty} p(fSx_{2n+1}, SSx_{2n+1}) = 0. \tag{2.24}$$

Letting $n \to \infty$ and using (2.23), (2.19) and Lemma 1.2 in

$$|p(fSx_{2n+1}, gx_{2n}) - p(z,Sz)| \leqslant p(fSx_{2n+1}, Sz) + p(z, gx_{2n}),$$

we get

$$\lim_{n\to\infty} p(fSx_{2n+1}, gx_{2n}) = p(Sz, z). \tag{2.25}$$

Letting $n \to \infty$ and using (2.21), (2.19) and Lemma 1.2 in

$$|p(SSx_{2n+1}, Tx_{2n}) - p(Sz,z)| \leqslant p(SSx_{2n+1}, Sz) + p(z, Tx_{2n}),$$

we get

$$\lim_{n\to\infty} p(SSx_{2n+1}, Tx_{2n}) = p(Sz, z). \tag{2.26}$$

Clearly $\alpha(SSx_{2n+1}, Tx_{2n}) = \alpha(Sy_{2n}, y_{2n-1}) \geqslant 1$, from (2.1.6)($a$). From (2.1.1), we have $x_{2n} \preceq gx_{2n} = Sx_{2n+1}$.

(2.27)

$$\begin{aligned}
\psi(p(Sz,z)) &= \psi(\lim_{n\to\infty} p(fSx_{2n+1}, gx_{2n})), from (2.25) \; and \; Lemma \; 1.2 \\
&\leqslant \lim_{n\to\infty} \alpha(SSx_{2n+1}, Tx_{2n}) \; \psi(p(fSx_{2n+1}, gx_{2n})) \\
&\leqslant \lim_{n\to\infty} \left[\phi(M(Sx_{2n+1}, x_{2n})) - \varphi(M(Sx_{2n+1}, x_{2n}))\right], from (2.1.3)
\end{aligned}$$

where

$$M(Sx_{2n+1},x_{2n}) \quad = \max\{\frac{p(Tx_{2n},gx_{2n})[1+p(SSx_{2n+1},fSx_{2n+1})]}{1+p(SSx_{2n+1},Tx_{2n})}, p(SSx_{2n+1},Tx_{2n})\}$$
$$\to p(Sz,z), \ from \ (2.7),(2.26).$$

Thus (2.27) becomes

$$\psi(p(Sz,z)) \leqslant \phi(p(Sz,z)) - \varphi(p(Sz,z)),$$

which in turn yields from (A) that $Sz = z$.

Also $\alpha(Sz, Tx_{2n}) = \alpha(z, y_{2n-1}) \geqslant 1$, from (2.1.6)($a$).

Since $x_{2n} \preceq gx_{2n}$ and $gx_{2n} \to z$,by (2.1.7), we have $x_{2n} \preceq z$. Using the continuity of ψ, (2.18) and Lemma 1.2 , we get

$$\begin{aligned}\psi(p(fz,z)) \quad &= \lim_{n\to\infty} \psi(p(fz,gx_{2n}))\\ &\leqslant \lim_{n\to\infty} \alpha(Sz,Tx_{2n})\ \psi(p(fz,gx_{2n}))\\ &\leqslant \lim_{n\to\infty} [\phi(M(z,x_{2n})) - \varphi(M(z,x_{2n}))]\,, from (2.1.3)\end{aligned}$$

where

$$\begin{aligned}M(z,x_{2n}) \quad &= \max\{\frac{p(Tx_{2n},gx_{2n})[1+p(z,fz)]}{1+p(z,Tx_{2n})}, p(z,Tx_{2n})\}\\ &\to 0 \text{ from (2.7), (2.19).}\end{aligned}$$

Hence $\psi(p(fz,z)) \leqslant \phi(0) - \varphi(0) = 0$. Thus $\psi(p(fz,z)) = 0$ so that $fz = z$. Since $f(X) \subseteq T(X)$, there exists $w \in X$ such that $z = fz = Tw$. Also we have $z = fz = Tw \preceq fTw \preceq w$, from (2.1.1).

From (2.1.6)(a), $\alpha(Sz,Tw) = \alpha(z,z) \geqslant 1$.

From (2.1.3), we have

$$\begin{aligned}\psi(p(z,gw)) \quad &= \psi(p(fz,gw))\\ &\leqslant \alpha(Sz,Tw)\ \psi(p(fz,gw))\\ &\leqslant \phi(M(z,w)) - \varphi(M(z,w)),\end{aligned}$$

where

$$\begin{aligned}M(z,w) \quad &= \max\{\frac{p(z,gw)[1+p(z,z)]}{1+p(z,z)}, p(z,z)\}\\ &= p(z,gw).\end{aligned}$$

Thus

$$\psi(p(z,gw)) \leqslant \phi(p(z,gw)) - \varphi(p(z,gw))$$

which in turn yields from (A) that $z = gw$.

Since the pair(g,T) is weakly compatible , we have $gz = gTw = Tgw = Tz$.

From (2.1.6)(a), we have $\alpha(Sz,Tz) = \alpha(z,Tz) \geqslant 1$.

From (2.1.3), we have

$$\begin{aligned}\psi(p(z,gz)) \quad &= \psi(p(fz,gz))\\ &\leqslant \alpha(Sz,Tz)\ \psi(p(fz,gz))\\ &\leqslant \phi(M(z,z)) - \varphi(M(z,z)),\end{aligned}$$

where

$$\begin{aligned}M(z,z) \quad &= \max\{\frac{p(Tz,gz)[1+p(z,z)]}{1+p(z,Tz)}, p(z,Tz)\}\\ &= p(z,gz), from(p_2).\end{aligned}$$

Thus

$$\psi(p(z,gz)) \leqslant \phi(p(z,gz)) - \varphi(p(z,gz))$$

which in turn yields from (A) that $z = gz = Tz$. Thus z is a common fixed point of f, g, S and T.

Suppose z' is another common common fixed point of f, g, S and T.
From (2.1.8), we have $\alpha(Sz, Tz') = \alpha(z, z') \geqslant 1$ and $z \preceq z'$.
From(2.1.3), we have

$$\begin{aligned} \psi(p(z, z')) &= \psi(p(fz, gz')) \\ &\leqslant \alpha(Sz, Tz')\ \psi(p(fz, gz')) \\ &\leqslant \phi(M(z, z')) - \varphi(M(z, z')), \end{aligned}$$

where

$$\begin{aligned} M(z, z') &= \max\{\tfrac{p(z', z')[1+p(z,z)]}{1+p(z,z')}, p(z, z')\} \\ &= p(z, z'), from(p_2). \end{aligned}$$

Thus

$$\psi(p(z, z')) \leqslant \phi(p(z, z')) - \varphi(p(z, z')) < \psi(p(z, z')), from(A)$$

which is a contradiction. Hence $z = z'$. Thus f, g, S and T have a unique common fixed point.

Similarly we can prove Theorem 2.1 when (2.1.6)(b) holds. □

Now we give an example to support Theorem 2.1.

EXAMPLE 2.1. Let $X = \mathcal{R}^+$, $p(x, y) = \max\{x, y\}$, $\forall\ x, y \in X$ and define $x \preceq y$ if $y \leqslant x$. Define $f, g, S, T : X \to X$ by $fx = \frac{x}{2}$, $gx = \frac{x}{4}$, $Sx = 8x$ and $Tx = 4x$.

Define $\alpha : X \times X \to \mathcal{R}^+$ by $\alpha(x, y) = \begin{cases} 1, & \text{if } x, y \in [0, 1], \\ 0, & \text{otherwise.} \end{cases}$

Define $\psi, \phi, \varphi : \mathcal{R}^+ \to \mathcal{R}^+$ by
$\psi(t) = 4t$, $\phi(t) = 7t$, $\varphi(t) = \frac{7}{2}t$, $\forall\ t \in \mathcal{R}^+$.
Clearly $\psi(t) - \phi(t) + \varphi(t) > 0$, $\forall\ t > 0$.
We have $fx = \frac{x}{2} \leqslant x \Rightarrow x \preceq fx$ and $gx = \frac{x}{4} \leqslant x \Rightarrow x \preceq gx$.
Also $fTx = 2x \geqslant x \Rightarrow fTx \preceq x$ and $gSx = 2x \geqslant x \Rightarrow gSx \preceq x$.
If $x > \frac{1}{8}$ and $y \in X$ then $\alpha(Sx, Ty) = 0$.
If $x \leqslant \frac{1}{8}$ and $y > \frac{1}{4}$ then $\alpha(Sx, Ty) = 0$.
In these cases,the condition (2.1.3) is clearly satisfied.
Suppose $x \leqslant \frac{1}{8}$ and $y \in [0, \frac{1}{4}]$ then $\alpha(Sx, Ty) = 1$.
Also

$$\begin{aligned} \alpha(Sx, Ty)\psi(p(fx, gy)) &= (1)4\max\left\{\tfrac{x}{2}, \tfrac{y}{4}\right\} \\ &= \max\left\{2x, y\right\}. \\ &= \tfrac{1}{4}p(Sx, Ty) \\ &\leqslant \tfrac{1}{4}M(x, y) \\ &\leqslant \phi(M(x, y)) - \varphi(M(x, y)) \end{aligned}$$

Thus (2.1.3) is satisfied.
One can easily verify the remaining conditions of Theorem 2.1. Clearly 0 is the common fixed point of f, g, S and T.

References

[1] B. Samet, M. Rajović, R. Lazović and R. Stojiljković, Common fixed point results for nonlinear contractions in ordered partial metric spaces, *Fixed Point Theory and Appl.*, **2011**, 2011:71,14 pages, doi:10.1186/1687-1812-2011-71.

[2] B. Samet, C. Vetro and P. Vetro, Fixed point theorems for $\alpha-\psi$-contractive type mappings, *Nonlinear Analysis: Theory, Methods and Applications*, **75**(4)(2012), 2154-2165.

[3] D. Ilić ,V. Pavlović and V. Rakočević, Some new extensions of Banach's contraction principle to partial metric spaces, *Appl. Math. Lett.*, **24**(8)(2011), 1326-1330, doi:10.1016/j.aml.2011.02.025.

[4] E. Karapınar and I. M. Erhan, Fixed point theorems for operators on partial metric spaces, *Appl. Math. Lett.*, **24**(11)(2011), 1894-1899.

[5] E. Karapınar, Generalizations of Caristi Kirk's theorem on partial metric spaces, *Fixed Point Theory and Appl.*, **2011**, 2011:4, 7 pages, doi:10.1186/1687-1812-2011-4.

[6] E. Karapinar, P. Kumam and P. Salimi, On $\alpha-\psi$-Meier-Keeler contractive mappings, *Fixed Point Theory and Appl.*, **2013**, 2013:94, 12 pages, doi:10-1186/1687-1812-2013-94.

[7] H. Aydi, Fixed point results for weakly contractive mappings in ordered partial metric spaces, *J. Adv. Math. Stud.*, **4**(2)(2011), 1-12.

[8] G. Jungck and B. E. Rhoades, Fixed points for set valued functions without continuity, *Indian J. Pure. Appl. Math.*, **29**(3)(1998), 227-238.

[9] I. Altun, F. Sola and H. Simsek, Generalized contractions on partial metric spaces, *Topology Appl.*, **157**(18)(2010), 2778-2785.

[10] I. Altun and A. Erduran, Fixed point theorems for monotone mappings on partial metric spaces, *Fixed Point Theory and Appl.*, **2011**(2011), Article ID 508730, 10 pages, doi:10.1155/2011/508730.

[11] K. P. R. Rao and G. N. V. Kishore, A unique common fixed point theorem for four maps under $\psi-\phi$ contractive condition in partial metric spaces, *Bull. Math. Anal. Appl.*, **3**(3)(2011), 56 - 63.

[12] M.Abbas, T.Nazir and S.Radenović, Common fixed points of four maps in partially ordered metric spaces, *Appl. Math. Lett.*, **24**(9)(2011), 1520-1526.

[13] O. Valero, On Banach fixed point theorems for partial metric spaces, *Appl. Gen. Topology*, **6**(2)(2005), 229-240.

[14] P. Shahi, J. Kaur and S. S. Bhatia, Coincidence and common fixed point results for generalized $\alpha-\psi$-contractive type mappings with applications, *Bull. Belg. Math. Soc. Simon Stevin*, **22**(2)(2015), 299-318.

[15] R. Heckmann, Approximation of metric spaces by partial metric spaces, *Appl. Categ. Structures*, **7**(1)(1999), 71-83.

[16] S. G. Matthews, Partial metric topology, *In Proceedings of the 8th Summer Conference on General Topology and Applications*, **728**(1994), 183-197.

[17] S. Romaguera, A Kirk type characterization of completeness for partial metric spaces, *Fixed Point Theory and Appl.*, **2010**(2010), Article ID 493298, 6 pages.

[18] T. Abdeljawad, E. Karapınar and K. Tas, Existence and uniqueness of a common fixed point on partial metric spaces, *Appl. Math. Lett.*, **24**(11)(2011), 1900 - 1904.

[19] T. Abdeljawad, Meier-Keeler α-contractive fixed and common fixed point theorems, *Fixed Point Theory and Appl.*, **2013**,(2013):19, 10 pages, doi:10.1186/1687-1812-2013-19.

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