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A NOTE ON soENERGY OF STARS, BISTAR AND DOUBLE STAR GRAPHS

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ABSTRACT. Let G be a finite non trivial connected graph.In the earlier paper idegree, odegree, oidegree of a minimal dominating set of G were introduced, oEnergy of a graph with respect to the minimal dominating set was calculated in terms of idegree and odegree. Algorithm to get the soEnergy was also introduced and soEnergy was calculated for some standard graphs in the earlier papers. In this paper soEnergy of stars, bistars and double stars with respect to the given dominating set are found out.

1. INTRODUCTION

Let $G = (V, E)$ be a finite non trivial connected graph. A set $D \subset V$ is a dominating set of G if every vertex in $V - D$ is adjacent to some vertex in D. A dominating set D of G is called a minimal dominating set if no proper subset of D is a dominating set. Star graph is a tree consisting of one vertex adjacent to all the others. Bistar is a graph obtained from K_2 by joining n pendent edges to both the ends of K_2. Double star is the graph obtained from K_2 by joining m pendent edges to one end and n pendent edges to the other end of K_2. In this paper soEnergy of stars, bistars and double stars are being calculated with respect to the given minimal dominating set .

2. PRELIMINARIES

2.1. Basic Definitions.

DEFINITION 2.1. Let G be a graph and S be a subset of $V(G)$. Let $v \in V - S$, the **idegree** of v with respect to S is the number of neighbours of $v \in V - S$ and it is denoted by $id_S(v)$.

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DEFINITION 2.2. Let G be a graph and S be a subset of $V(G)$. Let $v \in V - S$, the **odegree** of v with respect to S is the number of neighbours of $v \in V - S$, and is denoted as od_S(v).

DEFINITION 2.3. Let G be a graph and S be a subset of $V(G)$. Let $v \in V - S$, the **oidegree** of v with respect to S is $od_S(v) - id_S(v)$ if od > id and it is denoted by $oid_S(v)$

DEFINITION 2.4. Let G be a graph and S be a subset of $V(G)$. Let $v \in V - S$, the **iodegree** of v with respect to S is $(id_S(v) - od_S(v))$ if id > od and it is denoted by $io_S(v)$.

DEFINITION 2.5. Let G be a graph and D be a dominating set, **oEnergy** of a graph with respect to D denoted by $o\epsilon_D(G)$ is the summation of all oid if od > id or otherwise zero .

DEFINITION 2.6. Let G be a graph and D be a minmal dominating set, then **energy curve** is the curve obtained by joining the oEnergies of D_{i-1} and D_i for $1 \leqslant i \leqslant n$, taking the number of vertices of D_i along the x axis and the oEnergy along the y axis.

DEFINITION 2.7. Let G be a graph and D be a minimal dominating set, **soEnergy** of a graph with respect to D is

$$\sum_{i=0}^{|V-D|} o\epsilon_{D_i}(G)$$

where $D_{i+1} = D_i \cup V_{i+1}, V_{i+1}$ is a singleton vertex with minimum oidegree of $V - D_i$ and D_0 is a minimal dominating set where $0 \leqslant i \leqslant |V - D|$, it is denoted by $so\epsilon_D(G)$.

DEFINITION 2.8. Let G be a graph and $MDS(G)$ is the set of all minimal dominating set of G, then $Hardihood^+$ of a graph G is $max\{so\epsilon_{(MDS(G))}(G)\}$ is denoted as $HD^+(G)$.

DEFINITION 2.9. Let G be a graph and $MDS(G)$ is the set of all minimal dominating set of G, then $Hardihood^-$ of a graph G is $min\{so\epsilon_{(MDS(G))}(G)\}$ is denoted as $HD^-(G)$.

Given below is the algorithm to find the soEnergy of any given graph with respect to the given dominating set.

Algorithm

(1) Find a minimal dominating set D .
(2) Find the idegree, odegree and iodegree for the vertices in the graph induced by $< V - D >$.
(3) If ioderee > 0 proceed to step 5.
(4) If oidegree $\leqslant 0$ then put oenergy $= 0$ and goto step 6.
(5) Shift the vertex with minimum positive oidegree, which appears first, to the set D.

(6) If no such positive oidegree exists, shift a vertex with oidegree zero to the set D otherwise shift a vertex minimum iodegree to the set D.
(7) Find the idegree, odegree and iodegree and then oenergy for the vertices in the new $< V - D >$ with respect to new D set.
(8) Repeat steps 2 to 7 until $|V - D| = 0$.
(9) Find the sum of all the oEnergies at each step to get the soEnergy.
(10) Fix the x axis as the number of vertices of the set D_i and y axis as the oEnergy of the sets at each and every step. Plot the graph.

3. soENERGY OF STARS

The star graph $K_{1,n-1}$ of order n is a tree on n vertices, In a star graph one vertex has degree $n - 1$ and the remaining $n - 1$ vertices have vertex degree 1. A dominating set D of G is called a minimum dominating set if D is a dominating set with minimum cardinality of all the minimal dominating sets of G. A dominating set D of G is called an independent dominating set if the vertices in D are independent. A dominating set D of G is called a maximum independent dominating set if D is an independent dominating set with maximum cardinality. It is denoted by maxid.

THEOREM 3.1. *Let G be any graph with the given dominating set D with $id_D = 0$ and od_D are $d_1, d_2, d_3....d_n$ where $d_1 \leqslant d_2 \leqslant d_3 \leqslant, ..., \leqslant d_n$ for all the n vertices in D, then $so\epsilon_D(G) = nd_n + (n-1)d_{n-1} + ... + 2d_2 + d_1$.*

PROOF. Let G be a graph, D be a minimal dominating set with $|D| = n$. Given $id_D = 0$ for all the vertices in $V - D$ and od_D of the vertices of $V - D$ are $d_1, d_2, d_3....d_n$, then oid_D of the vertices in $V - D$ are $d_1, d_2, d_3....d_n$. Therefore

$$o\epsilon_D(G) = (d_1 + d_2 + d_3 + ... + d_n)$$

since vertex of the minimum oidegree is shifted to the D set,

$$o\epsilon_{D_1}(G) = (d_2 + d_3 + ... + d_n).$$

Proceeding in the similar way till $|V - D| = 0$ and $|D| = n$,we get
$so\epsilon_D(G) = (d_1 + d_2 + d_3 + ... + d_n) + (d_2 + d_3 + d_4 + ... + d_n) + ...(d_{n-1} + d_n) + d_n$
$= nd_n + (n-1)d_(n-2) + ... + 2d_2 + d_1$. □

For a star graph only two dominating sets are possible, (i) minimum dominating set; (ii) maximum independent dominating set.

THEOREM 3.2. *soEnergy of star graph $K_{1,n-1}$ of order n, $n \geqslant 2$ is:*
(i) $so\epsilon_D(K_{1,n-1}) = \frac{(n)(n-1)}{2}$ for $n \geqslant 2$, if D is a minimum dominating set.
(ii) $so\epsilon_{maxid}(K_{1,n-1}) = (n-1)$, $n \geqslant 2$ if maxid is a maximum independent dominating set .

PROOF. Let G be a star graph $K_{1,n-1}$ with n vertices. Let D be a minimal dominating set.

(i) Let D be a minimum dominating set. For a star graph $K_{1,n-1}$, $|D| = 1$ and degree of the vertex is $n-1$ if and only if D is a minimum dominating set. Then $|V-D| = n-1$. Therefore $id_D = 0$ and $od_D = 1$ for all the vertices in $V-D$, since D is a singleton set with degree $n-1$ and $V-D$ is the set of $n-1$ vertices of degree 1 and are adjacent only to the vertex in D. Hence $oid_D = 1$ for all the n-1 vertices. Since the $id_D = 0$ for all verices in $V-D$, applying Theorem 3.1. Thus $soEnergy = (n-1)+(n-2)+...+2+1+0$. Therefore $so\epsilon_D(K_{1,n-1}) = \frac{(n)(n-1)}{2}$ for $n \geqslant 2$, where D is a maximal dominating set.

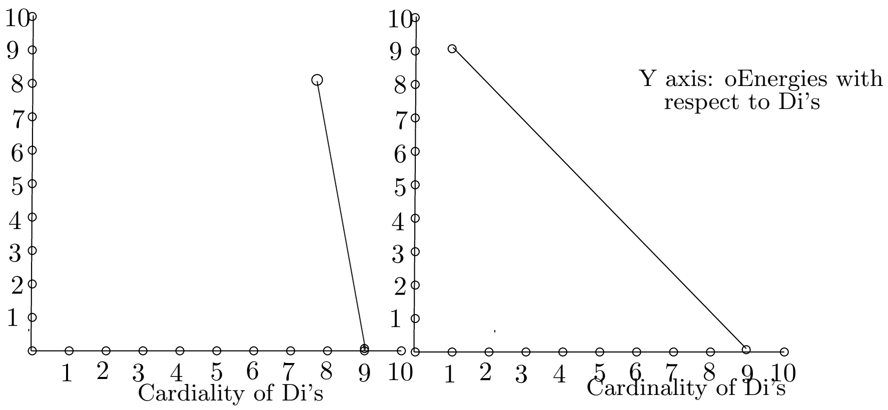
(ii) Let maxid be a maximum independent dominating set. For a star graph $K_{1,n-1}$,$|D| = n-1$ and degree of all the vertices are $n-1$ if and only if D is a maximum independent dominating set. Then $|V - maxid| = 1$. If $|V - maxid| = 1$, then $id_m axid = 0$ and $od_m axid = n-1$ for the vertex in V-maxid and $oid_D = n-1$ and finally energy vanishes in the final step. Therefore $so\epsilon_{maxid}(K_{1,n-1}) = n-1$ for $n \geqslant 2$, where maxid is a maximum independent dominating set. □

$so\epsilon_D K_{1,n-1}$ for $n \geqslant 2$ attains maximum value for minimum dominating set and minimum value for maximum independent dominating set .

COROLLARY 3.1. *Let $K_{1,n-1}$ for $n \geqslant 2$ be the graph. Then*
(i) $HD^+(K_{1,n-1})$ is $\frac{(n)(n-1)}{2}$ iff D is a maximal dominating set.
(ii) $HD^-(K_{1,n-1})$ is $(n-1)$ iff D is a maximum independent dominating set.

PROOF. Two soEnergies possible with $K_{1,n-1}$ graph are $\frac{(n)(n-1)}{2}$ and $(n-1)$. It is obvious that for $n \geqslant 2$ $\frac{(n)(n-1)}{2} \geqslant (n-1)$. Therefore $HD^+(K_{1,n-1}) = \frac{(n)(n-1)}{2}$ for $n \geqslant 2$ and $HD^-(K_{1,n-1}) = (n-1)$ for $n \geqslant 2$. Hence the proof. □

Figure 1: Energy curves of star graphs



Energy curves of $K_{1,8}$ with maximum and minimum independent dominating sets

4. soENERGT of BISTARS and DOUBLE STAR GRAPHS

The graph $B_{n,n}, n \geqslant 2$ is a bistar obtained from two disjoint copies of $K_{1,n}$ by joining the centre vertices by an edge. It have $2n+2$ vertices and $2n+1$ edges.

Double star is the graph obtained by joining the center of two stars $K_{1,n}$ and $K_{1,m}$ with an edge. Connected dominating set is a dominating set D such that $< D >$ is a connected graph. A dominating set D of G is called a minimum independent dominating set if D is a independent dominating set with minimum cardinality of G. Let us denote the connected dominating set by the notation CD, minimum independent dominating set by minid.

Definition 4.1. Let G be a $B_{n,m}$ and D be an independent dominating set of $B_{n,m}$, D is said to be n_independent set if D contains vertices adjacent to the vertex of degree n and the vertex of degree m.

Definition 4.2. Let G be a $B_{n,m}$ and D be an independent dominating set of $B_{n,m}$, D is said to be m_independent set if D contains vertices adjacent to the vertex of degree m and the vertex of degree n.

Three minimal dominating sets are possible for bistar and double star. soEnergy for three different dominating sets are calculated in the following result.

Theorem 4.1. *Let $B_{n,n}, n \geqslant 2$ be the bistar graph. Then*
(i) $so\epsilon_{CD}(B_{n,n}) = 2n + (2n-2) + (2n-2) + ... + 2 + 1 = n(2n+1)$, where CD is a connected dominating set.
(ii) $so\epsilon_{maxid}(B_{n,n}) = 3n - 1$ where maxid is a maximum independent dominating set D.
(iii) $so\epsilon_{minid}(B_{n,n}) = \frac{(3n+2)(n+1)}{2}$ where minid is a minimum independent dominating set.

Proof. Let G be a bistar $B_{n,n}$ and D be a minimal dominating set.

(i) Let CD be a connected dominating set. For a $B_{n,n}$, the minimal connected dominating set is P_2, a path with two vertices. Then $|CD| = 2$ and $|V - CD| = 2n$. Since the vertices in $V - CD$ are the pendent vertices they are not adjacent among themselves, $id_{V-CD} = 0$. As CD is a connected dominating set, vertices are adjacent to each other so $od_C D = 1$. Hence $iod_C D = 1$ for all vertices in $< V - CD >$. By Theorem 3.1

$$\begin{aligned} so\epsilon_{CD}(B_{n,n}) &= 2n + (2n-1) + (2n-2) + ... + 2 + 1 + 0 \\ &= \tfrac{2n(2n+1)}{2}. \\ &= n(2n+1). \end{aligned}$$

Hence $so\epsilon_{CD}(B_{n,n}) = 2n + (2n-2) + (2n-2) + ... + 2 + 1 = n(2n+1)$, where CD is a connected dominating set.

(ii) Let maxid be a maximum independent dominating set. Then for a $B_{n,n}$, $|maxid| = 2n$ and $|V - maxid| = 2$. As the V-maxid is P_2, a path with two vertices, the vertices are adjacent to each other. $id_m axid = 1$, $od_m axid = n$ and $io_m axid = n-1$ for the two vertices in $V - maxid \therefore o\epsilon_{maxid_0}(G) = 2(n-1)$ and in the second stage it is $n+1$ and at the third stage energy vanishes. $so\epsilon_{maxid}(B_{n,n}) = 2(n-1) + (n+1) = 3n - 1$, where maxid is a maximum independent dominating set . Hence $so\epsilon_{maxid}(B_{n,n}) = 3n - 1$ where maxid is a maximum independent dominating set D.

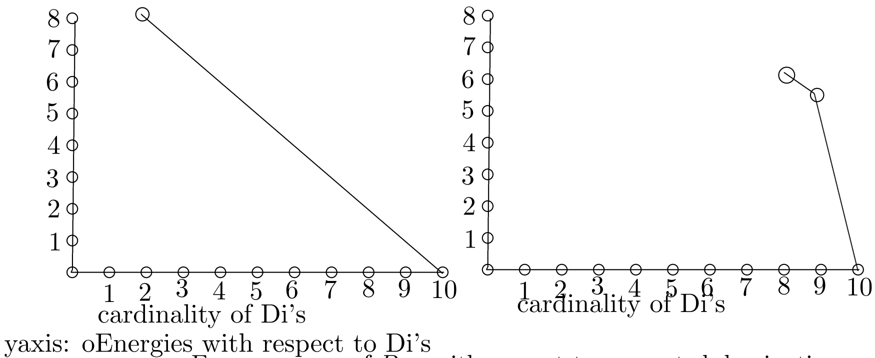
(iii) Let minid be a minimum independent dominating set. Then the vertices of V-minid have one vertex with degree $n+1$ and other with degree 1, i.e, $d_1 = n+1$,

$d_2 = 1, ..., d_{n+1} = 1$. Therefore By theorem 3.1,

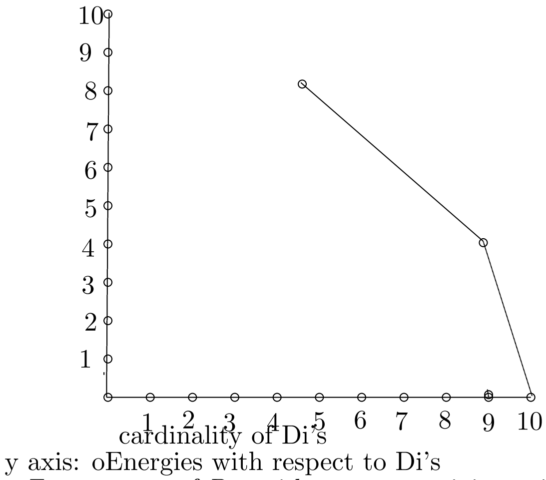
$$so\epsilon_{minid}(B_{n,n}) = ((n+1)+n) + ((n+1)+(n-1)) + ... + (n+1)$$
$$= \frac{(2n+1)(2n+2)}{2} - \frac{n(n+1)}{2}$$
$$= \frac{(3n+2)(n+1)}{2}$$

Hence $so\epsilon_{minid}(B_{n,n}) = \frac{(3n+2)(n+1)}{2}$ where minid is a minimum independent dominating set. □

Figure 2: Energy curves of BiStars



Energy curves of $B_{4,4}$ with respect to connected dominating and minimum independent dominating set



Energy curve of $B_{4,4}$ with respect to minimum independent dominating set

$so\epsilon_D(B_{n,n})$ attains maximum value when D is connected domination set and minimum independent dominating set for $n > 3$ and $n \leqslant 3$ respectively of n. It attains its minimum value when D is maximum independent set.

Corollary 4.1. *Let $B_{n,n}$ be the graph. Then*

(i) (a) $HD^+(B_{n,n})$ is $n(2n+1)$, when D is a connected dominaing set when $n > 3$;
(b) $HD^+(B_{n,n})$ is $\frac{(3n+2)(n+1)}{2}$ when D is a minimum independent dominating set when $n \leqslant 3$;
(ii) $HD^-(B_{n,n})$ is $3n-1$ when D is a maximum independent dominating set.

PROOF.
(i) Let CD be connected dominating set and minid be minimum independent dominating set. Then $so\epsilon_{CD}(B_{n,n})$ and $so\epsilon_{minid}(B_{n,n})$ are $2n+(2n-1)+(2n-2)+...+1$ and $(n+n)+(n+(n-1))+...+n)$. Thus

$$so\epsilon_{(CD)}(B_{n,n}) - so\epsilon_{(minid)}(B_{n,n}) = |1+2+3+...+n-1|.$$

The difference is higher when $n \leqslant 3$. Therefore $HD^+(B_{n,n})$ is

$$(2n+1)+(2n)+(2n-1)+...+(n+1)$$

this is possible when D is the minimum independent dominating set;
and $HD^+(B_{n,n})$ is

$$2n+(2n-2)+(2n-2)+...+2+1+0 = \frac{2n(2n+1)}{2},$$

when D is the connected dominaing set when $n > 3$.

(ii) Let maxid be a maximal independent dominating set. Then:
$so\epsilon_{maxid}B_{n,n}$ is $2(n-1)+(n+1)$;
$so\epsilon_{CD}B_{n,n}$ are $2n+(2n-1)+(2n-2)+...+1$;
$so\epsilon_{minid}B_{n,n}$ are $(n+n)+(n+(n-1))+...+n)$.
$2(n-1)+(n+1) \leqslant 2n+(2n-2)+(2n-2)+...+2+1+0$ and
$2(n-1)+(n+1) \leqslant (2n+1)+(2n)+(2n-1)+...+(n+1)$.
Therefore $HD^-(B_{n,n}) = 3n-1$, when D is a maximal independent dominating set. Hence the proof. □

COROLLARY 4.2. *Let $B_{n,m}, n, m \geqslant 2$ be the double star graph with $n < m$,*
(i) $so\epsilon_{CD}(B_{n,m}) = \frac{(n+m)(n+m+1)}{2}$, where D is a connected dominating set.
(ii) $so\epsilon_{maxid}(B_{n,m}) = (n-1)+(m-1)+(m-1)$ where D is a maximum independent dominating set D.
(iii) (a) $so\epsilon_D(B_{n,m}) = (m+n)+(m+n-1)+...+m+0$. where D is a n_independent dominating set.
(b) $so\epsilon_D(B_{n,m}) = (m+n)+(m+n-1)+...+n+0$. where D is a m_independent dominating set .

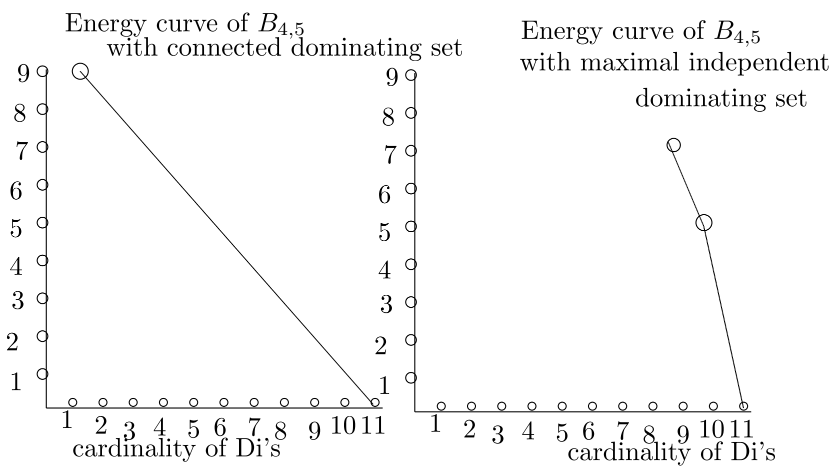
PROOF. Let D be a minimal dominating set of $B_{n,m}$.
(i) Let CD be a connected dominating set. By replacing the second n with m in $B_{n,n}$ in Theorem 4.1 the result is obtained. $so\epsilon_{CD}(B_{n,m}) = \frac{(n+m)(n+m+1)}{2}$, where CD is a connected dominating set.
(ii) Let maxid be a maximum independent dominating set. Then for a $B_{n,m}$, $|maxid| = n+m$ and $|V - maxid| = 2$. As the V-maxid is P_2, a path with two vertices, the vertices are adjacent to each other. So $id_{CD} = 1$.
$od_{CD} = n$ for the vertices that are adjacent with vertex of degree n, and $od_{CD} = m$ for the vertices that are adjacent with vertex of degree m. Hence $io_{CD} = n-1$ and

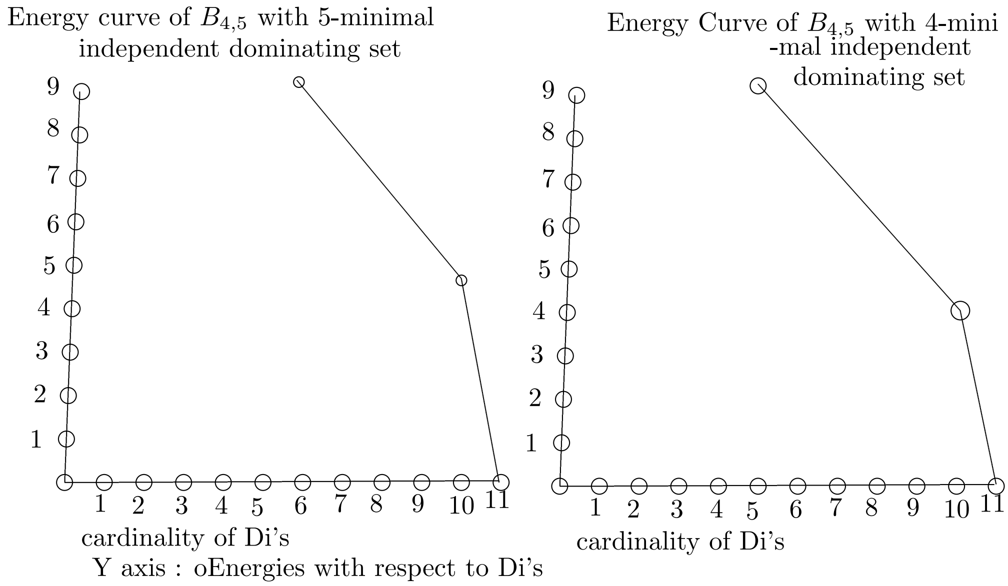
$m-1$ for the two vertices in $V - maxid$. Therefore:
$o\epsilon_{maxid}(G) = (n-1) + (m-1)$ and in the second stage it is $m-1$
$so\epsilon_{maxid}(B_{n,m}) = (n-1) + (m-1) + (m-1)$.
(iii) (a) Let D be a n_independent dominating set. Then the set contains n vertices of degree one and one vertex of degree m. Additionally the set $V - D$ contains the m vertices of degree one and one vertex of degree m and they are not adjacent to each other. So $id_D = 0$ for all the vertices in D. Further on, by Theorem 3.1, we conclude $so\epsilon_D(B_{n,n}) = (m+n) + (m+n-1) + ... + m + 0$.
(b) Similarly, if D is a m_independent dominating set then
$so\epsilon_D(B_{n,n}) = (m+n) + (m+n-1) + ... + n + 0$. □

Corollary 4.3. *Let $B_{n,m}$ be the graph. Then,*
(i) $HD^+(B_{n,m}) = (m+n) + (m+n-1) + ... + n + 0$ if D is the m_independent dominating set.
(ii) $HD^-(B_{n,m}) = \frac{(n+m)(n+m)+1)}{2}$, if D is the connected dominating set.

Proof. The four possible $so\epsilon$s of $B_{n,m}$ are:
(i) $(n+m) + (n+m-1) + ... + 1$;
(ii) $(n-1) + (m-1) + (m-1)$;
(iii) $(m+n) + (m+n-1) + ... + m + 0$;
(iv) $(m+n) + (m+n-1) + ... + n + 0$.
Among these four soe's
$(n+m) + (n+m-1) + ... + 1 \leqslant (n-1) + (m-1) + (m-1) \leqslant$
$(m+n) + (m+n-1) + ... + m + 0 \leqslant (m+n) + (m+n-1) + ... + n + 0$
Therefore $HD^+(B_{n,m}) = (m+n) + (m+n-1) + ... + n + 0$, it happens when D is the m_independent dominating set and $HD^-(B_{n,m}) = \frac{(n+m)(n+m)+1)}{2}$, happens when D is the connected dominating set. □

FIG 3: Energy curve of Double star





5. CONCLUSION

This paper attempts in developing the soEnergy concepts of a graph like star, bistar and dounle star with respect to the given minimal dominating set. The soEnergy of these graphs are studied and the maximum and minimum soEnergy were found out.

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