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On Divisibility of Almost Distributive Lattices

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Abstract. In this paper, the concepts of $*-$divisibility, $*-$prime elements, $*-$irreducible elements are introduced in an Almost Distributive Lattice(ADL) and studied extensively their properties. A definition has been introduced on a congruence relation in terms of multiplier ideals and derived a set of equivalent conditions for the corresponding quotient ADL which becomes a Boolean algebra. Finally, characterized the $*-$prime and $*-$irreducible elements with the corresponding multiplier ideals.

1. Introduction

The concept of an Almost Distributive Lattice (ADL) was introduced by U. M. Swamy and G. C. Rao [**8**] as a common abstraction to most of the existing ring theoretic generalizations of a Boolean algebra on one hand and the class of distributive lattices on the other. In [**6**], G.C.Rao and M.S.Rao introduced the concept of annulets in an ADL and characterized both generalized stone ADL and normal ADL in terms of their annulets. The concept of Quasi-complemented ADL was introduced by G.C. Rao et. al. in [**4**] and they proved that a uniquely quasi-complemented ADL is a pseudo-complemented ADL. And also, the authors derived that an ADL is quasi-complemented ADL if and only if every prime ideal of an ADL is maximal. In [**7**], M.S. Rao introduced the concept of divisibility in distributive lattices in terms of annihilator ideals. He established that a relation between $*-$prime and $*-$irreducible elements and corresponding ideals formed by their multiplies. In this paper, we extend the concepts of divisibility, $*-$prime elements, $*-$irreducible elements in to an Almost Distributive Lattice and also studied their important properties. We defined a congruence relation θ on an ADL and established a set of a equivalent conditions for quotient ADL L/θ which becomes a Boolean algebra. Characterized $*-$prime and $*-$irreducible elements in

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terms of prime and maximal ideals respectively. Finally, it is proved that every $*-$irreducible element of an ADL is a $*-$prime element.

2. Preliminaries

In this section, some important definitions and results are provided for better understanding in which those are frequently used.

Definition 2.1. ([**8**]) An Almost Distributive Lattice with zero or simply ADL is an algebra $(L, \vee, \wedge, 0)$ of type $(2, 2, 0)$ satisfying:
1. $(x \vee y) \wedge z = (x \wedge z) \vee (y \wedge z)$
2. $x \wedge (y \vee z) = (x \wedge y) \vee (x \wedge z)$
3. $(x \vee y) \wedge y = y$
4. $(x \vee y) \wedge x = x$
5. $x \vee (x \wedge y) = x$
6. $0 \wedge x = 0$
7. $x \vee 0 = x$, for all $x, y, z \in L$.

Every non-empty set X can be regarded as an ADL as follows. Let $x_0 \in X$. Define the binary operations $\vee, \wedge$ on X by

$$x \vee y = \begin{cases} x & \text{if } x \neq x_0 \\ y & \text{if } x = x_0 \end{cases} \qquad x \wedge y = \begin{cases} y & \text{if } x \neq x_0 \\ x_0 & \text{if } x = x_0. \end{cases}$$

Then $(X, \vee, \wedge, x_0)$ is an ADL (where x_0 is the zero) and is called a discrete ADL. If $(L, \vee, \wedge, 0)$ is an ADL, for any $a, b \in L$, define $a \leqslant b$ if and only if $a = a \wedge b$ (or equivalently, $a \vee b = b$), then $\leqslant$ is a partial ordering on L.

Theorem 2.1 ([**8**]). *If $(L, \vee, \wedge, 0)$ is an ADL, for any $a, b, c \in L$, we have the following:*

(1). $a \vee b = a \Leftrightarrow a \wedge b = b$
(2). $a \vee b = b \Leftrightarrow a \wedge b = a$
(3). $\wedge$ *is associative in* L
(4). $a \wedge b \wedge c = b \wedge a \wedge c$
(5). $(a \vee b) \wedge c = (b \vee a) \wedge c$
(6). $a \wedge b = 0 \Leftrightarrow b \wedge a = 0$
(7). $a \vee (b \wedge c) = (a \vee b) \wedge (a \vee c)$
(8). $a \wedge (a \vee b) = a$, $(a \wedge b) \vee b = b$ *and* $a \vee (b \wedge a) = a$
(9). $a \leqslant a \vee b$ *and* $a \wedge b \leqslant b$
(10). $a \wedge a = a$ *and* $a \vee a = a$
(11). $0 \vee a = a$ *and* $a \wedge 0 = 0$
(12). *If* $a \leqslant c$, $b \leqslant c$ *then* $a \wedge b = b \wedge a$ *and* $a \vee b = b \vee a$
(13). $a \vee b = (a \vee b) \vee a$.

It can be observed that an ADL L satisfies almost all the properties of a distributive lattice except the right distributivity of $\vee$ over $\wedge$, commutativity of $\vee$, commutativity of $\wedge$. Any one of these properties make an ADL L a distributive lattice. That is

THEOREM 2.2 ([**8**]). *Let $(L, \vee, \wedge, 0)$ be an ADL with 0. Then the following are equivalent:*

1). $(L, \vee, \wedge, 0)$ *is a distributive lattice*
2). $a \vee b = b \vee a$*, for all* $a, b \in L$
3). $a \wedge b = b \wedge a$*, for all* $a, b \in L$
4). $(a \wedge b) \vee c = (a \vee c) \wedge (b \vee c)$*, for all* $a, b, c \in L$.

As usual, an element $m \in L$ is called maximal if it is a maximal element in the partially ordered set $(L, \leqslant)$. That is, for any $a \in L$, $m \leqslant a \Rightarrow m = a$.

THEOREM 2.3 ([**8**]). *Let L be an ADL and $m \in L$. Then the following are equivalent:*

1). m *is maximal with respect to* $\leqslant$
2). $m \vee a = m$*, for all* $a \in L$
3). $m \wedge a = a$*, for all* $a \in L$
4). $a \vee m$ *is maximal, for all* $a \in L$.

As in distributive lattices ([**1**], [**2**]), a non-empty sub set I of an ADL L is called an ideal of L if $a \vee b \in I$ and $a \wedge x \in I$ for any $a, b \in I$ and $x \in L$. Also, a non-empty subset F of L is said to be a filter of L if $a \wedge b \in F$ and $x \vee a \in F$ for $a, b \in F$ and $x \in L$.

The set $I(L)$ of all ideals of L is a bounded distributive lattice with least element $\{0\}$ and greatest element L under set inclusion in which, for any $I, J \in I(L)$, $I \cap J$ is the infimum of I and J while the supremum is given by $I \vee J := \{a \vee b \mid a \in I, b \in J\}$. A proper ideal P of L is called a prime ideal if, for any $x, y \in L$, $x \wedge y \in P \Rightarrow x \in P$ or $y \in P$. A proper ideal M of L is said to be maximal if it is not properly contained in any proper ideal of L. It can be observed that every maximal ideal of L is a prime ideal. Every proper ideal of L is contained in a maximal ideal. For any subset S of L the smallest ideal containing S is given by $(S] := \{(\bigvee_{i=1}^{n} s_i) \wedge x \mid s_i \in S, x \in L \text{ and } n \in N\}$. If $S = \{s\}$, we write $(s]$ instead of $(S]$. Similarly, for any $S \subseteq L$, $[S) := \{x \vee (\bigwedge_{i=1}^{n} s_i) \mid s_i \in S, x \in L \text{ and } n \in N\}$. If $S = \{s\}$, we write $[s)$ instead of $[S)$.

THEOREM 2.4. [**8**] *For any x, y in L the following are equivalent:*

1). $(x] \subseteq (y]$
2). $y \wedge x = x$
3). $y \vee x = y$
4). $[y) \subseteq [x)$.

For any $x, y \in L$, it can be verified that $(x] \vee (y] = (x \vee y]$ and $(x] \wedge (y] = (x \wedge y]$. Hence the set $PI(L)$ of all principal ideals of L is a sublattice of the distributive lattice $I(L)$ of ideals of L.

DEFINITION 2.2 ([**6**]). For any $A \subseteq L$, the annihilator of A is defined as $A^* = \{x \in L \mid a \wedge x = 0 \text{ for all } a \in A\}$

If $A = \{a\}$, then we denote $(\{a\})^*$ by $(a)^*$.

THEOREM 2.5 ([**6**]). *For any $a, b \in L$, we have the following:*
(1). $(a] \subseteq (a)^{**}$
(2). $(a)^{***} = (a)^*$
(3). $a \leqslant b$ *implies* $(b)^* \subseteq (a)^*$
(4). $(a)^* \subseteq (b)^*$ *if and only if* $(b)^{**} \subseteq (a)^{**}$
(5). $(a \vee b)^* = (a)^* \cap (b)^*$
(6). $(a \wedge b)^{**} = (a)^{**} \cap (b)^{**}$.

DEFINITION 2.3 ([**3**]). An equivalence relation θ on an ADL L is called a congruence relation on L if $(a \wedge c, b \wedge d), (a \vee c, b \vee d) \in \theta$, for all $(a, b), (c, d) \in \theta$

DEFINITION 2.4 ([**3**]). For any congruence relation θ on an ADL L and $a \in L$, we define $[a]_\theta = \{b \in L \mid (a, b) \in \theta\}$ and it is called the congruence class containing a.

THEOREM 2.6 ([**3**]). *An equivalence relation θ on an ADL L is a congruence relation if and only if for any $(a, b) \in \theta,\ x \in L, (a \vee x, b \vee x), (x \vee a, x \vee b), (a \wedge x, b \wedge x), (x \wedge a, x \wedge b)$ are all in θ*

An element $a \in L$ is called dense [**4**] if $(a)^* = (0]$. The set D of all dense elements forms a filter provided $D \neq \emptyset$. A lattice L with 0 is called quasi-complemented [**4**] if for each $x \in L$, there exists $y \in L$ such that $x \wedge y = 0$ and $x \vee y$ is dense.

3. Divisibility in an ADL

In [**7**], M.S. Rao introduced the concepts of divisibility, $*-$prime, $*-$irreducible elements in distributive lattices in terms of annihilator ideals and proved their properties. In this section, we extend these concepts to an Almost Distributive Lattice, analogously and established a set of a equivalent conditions for quotient ADL L/θ to become a Boolean algebra. We characterized $*-$prime elements and $*-$irreducible elements in terms of prime ideals and maximal ideals respectively. In addition to this, it is proved that every $*-$irreducible element of an ADL is a $*-$prime element. Though many results look similar, the proofs are not similar because we do not have the properties like commutativity of $\vee$, commutativity of $\wedge$ and the right distributivity of $\vee$ over $\wedge$ in an ADL.
Now, we begin with following definition.

DEFINITION 3.1. Let L be an ADL and for any $a, b \in L$. An element a is said to be a $\star-$divisor of b or a divides b if $(b)^* = (a \wedge c)^*$ for some $c \in L$. In this case, we write it as $(a/b)_*$.

We prove the following result.

LEMMA 3.1. *Let L be an ADL. Then for any $a, b \in L$, we have $(a)^* = (b)^*$ implies that $(a \wedge x)^* = (b \wedge x)^*$ and $(a \vee x)^* = (b \vee x)^*$, for any $x \in L$.*

PROOF. Suppose that $(a)^* = (b)^*$. Let x be any element of L. Now, $t \in (a \wedge x)^* \Leftrightarrow t \wedge a \wedge x = 0 \Leftrightarrow t \wedge x \in (a)^* = (b)^* \Leftrightarrow t \wedge x \wedge b = 0 \Leftrightarrow t \in (b \wedge x)^*$. Therefore $(a \wedge x)^* = (b \wedge x)^*$. And now, $(a \vee x)^* = (a)^* \cap (x)^* = (b)^* \cap (x)^* = (b \vee x)^*$. Hence $(a \vee x)^* = (b \vee x)^*$. □

Now, we have the following properties of $\star-$divisibility.

LEMMA 3.2. *Let L be an ADL with maximal elements. Then for any three elements $a, b, c \in L$, we have the following:*
(1). $(a/0)_*$
(2). *If m is a maximal element of L then* $(m/a)_*$
(3). $(a/a)_*$
(4). $a \leqslant c \Rightarrow (c/a)_*$.
(5). $(a)^* = (b)^* \Rightarrow (a/b)_*$ *and* $(b/a)_*$
(6). $(a/b)_*$ *and* $(b/c)_* \Rightarrow (a/c)_*$
(7). $(a/b)_* \Rightarrow (a/b \wedge x)_*$ *for all* $x \in L$
(8). $(a/b)_* \Rightarrow (a \wedge x/b \wedge x)_*$ *and* $(a \vee x/b \vee x)_*$ *for all* $x \in L$.

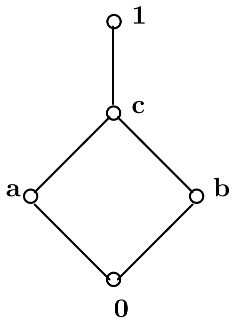
PROOF. (1), (2) and (3) are obviously true.
(4). Suppose $a \leqslant c$. Then $a = a \wedge c$. That implies $(a)^* = (a \wedge c)^*$. Therefore $(c/a)_*$.
(5). Suppose $(a)^* = (b)^*$. Then we have $(a)^* = (b)^* = (b \wedge b)^*$. Hence $(b/a)_*$. Similarly, we get $(a/b)_*$.
(6). Let $(a/b)_*$ and $(b/c)_*$. Then $(b)^* = (a \wedge x)^*$ and $(c)^* = (b \wedge y)^*$, for some $x, y \in L$. Now $d \in (c)^* = (b \wedge y)^* \Leftrightarrow d \wedge b \wedge y = 0 \Leftrightarrow d \wedge y \in (b)^* = (a \wedge x)^* \Leftrightarrow d \wedge y \wedge a \wedge x = 0 \Leftrightarrow d \in (a \wedge x \wedge y)^*$. Therefore $(c)^* = (a \wedge x \wedge y)^*$. Hence $(a/c)_*$.
(7). Let $(a/b)_*$. Then $(b)^* = (a \wedge r)^*$, for some $r \in L$. Now, for any $x \in L$, we get easily that $(b \wedge x)^* = (a \wedge r \wedge x)^*$. Therefore $(a/b \wedge x)_*$.
(8). Assume that $(a/b)_*$. Then $(b)^* = (a \wedge s)^*$, for some $s \in L$. Now, for any $x \in L$, we get easily that $(b \wedge x)^* = (a \wedge s \wedge x)^*$. Therefore $(a \wedge x/b \wedge x)^*$. Now, $(b \vee x)^* = (b)^* \cap (x)^* = (a \wedge s)^* \cap (x)^* = ((a \wedge s) \vee x)^* = (x \vee (a \wedge s))^* = ((x \vee a) \wedge (x \vee s))^* = ((a \vee x) \wedge (x \vee s))^*$. Therefore $(a \vee x/b \vee x)_*$. □

DEFINITION 3.2. For any element a of an ADL L, we define $(a)^\perp$ as the set of all multipliers of a. That is $(a)^\perp = \{x \in L \mid (a/x)_*\}$.

LEMMA 3.3. *Let L be an ADL with maximal elements. Then for any $a, b \in L$, we have the following:*
(1). $(0)^\perp = \{0\}$
(2). $(m)^\perp = L$, *where m is any maximal element of L.*
(3). $a \in (a)^\perp$
(4). $(a)^\perp$ *is an ideal of L.*
(5). $a \in (b)^\perp \Rightarrow (a)^\perp \subseteq (b)^\perp$
(6). $a \leqslant b \Rightarrow (a)^\perp \subseteq (b)^\perp$
(7). $(a)^* = (b)^* \Rightarrow (a)^\perp = (b)^\perp$
(8). $(a)^\perp \cap (b)^\perp = (a \wedge b)^\perp$
(9). *d is a dense element of L if and only if* $(d)^\perp = L$.

Proof. (1). Let $x \in (0)^{\perp}$. Then $(0/x)_*$. That implies $(x)^* = (c \wedge 0)^* = (0)^* = L$. So that $x \in (x)^*$. Therefore $x \wedge x = 0$. Hence $x = 0$. Thus $(0)^{\perp} = \{0\}$.
(2). Let m be any maximal element of an ADL L. Clearly we have $x = m \wedge x$, for all $x \in L$. That implies $(x)^* = (m \wedge x)^*$. Therefore $x \in (m)^{\perp}$ and hence $(m)^{\perp} = L$.
(3). Since $(a)^* = (a \wedge a)^*$, we get $(a/a)_*$. Hence $a \in (a)^{\perp}$.
(4). Let $x, y \in (a)^{\perp}$. Then $(a/x)_*$ and $(a/y)_*$. That implies $(x)^* = (r \wedge a)^*$ and $(y)^* = (s \wedge a)^*$, for some $r, s \in L$. Now $(x \vee y)^* = (x)^* \cap (y)^* = (r \wedge a)^* \cap (s \wedge a)^* = ((r \wedge a) \vee (s \wedge a))^* = ((r \vee s) \wedge a)^*$. Therefore $(a/x \vee y)_*$ and hence $x \vee y \in (a)^{\perp}$. Let $x \in (a)^{\perp}$ and $r \in L$. Then $(a/x)_*$. That implies $(x)^* = (s \wedge a)^*$, for some $s \in L$. Clearly, we get that $(x \wedge r)^* = (s \wedge a \wedge r)^*$. Therefore $(a/x \wedge r)_*$ and hence $x \wedge r \in (a)^{\perp}$. Thus $(a)^{\perp}$ is an ideal of L.
(5). Let $a \in (b)^{\perp}$. Then $(b/a)_*$. That implies $(a)^* = (s \wedge b)^*$, for some $s \in L$. Let $x \in (a)^{\perp}$. Then $(a/x)_*$ and hence $(x)^* = (r \wedge a)^*$, for some $r \in L$. Therefore $(x)^* = (r \wedge a)^* = (r \wedge s \wedge b)^*$. Hence $(b/x)_*$. Thus $x \in (b)^{\perp}$.
(6). Suppose $a \leqslant b$. Let $x \in (a)^{\perp}$. Then $(a/x)_*$. That implies $(x)^* = (r \wedge a)^* = (r \wedge a \wedge b)^*$ for some $r \in L$. Therefore $(b/x)_*$ and hence $x \in (b)^{\perp}$. Thus $(a)^{\perp} \subseteq (b)^{\perp}$.
(7). Suppose $(a)^* = (b)^*$. Let $x \in (a)^{\perp}$. Then $(a/x)_*$. This implies $(x)^* = (r \wedge a)^* = (r \wedge b)^*$, for some $r \in L$. Therefore $(b/x)_*$ and hence $x \in (b)^{\perp}$. Similarly, we verify that $(b)^{\perp} \subseteq (a)^{\perp}$.
(8). Clearly, we have $(a \wedge b)^{\perp} \subseteq (a)^{\perp} \cap (b)^{\perp}$. Let $x \in (a)^{\perp} \cap (b)^{\perp}$. Then $(a/x)_*$ and $(b/x)_*$. Hence $(x)^* = (r \wedge a)^*$ and $(x)^* = (s \wedge b)^*$, for some $r, s \in L$. Now, $(x)^{**} = (x)^{**} \cap (x)^{**} = (r \wedge a)^{**} \cap (s \wedge b)^{**} = ((r \wedge s) \wedge (a \wedge b))^{**}$. That implies $(x)^* = (r \wedge s \wedge a \wedge b)^*$. Thus $((a \wedge b)/x)_*$. Therefore $x \in (a \wedge b)^{\perp}$ and hence $(a)^{\perp} \cap (b)^{\perp} = (a \wedge b)^{\perp}$.
(9). Let m be any maximal element of L. Assume that d is a dense element of L. Then $(d)^* = \{0\} = (m)^*$. Now, $d = m \wedge d \Rightarrow (d)^* = (m \wedge d)^* \Rightarrow (m)^* = (m \wedge d)^*$. Therefore $(d/m)_*$ and hence $m \in (d)^{\perp}$. Thus $(d)^{\perp} = L$. Conversely assume that $(d)^{\perp} = L$. Then maximal element $m \in (d)^{\perp}$. That implies $(d/m)_*$. Therefore $(m)^* = (d \wedge c)^*$. Implies that $\{0\} = (d \wedge c)^*$. Therefore $d \wedge c$ is maximal element and hence d is maximal element. Thus $(d)^* = \{0\}$. □

Let us denote the set of all ideals of the form $(x)^{\perp}$ for all $x \in L$ by $\mathcal{I}^{\perp}(L)$. In general, $\mathcal{I}^{\perp}(L)$ is not a sublattice of $\mathcal{I}(L)$ of all ideals of L. For, consider the following distributive lattice $L = \{0, a, b, c, 1\}$ whose Hasse diagram is given by:



Then clearly $(a)^\perp = \{0, a\}$ and $(b)^\perp = \{0, b\}$. Hence $(a)^\perp \vee (b)^\perp = \{0, a\} \vee \{0, b\} = \{0, a, b, c\}$. But $(a \vee b)^\perp = (c)^\perp = L$(because c is a dense element). Therefore $(a)^\perp \vee (b)^\perp \neq (a \vee b)^\perp$. Thus $\mathcal{I}^\perp(L)$ is not a sublattice of $\mathcal{I}(L)$.

We have the following theorem.

THEOREM 3.1. *For any ADL L, the set $\mathcal{I}^\perp(L)$ forms a complete distributive lattice on its own.*

PROOF. For any $a, b \in L$, define as $(a)^\perp \cap (b)^\perp = (a \wedge b)^\perp$ and $(a)^\perp \sqcup (b)^\perp = (a \vee b)^\perp$. Clearly, $(a \wedge b)^\perp$ is the infimum of both $(a)^\perp$ and $(b)^\perp$ in $\mathcal{I}^\perp(L)$. We have always $(a)^\perp, (b)^\perp \subseteq (a \vee b)^\perp$. Suppose $(a)^\perp \subseteq (c)^\perp$ and $(b)^\perp \subseteq (c)^\perp$ for some $c \in L$. Then we get $a, b \in (c)^\perp$. Since $(c)^\perp$ is an ideal, it gives $a \vee b \in (c)^\perp$. Hence $(a \vee b)^\perp \subseteq (c)^\perp$. Thus $(a \vee b)^\perp$ is the supremum of both $(a)^\perp$ and $(b)^\perp$ in $\mathcal{I}^\perp(L)$. Therefore $\mathcal{I}^\perp(L)$ is a lattice. We now prove the distributivity of these ideals. For any $(a)^\perp, (b)^\perp, (c)^\perp \in \mathcal{I}^\perp(L)$, $(a)^\perp \sqcup \{(b)^\perp \cap (c)^\perp\} = (a)^\perp \sqcup (b \wedge c)^\perp = \{a \vee (b \wedge c)\}^\perp = \{(a \vee b) \wedge (a \vee c)\}^\perp = (a \vee b)^\perp \cap (a \vee c)^\perp = \{(a)^\perp \sqcup (b)^\perp\} \cap \{(a)^\perp \sqcup (c)^\perp\}$. Therefore $(\mathcal{I}^\perp(L), \cap, \sqcup)$ is a distributive lattice. Let a, b be two elements in L. Then $(a)^\perp, (b)^\perp \in \mathcal{I}^\perp(L)$. Define $(a)^\perp \leqslant (b)^\perp \Leftrightarrow (a)^\perp \subseteq (b)^\perp$. Clearly $(\mathcal{I}^\perp(L), \leqslant)$ is a partially ordered set. Clearly $\{0\}$ and L are the bounds for $\mathcal{I}^\perp(L)$. By lemma 3.3(8), we get that $\mathcal{I}^\perp(L)$ is bounded and complete distributive lattice. □

We have the following definition.

DEFINITION 3.3. Let L be an ADL. For any $a, b \in L$, define a relation θ on L as follows:
$(a, b) \in \theta$ if and only if $(a)^\perp = (b)^\perp$.

The following result can be verified easily.

LEMMA 3.4. *Let L be an ADL. Then the relation θ defined above is a congruence on L.*

Let θ be any congruence relation on an ADL L. For any $x \in L$, $[x]_\theta = \{y \in L \mid (x, y) \in \theta\}$. Write $L/\theta = \{[x]_\theta \mid x \in L\}$. Define binary operations $\vee$, $\wedge$ on L/θ by $[x]_\theta \wedge [y]_\theta = [x \wedge y]_\theta$ and $[x]_\theta \vee [y]_\theta = [x \vee y]_\theta$, then it can be verified easily that $(L/\theta, \vee, \wedge)$ is an ADL. Let ρ be the natural homomorphism from L onto L/θ defined by $\rho(x) = [x]_\theta$, for all $x \in L$.

We prove the following lemma.

LEMMA 3.5. *Let θ be any congruence relation on an ADL L. Then* $(0]$ *is the smallest congruence class and D is the unit congruence class of L/θ*

PROOF. Clearly, $(0]$ is the smallest congruence of L/θ. Let $x, y \in D$. Then $(x)^* = (y)^* = \{0\}$. By lemma-3.3(7), we get that $(x)^\perp = (y)^\perp$. Therefore $(x, y) \in \theta$. Thus D is a congruence class of L/θ. Now, let $a \in D$ and $x \in L$. Since D is a filter, we get $a \vee x \in D$. Hence $[x]_\theta \vee [a]_\theta = [a \vee x]_\theta = D$. Thus D is the unit congruence class of L/θ. □

From [**4**], recall that an Almost Distributive Lattice L is called quasi - complemented if for each $x \in L$, there is an element $y \in L$ such that $x \wedge y = 0$ and $x \vee y$ is a dense element.

Now, we establish a set of equivalent conditions for L/θ to become a Boolean algebra which leads to a characterization of quasi-complemented ADL.

THEOREM 3.2. *Let L be an ADL. Then the following conditions are equivalent:*
(1). *L is a quasi-complemented ADL*
(2). *L/θ is a Boolean algebra*
(3). *$\mathcal{I}^{\perp}(L)$ is a Boolean algebra*

PROOF. (1) $\Longrightarrow$ (2): Assume that L is a quasi-complemented ADL. Let $[x]_\theta \in L/\theta$. Since L is a quasi-complemented ADL and $x \in L$, there exists $x' \in L$ such that $x \wedge x' = 0$ and $x \vee x'$ is dense. Therefore $[x]_\theta \cap [x']_\theta = [x \wedge x']_\theta = [0]_\theta$ and also $[x]_\theta \vee [x']_\theta = [x \vee x']_\theta = D$. Hence L/θ. is a Boolean algebra.
(2) $\Longrightarrow$ (3): Assume that L/θ is a Boolean algebra. Define a mapping $\Phi : L/\theta \longrightarrow \mathcal{I}^{\perp}(L)$ by $\Phi([x]_\theta) = (x)^{\perp}$ for all $[x]_\theta \in L/\theta$. Clearly, Φ is well defined. Let $[x]_\theta, [y]_\theta \in L/\theta$, Suppose $\Phi([x]_\theta) = \Phi([y]_\theta)$. Then $(x)^{\perp} = (y)^{\perp}$. This implies $(x, y) \in \theta$. Thus $[x]_\theta = [y]_\theta$. Therefore Φ is injective. Let $(x)^{\perp} \in \mathcal{I}^{\perp}(L)$, where $x \in L$. Now for this x, $\rho(x) = [x]_\theta \in L/\theta$ such that $\Phi([x]_\theta) = (x)^{\perp}$. Therefore Φ is surjective and hence it is bijective. Let $[x]_\theta, [y]_\theta \in L/\theta$ where $x, y \in L$. Then $\Phi([x]_\theta \cap [y]_\theta) = \Phi([x \wedge y]_\theta) = (x \wedge y)^{\perp} = (x)^{\perp} \cap (y)^{\perp} = \Phi([x]_\theta) \cap \Phi([y]_\theta)$. Again $\Phi([x]_\theta \vee [y]_\theta) = \Phi([x \vee y]_\theta) = (x \vee y)^{\perp} = (x)^{\perp} \sqcup (y)^{\perp} = \Phi([x]_\theta) \sqcup \Phi([y]_\theta)$. Thus L/θ is isomorphic to $\mathcal{I}^{\perp}(L)$. Therefore $\mathcal{I}^{\perp}(L)$ is a Boolean algebra.
(3) $\Longrightarrow$ (1): Assume that $\mathcal{I}^{\perp}(L)$ is a Boolean algebra. Let $x \in L$. Then $(x)^{\perp} \in \mathcal{I}^{\perp}(L)$. Since $\mathcal{I}^{\perp}(L)$ is a Boolean algebra, there exists $(y)^{\perp} \in \mathcal{I}^{\perp}(L)$ such that $(x \wedge y)^{\perp} = (x)^{\perp} \cap (y)^{\perp} = (0)^{\perp}$ and $(x \vee y)^{\perp} = (x)^{\perp} \vee (y)^{\perp} = L$. Hence $x \wedge y = 0$ and $x \vee y$ is dense. Therefore L is quasi-complemented. □

Now, we have the following definition.

DEFINITION 3.4. A non-zero element a of an ADL L is called $\star-$prime if $(a/b \wedge c)_*$ implies that $(a/b)_*$ or $(a/c)_*$

We characterized the $\star-$prime elements in the following result.

THEOREM 3.3. *Let a be a non-dense element of an ADL L. Then a is a $\star-$prime element of L if and only if $(a)^{\perp}$ is a prime ideal of L.*

PROOF. Assume that a is $\star-$prime. Let $x, y \in L$ such that $x \wedge y \in (a)^{\perp}$. Then $(a/x \wedge y)_*$. Since a is $\star-$prime, we get either $(a/x)_*$ or $(a/y)_*$. That implies $x \in (a)^{\perp}$ or $y \in (a)^{\perp}$. Therefore $(a)^{\perp}$ is prime ideal of L. Conversely, assume that $(a)^{\perp}$ is a prime ideal of L. Let $x, y \in L$ with $(a/x \wedge y)_*$. Then $x \wedge y \in (a)^{\perp}$. Since $(a)^{\perp}$ is prime, we get either $x \in (a)^{\perp}$ or $y \in (a)^{\perp}$. Hence $(a/x)_*$ or $(a/y)_*$. Therefore a is a $\star-$prime element of L. □

DEFINITION 3.5. A non-zero element a of an ADL L is called $\star-$irreducible if $(a)^* = (b \wedge c)^*$, then either $b \in D$ or $c \in D$.

Now, we have the following lemma.

LEMMA 3.6. *Every dense element of L is a $\star-$irreducible element.*

PROOF. Let d be a dense element of L. Then $(d)^* = (0]$. Suppose $(d)^* = (b \wedge c)^*$, for some $b, c \in L$. Then $(b \wedge c)^* = (0]$. Hence $(b)^* = (0]$ or $(c)^* = (0]$. Thus d is $\star-$irreducible. □

We prove the following theorem.

THEOREM 3.4. *Let a be a non-dense element of an ADL L with maximal elements. Then the following conditions are equivalent:*
(1). *a is $\star-$irreducible.*
(2). *i) $(a)^\perp$ is a maximal among all proper ideals of the form $(x)^\perp$.*
ii) For any $x \in L$, $(a)^* = (a \wedge x)^*$ implies $(x)^* = (0]$.

PROOF. Let m be any maximal element of an ADL L.
$(1) \Longrightarrow (2)$(i): Assume that a is a $\star-$irreducible element. Suppose $(a)^\perp \subseteq (b)^\perp \neq L$ for some a non-zero element b of L. We have $a \in (a)^\perp \subseteq (b)^\perp$. Then $(b/a)_*$. So that there exists $c \in L$ such that $(a)^* = (c \wedge b)^*$. Since a is $\star-$irreducible, we get that either $(b)^* = (0]$ or $(c)^* = (0]$. Since $(b)^\perp \neq L$, by lemma-3.3(9), we get that $(b)^* \neq (0]$. Hence $(c)^* = (0]$. Now, $(c)^* = (0] = (m)^* \Rightarrow (b \wedge c)^* = (b \wedge m)^* \Rightarrow (b \wedge c)^* = (b)^* \Rightarrow (a)^* = (b)^* \Rightarrow (a)^\perp = (b)^\perp$. Therefore $(a)^\perp$ is maximal among all ideals of the form $(x)^\perp$.
$(1) \Longrightarrow (2)$(ii): Suppose $(a)^* = (a \wedge x)^*$ for $x \in L$. Since a is $\star-$irreducible, we get that either $(a)^* = (0]$ or $(x)^* = (0]$. Since a is non-dense, we must have $(x)^* = (0]$.
$(2) \Longrightarrow (1)$: Assume the conditions (2)(i) and 2(ii). Suppose $(a)^* = (c \wedge d)^*$ for some $c, d \in L$. Hence $(d/a)_*$. So we get $a \in (d)^\perp$ and hence $(a)^\perp \subseteq (d)^\perp$. Since the ideal $(a)^\perp$ is maximal, we get that either $(a)^\perp = (d)^\perp$ or $(d)^\perp = L$. Suppose $(a)^\perp = (d)^\perp$. Then we get $d \in (a)^\perp \Rightarrow (a/d)_* \Rightarrow (d)^* = (r \wedge a)^*$ for some $r \in L \Rightarrow (c \wedge d)^* = (c \wedge r \wedge a)^* \Rightarrow (a)^* = (c \wedge r \wedge a)^* \Rightarrow (c \wedge r)^* = (0]$ by (2)(ii) $\Rightarrow (c)^* = (0]$. Suppose $(d)^\perp = L$. Let m be any maximal element of L. Then we have $m \in (d)^\perp$. Hence $(d/m)_*$. Then there exists some $s \in L$ such that $(m)^* = (s \wedge d)^*$. Thus $(s \wedge d)^* = \{0\}$ and hence $(d)^* = (0]$. Therefore a is a $\star-$irreducible element. □

We conclude this paper with the following result.

THEOREM 3.5. *Let L be an ADL. Then every $\star-$irreducible element of L is a $\star-$prime element.*

PROOF. If a is a dense element of an ADL L, then we are through. Suppose a is non-dense. Assume that a is a $\star-$irreducible element of L. Then by above theorem, $(a)^\perp$ is a maximal among all ideals of the form $(r)^\perp$. Choose $x, y \in L$ such that $x \notin (a)^\perp$ and $y \notin (a)^\perp$. Hence $(a)^\perp \subset (a)^\perp \vee (x] \subseteq (a)^\perp \vee (x)^\perp \subseteq (a)^\perp \sqcup (x)^\perp$ and also $(a)^\perp \subset (a)^\perp \sqcup (y)^\perp$. By the maximality of $(a)^\perp$, we get that $(a)^\perp \sqcup (x)^\perp = L$ and $(a)^\perp \sqcup (y)^\perp = L$. Now, $L = L \cap L = \{(a)^\perp \sqcup (x)^\perp\} \cap \{(a)^\perp \sqcup (y)^\perp\} = (a)^\perp \sqcup \{(x)^\perp \cap (y)^\perp\} = (a)^\perp \sqcup (x \wedge y)^\perp$. If $x \wedge y \in (a)^\perp$, then $(x \wedge y)^\perp \subseteq (a)^\perp$. Hence $(a)^\perp = L$. Which is a contradiction. Thus $(a)^\perp$ is a prime ideal. Therefore by theorem 3.3, a is a $\star-$prime element of L. □

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