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BLOCK LINE CUT VERTEX DIGRAPHS OF DIGRAPHS

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ABSTRACT. In this paper, the digraph valued function(digraph operator), namely the block line cut vertex digraph $BLC(D)$ of a digraph D is defined, and the problem of reconstructing a digraph from its block line cut vertex digraph is presented. Outer planarity, maximal outer planarity, and minimally non-outer planarity properties of these digraphs are discussed.

1. Introduction

Notations and definitions not introduced here can be found in [2,3]. For a simple graph G with vertex set $V(G) = \{v_1, v_2, \ldots, v_n\}$, detailed by V.R. Kulli et al.[5] gave the following definition. The *block line cut vertex graph* of G, written $Bn(G)$, is the graph whose vertices are the edges, cut vertices, and blocks of G, with two vertices of $Bn(G)$ adjacent whenever the corresponding members of G are adjacent or incident, where the edges, cut vertices, and blocks of G are called its *members*.

In this paper, we extend the definition of the block line cut vertex graph of a graph to a directed graph. M.Aigner [1] defines the *line digraph* of a digraph as follows. Let D be a digraph with n vertices $v_1, v_2, \ldots, v_n$ and m arcs, and $L(D)$ its associated *line digraph* with $n^{'}$ vertices and $m^{'}$ arcs. We immediately have $n^{'} = m$ and $m^{'} = \sum_{i=1}^{n} d^-(v_i) \cdot d^+(v_i)$. Furthermore, the in-respectively out-degree of a vertex $v^{'} = (v_i, v_j)$ in $L(D)$ are $d^-(v^{'}) = d^-(v_i)$, $d^+(v^{'}) = d^+(v_j)$. Also, a digraph D is said to be a *line digraph* if it is isomorphic to the line digraph of a certain digraph H[7].

We need some concepts and notations on directed graphs. A *directed graph*(or just *digraph*) D consists of a finite non-empty set $V(D)$ of elements called *vertices* and a finite set $A(D)$ of ordered pair of distinct vertices called *arcs*. Here, $V(D)$ is

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the *vertex set* and $A(D)$ is the *arc set* of D. For an arc (u, v) or uv the first vertex u is its *tail* and the second vertex v is its *head*. The *out-degree* of a vertex v, written $d^+(v)$, is the number of arcs going out from v and the *in-degree* of a vertex v, written $d^-(v)$, is the number of arcs coming into v. The *total degree* of a vertex v, written $td(v)$, is the number of arcs incident with v, i.e., $td(v) = d^-(v) + d^+(v)$. A vertex v for which $d^+(v) = d^-(v) = 0$ is called an *isolate*. A vertex v is called a *transmitter* or a *receiver* according as $d^+(v) > 0, d^-(v) = 0$ or $d^+(v) = 0, d^-(v) > 0$. An *out-star* (*in-star*) in a digraph D is a star in the underlying undirected graph of D such that all arcs are directed out of (into) the center. The out-star and in-star of order k is denoted by S_k^+ and S_k^-, respectively.

A *cut set* of a digraph D is defined as a minimal set of vertices whose removal increases the number of connected components of D. A cut set of size one is called a *cut vertex*. A *block* of a digraph D is a maximal connected subdigraph B of D such that no vertex of B is a cut vertex of D. A *tournament* is a digraph whose underlying graph is a complete graph. A tournament of order n is denoted by T_n.

Since most of the results and definitions for undirected planar graphs are valid for planar digraphs also, the following definitions hold good for planar digraphs. A *planar drawing* of a digraph D is a drawing of D in which no two distinct arcs intersect. A digraph is said to be *planar* if it admits a planar drawing. If D is a planar digraph, then the *inner vertex number* $i(D)$ of D is the minimum number of vertices not belonging to the boundary of the exterior region in any embedding of D in the plane. A digraph D is an *outerplanar* if $i(D) = 0$ and *minimally non-outerplanar* if $i(D) = 1$ ([4]).

2. Definition of BLC(D):

For a connected digraph D, the *block line cut vertex digraph* $Q = BLC(D)$ has vertex set $V(Q) = A(D) \cup C(D) \cup B(D)$ and the arc set

$$A(Q) = \begin{cases} ab : a, b \in A(D), \text{the head of } a \text{ coincides with the tail of } b, \\ Cd : C \in C(D), d \in A(D), \text{the tail of } d \text{ is } C, \\ dC : C \in C(D), d \in A(D), \text{the head of } d \text{ is } C, \\ Be : B \in B(D), e \in A(D), \text{the arc } e \text{ lie on block } B. \end{cases}$$

Here $C(D)$ is the cut vertex set and $B(D)$ is the block set of D.

For a connected digraph D with $V(Q) = A(D) \cup C(D)$, the first three conditions of $A(Q)$ is the *line cut vertex digraph* of D and denoted by $LC(D)$.

Clearly, $LC(D) \subseteq BLC(D)$, where $\subseteq$ is the subdigraph notation.

3. Decomposition and Reconstruction

One of the major challenges in the study of digraph operators is to reproduce the original digraph from the digraph operator, i.e., when is a digraph the block line cut vertex digraph of a certain digraph D and is D reconstructible from $BLC(D)$?

A digraph D is a *complete bipartite digraph* if its vertex set can be partitioned into two sets A, B in such a way that every arc has its initial vertex in A and its terminal vertex in B and any two vertices $a \in A$ and $b \in B$ are joined by an arc. An arc (u, v) of D is said to be an *end arc* if u is the transmitter and v is the receiver.

Let D be a digraph with vertex set $V(D) = \{v_1, v_2, \ldots, v_n\}$, cut vertex set $C(D) = \{C_1, C_2, \ldots, C_r\}$, and block set $B(D) = \{B_1, B_2, \ldots, B_s\}$. We consider the following four cases.

Case 1: Let v be a vertex of D with $d_D^-(v) = \alpha$ and $d_D^+(v) = \beta$. Then α arcs coming into v and the β arcs going out from v give rise to a complete bipartite subdigraph with α tails and β heads and $\alpha \cdot \beta$ arcs joining each tail with each head. This is the decomposition of $L(D)$ into mutually arc disjoint complete bipartite subdigraphs.

Case 2: Let C be a cut vertex of D with $d_D^-(C) = \alpha^{'}$. Then $\alpha^{'}$ arcs coming into C give rise to a complete bipartite subdigraph with $\alpha^{'}$ tails and a single head(i.e., C) and $\alpha^{'}$ arcs joining each tail with C.

Case 3: Let C be a cut vertex of D with $d_D^+(C) = \beta^{'}$. Then $\beta^{'}$ arcs going out from C give rise to a complete bipartite subdigraph with a single tail (i.e., C) and $\beta^{'}$ heads and $\beta^{'}$ arcs joining C with each head.

Case 4: Let B be a block of D. Then the arcs, say γ that lie on B give rise to a complete bipartite subdigraph with a single tail(i.e., B) and γ heads and γ arcs joining B with each head.

Hence by all above cases, $H = BLC(D)$ is decomposed into mutually arc-disjoint complete bipartite subdigraphs with $V(H) = A(D) \cup C(D) \cup B(D)$ and arc sets (i) $\cup_{i=1}^{n} X_i \times Y_i$, where X_i and Y_i be the sets of in-coming and out-going arcs at v_i, respectively. (ii) $\cup_{j=1}^{r} \cup_{k=1}^{r} Z_j^{'} \times C_k$ such that $Z_j^{'} \times C_k = 0$ for $j \neq k$, (iii) $\cup_{k=1}^{r} \cup_{j=1}^{r} C_k \times Z_j$ such that $C_k \times Z_j = 0$ for $k \neq j$, where $Z_j^{'}$ and Z_j be the sets of in-coming and out-going arcs at C_j, respectively. (iv) $\cup_{l=1}^{s} \cup_{l'=1}^{s} B_l \times N_{l'}$ such that $B_l \times N_{l'} = 0$ for $l \neq l^{'}$, where $N_{l'}$ is the set of arcs that lie on B_l of D.

Conversely, let H be a digraph of the type described above. It should be noted that the subdigraphs obtained by Case 2, Case 3 and Case 4 are used to identify the cut vertex(es) and blocks of D. So, we first reconstruct D without cut vertex(es) and blocks. For that, let us denote each of the complete bipartite subdigraphs obtained by Case 1 by $T_1, T_2, \ldots, T_p$. Let $V(D) = \{t_0, t_1, \ldots, t_p, t_{p+1}\}$. On the other hand, if D has end arcs, then $V(D) = \{t_0, t_1, \ldots, t_p, t_{p+1}, t_1^{'}, t_2^{'}, t_3^{'}, \ldots\}$, where $t_1^{'}, t_2^{'}, t_3^{'}, \ldots$ are the vertices corresponding to end arcs $e_1^{'}, e_2^{'}, e_3^{'}, \ldots$ of D, respectively. The arcs of D are obtained by the following procedure. For each vertex $v \in L(D)$, we draw an arc, say a_v to D as follows.

Step 1: If $d_{L(D)}^+(v) > 0$, $d_{L(D)}^-(v) = 0$, then $a_v = (t_0, t_i)$, where i is the index(or base) of T_i such that $v \in X_i$;

Step 2: If $d_{L(D)}^+(v) = 0$, $d_{L(D)}^-(v) > 0$, then $a_v = (t_j, t_{p+1})$, where j is the index of T_j such that $v \in Y_j$;

Step 3: If $d^+_{L(D)}(v) > 0$, $d^-_{L(D)}(v) > 0$, then $a_v = (t_i, t_j)$, where i and j are the indices of T_i and T_j such that $v \in X_j \cap Y_i$.

If D has an end arc, then the corresponding vertex in $L(D)$ is an isolate. Then,

Step 4: Let e_1 and $e_1^{'}$ be an arc and end arc of D, respectively, and let v be a vertex of $e_1^{'}$ such that $d^-(v) > 0$, $d^+(v) = 0$. Then $a_v = (t_1^{'}, t_{p+1})$.

Step 5: Let e_1 and $e_1^{'}$ be an arc and end arc of D, respectively, and let v be a vertex of $e_1^{'}$ such that $d^+(v) > 0$, $d^-(v) = 0$. Then $a_v = (t_0, t_1^{'})$.

We now mark the cut vertices of D as follows. From Case 2 and Case 3, we observe that for every cut vertex C, there exists at most two complete bipartite subdigraphs, one containing C as the tail, and other as head. Let it be $C_j^{'}$ and $C_j^{''}$, $1 \leqslant j \leqslant r$ such that $C_j^{'}$ contains C as the tail and $C_j^{''}$ contains C as the head. If the heads of $C_j^{'}$ and tails of $C_j^{''}$ are the heads and tails of a single T_i, $1 \leqslant i \leqslant p$, then the vertex t_i is a cut vertex in D, where i is the index of T_i. If the (original digraph) D has an end arc, then a vertex of an end arc whose total degree at least two is a cut vertex in the reconstruction.

Finally, we mark the blocks of D as follows. Let us denote each of the complete bipartite subdigraphs obtained by Case 4 by $B_1^{'}, B_2^{'}, \ldots, B_s^{'}$. Now, for all l such that $1 \leqslant l \leqslant s$, if all heads of $B_l^{'}$ are the tails(or heads) of some T_i, then the arcs joining the vertices t_i forms a block in the reconstruction, where i is the indices of T_i. Furthermore, an arc whose one of the end vertices having total degree one is a block. The digraph D thus constructed apparently has H as its block line cut vertex digraph. Hence we have the following Theorem.

THEOREM 3.1. *H is the block line cut vertex digraph of a certain digraph D if and only if $V(H) = A(D) \cup C(D) \cup B(D)$ and arc sets $A(H)$ equals : (i) $\cup_{i=1}^{n} X_i \times Y_i$, (ii) $\cup_{j=1}^{r} \cup_{k=1}^{r} Z_j^{'} \times C_k$ such that $Z_j^{'} \times C_k = 0$ for $j \neq k$, (iii) $\cup_{k=1}^{r} \cup_{j=1}^{r} C_k \times Z_j$ such that $C_k \times Z_j = 0$ for $k \neq j$, (iv) $\cup_{l=1}^{s} \cup_{l'=1}^{s} B_l \times N_{l'}$ such that $B_l \times N_{l'} = 0$ for $l \neq l^{'}$.*

The following existing theorems are required to prove further results:

Theorem A ([2]): Every maximal outerplanar graph G with n vertices has $(2n-3)$ edges.

Theorem B ([3]): A directed multi digraph D is Eulerian if and only if D is connected and $d^-_D(v) = d^+_D(v)$, for every vertex $v \in D$.

4. Properties of the block line cut vertex digraph

In this section, we establish some basic relationships between a digraph and its block line cut vertex digraph. The first Theorem is clear, and we omit the proof.

THEOREM 4.1. *Let D be a digraph with vertex set $V(D) = \{v_1, v_2, \ldots, v_n\}$, cut vertex set $C(D) = \{C_1, C_2, \ldots, C_r\}$, and block set $B(D) = \{B_1, B_2, \ldots, B_s\}$. Then* *the order and size of* $BLC(D)$ *are* $m+\sum_{j=1}^{r} C_j + \sum_{k=1}^{s} B_k$ *and*

$m+\sum_{i=1}^{n} d^-(v_i)\cdot d^+(v_i) + \sum_{j=1}^{r}\{d^-(C_j)+d^+(C_j)\}$, *respectively, where* m *is the size of* D.

Theorem 4.2. *The block line cut vertex digraph* $BLC(D)$ *of a digraph* D *is always non-Eulerian.*

Proof. For every block vertex $B \in BLC(D)$, $d^-_{BLC(D)}(B) = 0$, $d^+_{BLC(D)}(B) > 0$. Hence $d^-_{BLC(D)}(B) \neq d^+_{BLC(D)}(B)$. By Theorem[B], $BLC(D)$ is non-Eulerian. □

Theorem 4.3. *For a connected digraph* D, $BLC(D)$ *is an outerplanar if*
(a) D *is a directed path* $\vec{P_n}$ *on* $n \geqslant 3$ *vertices.*
(b) D *is an in-star* $($*out-star*$)$ *of order* $n \geqslant 3$.

Proof. **Case 1**: Suppose D is a directed path $\vec{P_n}$ on $n \geqslant 3$ vertices. Then every block of $BLC(D)$ is either T_2 or T_3 such that $i(BLC(D)) = 0$. Thus, $BLC(D)$ is an outerplanar.
Case 2: Suppose D is an in-star(out-star) with vertex set $V(D) = \{v_1, v_2, \ldots, v_n\}$ and arc set $A(D) = \{e_1, e_2, \ldots, e_{n-1}\}$, $n \geqslant 3$. Then $L(D)$ is totally disconnected of order $(n-1)$. The number of cut vertex of D is exactly one. Then $LC(D)$ is an in-star(out-star) of order n such that the center of $LC(D)$ is the cut vertex of D. Finally, since every arc of an in-star(out-star) is a block, the arcs incident out of corresponding block vertices reaches the vertices of $L(D)$ gives $BLC(D)$ such that $i(BLC(D)) = 0$. This completes the proof. □

Theorem 4.4. *For any connected digraph* D, $BLC(D)$ *is not maximal outerplanar.*

Proof. We prove this by the method of contradiction. Suppose $BLC(D)$ is maximal outerplanar. We consider the following two cases.

Case 1: Let D be a directed path on $n \geqslant 3$ vertices. By Theorem 4.1, the order and size of $BLC(D)$ are $3\phi+2$ and $4\phi+1$, respectively, where $\phi = (n-2), n \geqslant 3$. But, $4\phi+1 < 6\phi+1 = 2(3\phi+2)-3$. By Theorem [A], $BLC(D)$ is not maximal outerplanar, a contradiction.

Case 2: Let D be an in-star(out-star) of order $n \geqslant 3$. By Theorem 4.1, the order and size of $BLC(D)$ are $2\phi+3$ and $2\phi+2$, respectively, where $\phi = (n-2), n \geqslant 3$. But, $2\phi+2 < 4\phi+3 = 2(2\phi+3)-3$. By Theorem[A], $BLC(D)$ is not maximal outerplanar, a contradiction. This completes the proof. □

Theorem 4.5. *For a digraph* $D = C_3 \cup \{e\}$, *i.e., the 3-directed cycle with a pendant arc,* $BLC(D)$ *is minimally non-outerplanar.*

Proof. Suppose $D = C_3 \cup \{e\}$. Let $V(D) = \{a, b, c, d\}$ and

$$A(D) = \{(a, b), (b, c), (c, a), (c, d)\}.$$

Then $A(L(D)) = \{(ab, bc), (bc, ca), (bc, cd), (ca, ab)\}$. Now, c is the cut vertex of D such that c is the tail of arcs $T = \{(c, a), (c, d)\}$ and head of an arc $H = (b, c)$. Then the arcs incident into c from the vertices corresponding to H, and the arcs incident out of c reaches the vertices corresponding to arcs of T in $L(D)$ gives $LC(D)$ such that $i(LC(D)) = 0$. Let $B_1 = \{(a, b), (b, c), (c, a)\}$ and $B_2 = (c, d)$ be two blocks of D. Then the arcs incident out of block vertices B_1 and B_2 in $LC(D)$ gives $BLC(D)$ such that $i(BLC(D)) = 1$. Hence $BLC(D)$ is minimally non-outerplanar. □

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