BULLETIN OF THE INTERNATIONAL MATHEMATICAL VIRTUAL INSTITUTE
ISSN (p) 2303-4874, ISSN (o) 2303-4955
www.imvibl.org /JOURNALS / BULLETIN
Vol. 5(2015), 133-140

Former
BULLETIN OF THE SOCIETY OF MATHEMATICIANS BANJA LUKA
ISSN 0354-5792 (o), ISSN 1986-521X (p)

Gradings on Semidihedral Blocks With Two Simple Modules

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Abstract. In this paper we show that tame blocks of group algebras with semidihedral defect groups and two isomorphism classes of simple modules can be non-trivially graded. We prove this by using the transfer of gradings via derived equivalences.

1. Introduction and preliminaries

This paper is a continuation of a series of papers [**6, 7, 8**] in which we study existence of gradings on blocks of group algebras. In our previous paper [**8**] we proved that tame blocks of group algebras with semidihedral defect groups and three isomorphism classes of simple modules can be non-trivially graded. We use the same techniques of transfer of gradings via derived equivalences in order to construct non-trivial gradings on semidihedral blocks with two simple modules.

Let A be an algebra over a field k. We say that A is a graded algebra if A is the direct sum of subspaces $A = \bigoplus_{i \in \mathbb{Z}} A_i$, such that $A_i A_j \subset A_{i+j}$, $i, j \in \mathbb{Z}$. The subspace A_i is said to be the homogeneous subspace of degree i. It is obvious that we can always trivially grade A by setting $A_0 = A$. In this paper we study the problem of existence of non-trivial gradings on blocks of group algebras with semidihedral defect groups and two simple modules. We refer the reader to [**1**] for details on defect groups of blocks of group algebras. Tame blocks of group algebras appear only for group algebras over fields of characteristic 2, so for the remainder of this paper we will assume that the field we work over is of characteristic 2. All algebras in this paper are finite dimensional algebras over the field k, and all modules will be

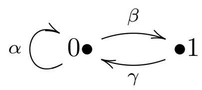
2010 *Mathematics Subject Classification.* Primary 16A03; Secondary 16W50, 20C20.
Key words and phrases. Graded algebras, derived equivalence, semidihedral defect groups.

left modules. The category of finite dimensional A–modules is denoted by A–mod and the full subcategory of finite dimensional projective A–modules is denoted by P_A. The derived category of bounded complexes over A–mod is denoted by $D^b(A)$, and the homotopy category of bounded complexes over P_A will be denoted by $K^b(P_A)$.

We refer the reader to [**5, 8**] for introductory remarks about graded algebras and modules. The aim of this paper is to show how one can use transfer of gradings via derived equivalences to grade tame blocks with semidihedral defect groups and two simple modules. These complex methods of constructing gradings on associative algebras have previously been studied in [**12, 5, 6, 7**].

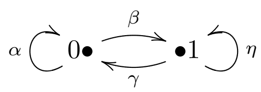
1.1. Semidihedral blocks with two simple modules. Any block with a semidihedral defect group and two isomorphism classes of simple modules is Morita equivalent to some algebra from the following list (cf. [9]).

(1) For any integer $r \geqslant 2$ and any $c \in k$ let $SD(2A)_1 := SD(2A)_1^{r,c}$ be the algebra defined by the quiver and relations



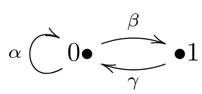
$$\alpha^2 = c(\alpha\beta\gamma)^r,\ \beta\gamma\beta = (\alpha\beta\gamma)^{r-1}\alpha\beta,$$
$$\gamma\beta\gamma = (\gamma\alpha\beta)^{r-1}\gamma\alpha,\ (\alpha\beta\gamma)^r\alpha = 0.$$

(2) For any integer $r \geqslant 3$ and any $c \in k$ let $SD(2B)_2 := SD(2B)_2^{r,c}$ be the algebra defined by the quiver and relations



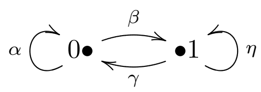
$$\beta\eta = \alpha\beta\gamma\alpha\beta,\ \eta\gamma = \gamma\alpha\beta\gamma\alpha,$$
$$\beta\eta^2 = 0,\ \gamma\beta = \eta^{r-1},$$
$$\alpha^2 = c(\alpha\beta\gamma)^2,\ \eta^2\gamma = 0.$$

(3) For any integer $r \geqslant 2$ and any $c \in k$ let $SD(2A)_2 := SD(2A)_2^{r,c}$ be the algebra defined by the quiver and relations



$$\gamma\beta = 0,\ (\alpha\beta\gamma)^r = (\beta\gamma\alpha)^r,$$
$$\alpha^2 = (\beta\gamma\alpha)^{r-1}\beta\gamma + c(\alpha\beta\gamma)^r.$$

(4) For any integer $r \geqslant 2$ and any $c \in k$ let $SD(2B)_1 := SD(2B)_1^{r,c}$ be the algebra defined by the quiver and relations



$$\gamma\beta = \eta\gamma = \beta\eta = 0,\ \eta^r = \gamma\alpha\beta,$$
$$\alpha^2 = \beta\gamma + c(\beta\gamma\alpha),\ \alpha\beta\gamma = \beta\gamma\alpha.$$

We recommend [**2**] and [**4**] as a good introduction to path algebras of quivers.

1.2. Derived equivalence classes. If B is a block with a semidihedral defect group of order 2^n and with two isomorphism classes of simple modules, then the center of B either has dimension $2^{n-2}+4$ or $2^{n-2}+3$ (cf. [9]). If the dimension of the center is $2^{n-2}+4$, then B is Morita equivalent to $SD(2A)_1^{2^{n-2},c}$ or to $SD(2B)_2^{2^{n-2},c}$ for some $c \in k$. If the dimension of the center is $2^{n-2}+3$, then B is Morita equivalent to $SD(2A)_2^{2^{n-2},c}$ or to $SD(2B)_1^{2^{n-2},c}$ for some $c \in k$ (cf. [9]). Since the center of an algebra is invariant under derived equivalence, these two cases lead to different classes of derived equivalent blocks. For fixed r and c, Holm proved in [10] that $SD(2A)_1^{r,c}$ and $SD(2B)_2^{r,c}$ are derived equivalent, and that $SD(2A)_2^{r,c}$ and $SD(2B)_1^{r,c}$ are derived equivalent. It is not known if for different scalars c, these algebras belong to the same derived equivalence class.

2. Transfer of gradings via derived equivalences

The rest of the paper is devoted to proving the following theorem.

THEOREM 2.1. *Let A be a tame block of group algebras with semidihedral defect groups and two isomorphism classes of simple modules. There exists a non-trivial grading on A.*

We recall from [**8**] the procedure of transfer of gradings via derived equivalences. Let A and B be two symmetric algebras over a field k and let us assume that A is a graded algebra, and that A and B are derived equivalent. For a given tilting complex T of A-modules, which is a bounded complex of finitely generated projective A-modules, there exists a structure of a complex of graded A-modules T' on T. If T is a tilting complex that tilts from A to B, then $\mathrm{End}_{K^b(P_A)}(T) \cong B^{op}$. Viewing T as a graded complex T', and by computing its endomorphism ring as a graded object, we get a graded algebra which is isomorphic to the opposite algebra of the algebra B. The choice of a grading on T' is unique up to shifting the grading of each indecomposable summand of T', because any two different gradings on an indecomposable module (bounded complex) differ only by a shift (see [3, Lemma 2.5.3]). We refer the reader to [**11, 6, 7**] for details on categories of graded modules and their derived categories, and [**13**] for basics of homological algebra.

We will use tilting complexes given in [10] to transfer gradings from $SD(2A)_1^{r,c}$ to $SD(2B)_2^{r,c}$, and from $SD(2A)_2^{r,c}$ to $SD(2B)_1^{r,c}$.

For the remainder of this paper, if we say that an algebra given by a quiver and relations is graded, we will assume that it is graded in such a way that the arrows and the vertices of its quiver are homogeneous.

Let us now fix an integer r and an element c from the field k, and let us assume that $SD(2A)_1$ is graded in such a way that $d_1 = \deg(\alpha)$, $d_2 = \deg(\beta)$ and $d_3 = \deg(\gamma)$. From the relations of $SD(2A)_1$ it follows that $2d_2 + 2d_3 = r(d_1 + d_2 + d_3)$. We write Σ for $d_1 + d_2 + d_3$.

The graded radical layers of the projective indecomposable $SD(2A)_1$-modules are (numbers to the left and right of the radical layers denote degrees of the corresponding layers):

$$\begin{array}{ccccc}
 & & S_0 & & \\
d_1 & S_0 & & S_1 & d_3 \\
d_1 + d_3 & S_1 & & S_0 & d_3 + d_2 \\
\Sigma & S_0 & & S_0 & \Sigma \\
 & \vdots & & \vdots & \\
(r-1)\Sigma + d_1 & S_0 & & S_1 & (r-1)\Sigma + d_3 \\
r\Sigma - d_2 & S_1 & & S_0 & r\Sigma - d_1 \\
r\Sigma & & S_0 & &
\end{array},$$

$$\begin{array}{ccccc}
 & & S_1 & & \\
d_2 & & S_0 & & \\
d_1 + d_2 & S_0 & & & \\
\Sigma & S_1 & & & \\
 & \vdots & & S_1 & d_2 + d_3 \\
(r-1)\Sigma + d_2 & S_0 & & & \\
2d_2 + d_3 & & S_0 & & \\
2d_2 + 2d_3 & & S_1 & &
\end{array}.$$

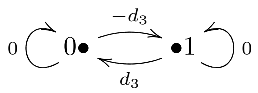
A tilting complex T that tilts from $SD(2A)_1$ to $SD(2B)_2$ is given by $T := T_0 \oplus T_1$, where T_1 is the stalk complex with P_1 in degree 0, and T_0 is given by

$$0 \longrightarrow P_1\langle -d_3\rangle \oplus P_1\langle -d_1 - d_3\rangle \xrightarrow{(\gamma,\gamma\alpha)} P_0 \longrightarrow 0,$$

with non-zero terms in degrees 0 and 1.

If $c \neq 0$, then the relations of $SD(2A)_1$ force that $\Sigma = 0 = d_1 = d_2 + d_3$. Also, the relations of $SD(2B)_2$ imply that $\deg(\beta) + \deg(\gamma) = 0$ and $\deg(\eta) = \deg(\alpha) = 0$ in the quiver of $SD(2B)_2$. Only $\deg(\gamma)$ is left to be determined. Since the only summand of $\mathrm{Homgr}_{K^b(P_{SD(2A)_1})}(T_1, T_0)$ is $k\langle -d_3\rangle$, we conclude that $\deg(\gamma) = d_3$ and $\deg(\beta) = -d_3$ in the quiver of $SD(2B)_2$.

Therefore, when $c \neq 0$ the resulting graded quiver of $SD(2B)_2^{r,c}$ is given by



We note here that the resulting grading is never a positive grading, because d_3 and $-d_3$ can not both be positive. The same holds for the grading on $SD(2A)_1$.

In the case when $c = 0$ we have fewer relations and a bit more work to do.

From

$$\mathrm{Endgr}_{K^b(P_{SD(2A)_1})}(T_1) \cong \mathrm{Endgr}_{SD(2A)_1}(P_1) \cong \bigoplus_{t=0}^{r} k\langle -t\Sigma\rangle \oplus k\langle d_2 + d_3\rangle$$

we conclude that $\deg(\eta) = \Sigma$ and $\deg(\gamma) + \deg(\alpha) + \deg(\beta) = d_2 + d_3$, because $\gamma\alpha\beta$ is the only path starting and ending at vertex 1 which can not be written as a linear combination of powers of η. From this and the relations of $SD(2B)_2$ we have that $\deg(\alpha) = \deg(\eta) - (\deg(\alpha) + \deg(\beta) + \deg(\gamma))$. Hence, it follows that $\deg(\alpha) = \Sigma - d_2 - d_3 = d_1$.

From the relations of $SD(2B)_2$ we have that

$$\deg(\beta) + \deg(\gamma) = (r-1)\Sigma = d_2 + d_3 - d_1.$$

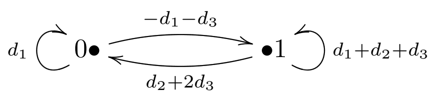
There are four copies of S_1 as a composition factor of $\ker(\gamma, \gamma\alpha)$. From this it follows that the graded vector space $\mathrm{Homgr}_{K^b(P_{SD(2A)_1})}(T_1, T_0)$ is isomorphic to the sum $k\langle -d_2 - 2d_3\rangle \oplus k\langle -2d_2 - 3d_3\rangle \oplus k\langle -d_1 - d_2 - 2d_3\rangle \oplus k\langle -2d_2 - 3d_3 - d_1\rangle$. Hence, we conclude that

$$\{\deg(\gamma), \deg(\gamma\alpha), \deg(\gamma\alpha\beta\gamma), \deg(\gamma\alpha\beta\gamma\alpha)\} =$$

$$= \{d_2 + 2d_3, 2d_2 + 3d_3, d_1 + d_2 + 2d_3, d_1 + 2d_2 + 3d_3\}.$$

Combining this with the previous paragraph gives us that $\deg(\gamma) = d_2 + 2d_3$, $\deg(\alpha) = d_1$, $\deg(\beta) = -d_1 - d_3$ and $\deg(\eta) = d_1 + d_2 + d_3$.

With respect to this grading, the graded quiver of $SD(2B)_2$ is given by



Notice that the previous graded quiver that we got with the additional assumption that $c \neq 0$, is just a special case of the last graded quiver. We get it from the latter quiver by setting $d_1 = 0$ and $d_2 = -d_3$.

Also, the resulting grading is positive for an appropriate choice of degrees d_i. If $-d_1 - d_3 > 0$, then $d_1 + d_3 < 0$, which forces $d_2 > -d_1 - d_3$ and $d_2 > -2d_3$.

We proceed by transferring gradings from $SD(2A)_2$ to $SD(2B)_1$. Let us fix an integer r and an element c from the field k. Let us assume that $SD(2A)_2$ is graded in such a way that $d_1 = \deg(\alpha)$, $d_2 = \deg(\beta)$ and $d_3 = \deg(\gamma)$. Since the relations of $SD(2A)_1$ are homogeneous, it follows that $3d_1 = r(d_1 + d_2 + d_3)$. We write Σ for $d_1 + d_2 + d_3$.

The graded radical layers of the projective indecomposable $SD(2A)_2$-modules are:

$$\begin{array}{cccc}
 & \multicolumn{2}{c}{S_0} & \\
d_1 & S_0 & S_1 & d_3 \\
d_1 + d_3 & S_1 & S_0 & d_2 + d_3 \\
\Sigma & S_0 & S_0 & \Sigma \\
 & \vdots & \vdots & \\
(r-1)\Sigma + d_1 & S_0 & S_1 & (r-1)\Sigma + d_3 \\
(r-1)\Sigma + d_1 + d_3 & S_1 & S_0 & (r-1)\Sigma + d_3 + d_2, \\
r\Sigma & \multicolumn{2}{c}{S_0} &
\end{array},$$

$$\begin{array}{cc}
S_1 & \\
S_0 & d_2 \\
S_0 & d_1 + d_2 \\
S_1 & \Sigma \\
\vdots & \\
S_0 & (r-1)\Sigma + d_2 \\
S_0 & (r-1)\Sigma + d_2 + d_1 \\
S_1 & r\Sigma
\end{array}.$$

A tilting complex T that tilts from $SD(2A)_2$ to $SD(2B)_1$ is given by $T := T_0 \oplus T_1$, where T_1 is the stalk complex with P_1 in degree 0, and T_0 is given by

$$0 \longrightarrow P_1\langle -d_3\rangle \oplus P_1\langle -d_1 - d_3\rangle \xrightarrow{(\gamma,\gamma\alpha)} P_0 \longrightarrow 0,$$

with non-zero terms in degrees 0 and 1.

From

$$\mathrm{Endgr}_{K^b(P_{SD(2A)_2})}(T_1) \cong \mathrm{Endgr}_{SD(2A)_2}(P_1) \cong \bigoplus_{t=0}^{r} k\langle -t\Sigma\rangle$$

we conclude that $\deg(\eta) = \Sigma$ in the quiver of $SD(2B)_1$.

Also, $\mathrm{Homgr}_{K^b(P_{SD(2A)_2})}(T_1, T_0) \cong \mathrm{Homgr}_{SD(2A)_2}(P_1, \ker(\gamma, \gamma\alpha))$. The latter graded vector space is isomorphic to $k\langle -(d_3 + r\Sigma)\rangle \oplus k\langle -(d_1 + d_3 + r\Sigma\rangle)$. This means that the paths in the quiver of $SD(2B)_2$ that start at vertex 1 and end at vertex

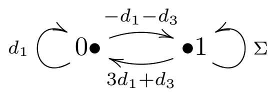
0 generate a 2-dimensional graded vector space with 1-dimensional components in degrees $d_3 + r\Sigma$ and $d_1 + d_3 + r\Sigma$. In other words

$$\{\deg(\gamma), \deg(\gamma\alpha)\} = \{d_3 + r\Sigma, d_1 + d_3 + r\Sigma\}.$$

We conclude easily that $\mathrm{Homgr}_{K^b(P_{SD(2A)_2})}(T_0, T_1) \cong k\langle d_3\rangle \oplus k\langle d_1 + d_3\rangle$. This gives us that $\{\deg(\beta), \deg(\alpha\beta)\} = \{-d_3, -d_1 - d_3\}$.

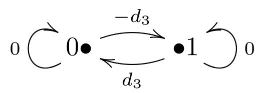
From the relations of $SD(2B)_1$ we have that $\deg(\alpha) = r\deg(\eta)/3 = r\Sigma/3 = d_1$. It follows that $\deg(\gamma) = r\Sigma + d_3 = 3d_1 + d_3$ and $\deg(\beta) = -d_1 - d_3$.

Therefore, the resulting graded quiver of $SD(2B)_1$ is given by



If the resulting grading on $SD(2B)_1$ is positive, then it must hold that $d_1 > 0$, $-d_3 > d_1$ and $3d_1 > -d_3$.

Note that we have done our calculations without using the additional assumption that $c \neq 0$. If $c \neq 0$, then from the relations of $SD(2A)_2$ we have that $d_1 = \Sigma = 0$, $d_2 = -d_3$. It follows that $\deg(\alpha) = \deg(\eta) = 0$ and $\deg(\gamma) = -\deg(\beta) = d_3$ in the quiver of $SD(2B)_1$. Under this additional assumption, the resulting graded quiver of $SD(2B)_1$ is given by



Obviously, in this case the resulting grading is never positive.

We see that for each semidihedral block the resulting grading obtained by transfer of gradings via derived equivalences is a non-trivial grading for an appropriate choice of the homogeneous degrees of the arrows of an appropriate quiver. Thus, we have proved Theorem 2.1.

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Received by editors 02.06.2015; Available online 20.07.2015.

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