Complementary Connected Vertex Edge Domination

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Abstract. Let $D$ be a vertex edge dominating set of $G$. If $\langle V - D \rangle$ is connected, then $D$ is called a complementary connected vertex edge dominating set (ccved-set) of $G$. The complementary connected vertex edge domination number $\gamma_{ccved}(G)$ of $G$ is the minimum cardinality of a ccved-set of $G$. Bounds for this variant of vertex edge domination in terms of various graph theoretic parameters are obtained. The graphs attaining these bounds are characterized in some cases. Also graphs having ccved-numbers as $1, p - 1, p - 2, p - 3$ are characterized. Complementary connected vertex edge domination numbers for some of the standard graphs are given.

1. Introduction and Preliminaries.

In this paper all our graphs will be finite, undirected and without loops or multiple edges having $p$ vertices and $q$ edges. Any undefined term in this paper, may be found in Harary [1].

If in a graph $G = (V, E)$, each vertex in $V - D (D \subseteq V)$ is adjacent to a vertex in $D$, then $D$ is said to be a dominating set of $G$. The minimum cardinality of a dominating set of $G$ is said to be a domination number of $G$ and is denoted by $\gamma(G)$ [2]. If each edge in $E - F$ is adjacent to an edge in $F$ for some $F \subseteq E$, then $F$ is said to be an edge dominating set of $G$. The edge domination number $\gamma'(G)$ is the cardinality of a minimum edge dominating set of $G$ [2].

A set $D$ of vertices in a graph $G$ is said to vertex edge dominate $G$, if for each edge in $G$ one of the end vertices is from $D$ or one of the end vertices is adjacent to a vertex in $D$. The smallest cardinality of any such vertex edge dominating set is said to be vertex edge domination number of $G$ and is denoted by $\gamma_{ve}(G)$ [3]. A vertex edge dominating set $D'$ is said to be a minimal vertex edge dominating set of $G$ if

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and only if there is no vertex edge dominating set $D''$ of $G$ such that $D'' \subset D'$ [3]. 

If $D$ is a vertex edge dominating set of $G$ such that $D$ is connected, then $D$ is said to be a connected vertex edge dominating set of $G$. The minimum cardinality of a connected vertex edge dominating set of $G$ is said to be the connected vertex edge domination number of $G$ and is denoted by $\gamma_{cve}(G)$ [5]. If $D$ is a vertex edge dominating set of $G$ such that $<D>$ is a tree, then $D$ is said to be a complementary tree vertex edge dominating set of $G$. The minimum cardinality of a complementary tree vertex edge dominating set of $G$ is said to be the complementary tree vertex edge domination number of $G$ and is denoted by $\gamma_{ctve}(G)$ [7]. If $D$ is a vertex edge dominating set such that $V - D$ is not a vertex edge dominating set, then $D$ is said to be complementary null vertex edge dominating set (cnved - set). The minimum cardinality of the complementary null vertex edge dominating set of $G$ is said to be the complementary null vertex edge domination number of $G$ and is denoted by $\gamma_{cnve}(G)$ [8].

A graph $G$ is said to be semi complete if and only if there is a path of length two between any pair of vertices in $G$ [4]. A graph $G$ is said to be unicyclic if and only if it has exactly one cycle. The friendship graph $F_p$ is the graph obtained by joining $p$ copies of $C_3$ to a common vertex. The clique number $\omega(G)$ of a graph $G$ is the maximum size of the clique in $G$.

In this paper, we define a new variant of vertex edge domination namely complementary connected vertex edge domination whose definition is as follows.

Let $D$ be a vertex edge dominating set of $G$. Then $D$ is said to be complementary connected vertex edge dominating set if and only if $<V - D>$ is connected.

The complementary connected vertex edge domination number $\gamma_{cve}(G)$ of $G$ is the cardinality of a minimum vertex edge domination number of $G$. By a $\gamma_{cve}(G)$ set we mean a minimum complementary connected vertex edge dominating set.

Throughout this paper complementary connected vertex edge domination set is abbreviated as ccved - set. Hereafter, we assume that $G$ is a connected graph.

2. Main Results.

Now, we give the characterization result for a proper subset $D$ of $V$ to be a ccved-set.

**Theorem 2.1.** A subset $D$ of $V$ is a ccved - set for $G$ if and only if the following conditions hold:

1. $\{xy \in E(G): \text{atleast one of } x, y \text{ is in } D\}$ is an edge dominating set of $G$.
2. $D$ is not a vertex cut in $G$.

**Proof.** The proof is trivial. \qed

**Theorem 2.2.** A ccved - set $D$ of $G$ is minimal if and only if for each $v$ in $D$ one of the following conditions holds:

1. For all $u$ in $V - D$ adjacent to $v$, $N(u) \cap D = \{v\}$.
2. There is an edge $v_1v_2$ in $E - F$ for which $(N(v_1) \cup N(v_2)) \cap D = \{v\}$, where $F = \{xy \in E(G): \text{atleast one of } x, y \text{ is in } D\}$.
3. $<V - D> \cup \beta >$ is disconnected.
Proof. Assume that $D$ is a minimal ccved set of $G$.
Suppose that there is a vertex $v$ in $D$ which does not satisfy any of the conditions. By (i) and (ii), $D - \{v\} (= D')$ is a vertex edge dominating set for $G$. By (iii), $< V - D' >$ is connected. This implies $D'$ is a ccved set of $G$, contradicting our assumption.

Conversely, suppose that $D$ is a ccved set and for each $v$ in $G$, one of the three conditions holds. Suppose $D$ is not minimal ccved set. Then, $D - \{v\}$ is a ccved set. This implies, for all $u$ in $V - D$ adjacent to $v$, $N(u) \cap D \neq \{v\}$, a contradiction to (i). If $D - \{v\}$ is a ccved set, then, there is no edge $v_1v_2$ in $E - F$ for which $(N(v_1) \cup N(v_2)) \cap D = \{v\}$. This implies a contradiction to (ii). Also, since $D - \{v\}$ is a ccved set, $< V (D - \{v\}) >$ is connected, a contradiction to (iii).

**Proposition 2.1.** For a graph $G$, $1 \leq \gamma_{ccve}(G) \leq p - 1$.

Proof. The proof follows from the fact that for a complete graph $K_2$ both the bounds hold.

**Note:**
For characterizing the graphs having $\gamma_{ccve}(G) = 1$, we define a family $\mathcal{F}$ of graphs as follows.
A graph $G$ of order $p \geq 4$, $\delta(G) \geq 2$ is in $\mathcal{F}$ if and only if there is a vertex $v$ in $G$ satisfying the following properties:
1. Each edge in $G$ lies on a $n$-cycle through $v$ for some $n \leq 4$.
2. Any pair of vertices lie on a cycle through $v$.

**Theorem 2.3.** For a graph $G$ with $p \geq 4$, $\delta(G) \geq 2$, $\gamma_{ccve}(G) = 1$ if and only if $G \in \mathcal{F}$.

Proof. Assume that $\gamma_{ccve}(G) = 1$. Then, there is a $v$ in $G$ vertex edge dominating the edges in $G$ and $G - v$ is connected.
Let $v_1v_2$ be an edge in $G$.

**Case 1:** $v_1 = v$ or $v_2 = v$.
W.l.g assume that $v_1 = v$. By our assumption $\text{deg}(v_2) \geq 2$. So, there is $v_3$ in $G$ such that $v_2v_3$ is an edge in $G$. If $v_3$ is not adjacent to $v$ or to a vertex adjacent to $v$, then there is an edge which is not vertex edge dominated by $v$, a contradiction to our assumption. Then in either case $v_2v_3$ lies on an $n$-cycle for some $n \leq 4$.

**Case 2:** $v_1 \neq v, v_2 \neq v$.
By the construction in Case 1, we get $v_1v_2$ lies on an $n$-cycle for some $n \leq 4$.

So, in any case (1) holds.
Let $v_1, v_2$ be a pair of vertices in $G$.

**Case 1:** $v_1v_2$ is an edge in $G$.
By (1), $v_1, v_2$ lie on a cycle.
**Case 2:** $v_1v_2$ is not an edge in $G$.
For $G$ is connected and $\{v\}$ is a ccved set of $G$, through $v$ there exists
For any complete bipartite graph $G$, there exists a path of length at most 4. For $\deg(v_1), \deg(v_2) \geq 2$, there exists a vertex $v_3$ such that $v_1v_3, v_2v_3$ are edges in $G$. If $v_3 = v_4$, then we are through. Suppose not. By (1), each of $v_1v_3, v_2v_4$ lie on a cycle through $v$. Since $G = \{v\}$ is connected, $v_1, v_2$ lie on a cycle through $v$.

Hence (2) holds.

Converse is clear.

**Corollary 2.1.** For any complete bipartite graph $K_{m,n}(m,n \geq 2)$, $\gamma_{ccve}(K_{m,n}) = 1$.

**Proof.** The proof follows from the fact that in $K_{m,n}(m,n \geq 2)$ any edge (pair of vertices) lie on a cycle of length 4. Also observe that $\gamma_{ccve}(K_{m,n}) = 1$ for $m + n \leq 3$.

**Theorem 2.4.** $\gamma_{ccve}(G) = p - 1$ if and only if $G = K_2$.

**Proof.** Suppose that $\gamma_{ccve}(G) = p - 1$.

If $\text{diam}(G) \geq 2$, then $V - \{v_1, v_2\}$ is a ccved-set of G for an arbitrary edge $v_1v_2$ in $G$. This implies that $\gamma_{ccve}(G) \leq p - 2$, which is a contradiction to our assumption. Hence $\text{diam}(G) = 1$.

$\Rightarrow G \cong K_p$ for some $p \geq 2$.

If $p > 2$, then $\gamma_{ccve}(G) = 1 \neq p - 1$, a contradiction to our assumption. Hence $G = K_2$.

The converse is clear.

**Theorem 2.5.** For a graph $G$, $\gamma_{ccve}(G) = p - 2$ if and only if $G = P_3$ or $K_3$ or $P_4$.

**Proof.** Assume that $\gamma_{ccve}(G) = p - 2$.

Suppose that $\text{diam}(G) \geq 4$. Let $v_1v_2v_3\ldots v_kv_{k+1}$ be a diammetral path in $G$. Clearly $k \geq 4$. Since $v_1v_2v_3v_4$ is connected, $V - \{v_1, v_2, v_3, v_4\}$ is a ccved-set of cardinality $p - 3$ a contradiction. So, $\text{diam}(G) \leq 3$.

Suppose $\text{diam}(G) = 3$. Then there is a diammetral path of length 3, say $v_1v_2v_3v_4$. Assume that there is a $v_5$ in $V - \{v_1, v_2, v_3, v_4\}$ adjacent to one of the vertices in $v_1, v_2, v_3, v_4$. If $v_5$ is adjacent with $v_4$ or $v_1$, then $V - \{v_2, v_3, v_4\}$ is a ccved-set, a contradiction. If $v_5$ is adjacent to $v_2$ or $v_3$, then $V - \{v_1, v_2, v_3\}, V - \{v_3, v_4, v_5\}$ is a ccved-set, respectively, a contradiction. Hence $G = v_1v_2v_3v_4$.

Suppose that $\text{diam}(G) = 2$, by the above construction we get that $G = P_3$.

Suppose that $\text{diam}(G) = 1$. Then $G = K_n$ for $n \geq 2$. Except for $n = 3, \gamma_{ccve}(G) \neq 3$. Hence $G = K_3$.

The converse part is clear.

**Corollary 2.2.** For a semi complete graph $G$, $\gamma_{ccve}(G) = p - 2$ if and only if $G = K_3$.

**Proof.** Since $K_3$ is the only semi complete graph among the class of graphs having $\gamma_{ccve}(G) = p - 2$, the proof follows.
Note: For a tree \( T \), \( \gamma_{ctve}(T) = \gamma_{ccve}(T) \).

Now, we give the necessary and sufficient condition for a ccved - set to be a ctved - set.

**Theorem 2.6.** A ccved - set \( D \) of \( G \) is a ctved - set if and only if each cycle in \( G \) has a vertex from \( D \).

**Proof.** The proof is trivial. \( \square \)

**Theorem 2.7.** For any spanning subgraph \( H \) of \( G \), \( \gamma_{ccve}(G) \leq \gamma_{ccve}(H) \).

**Theorem 2.8.** For any connected \((p,q)\) graph \( G \),
\[ \frac{3}{2}(p + k - 1) - q \leq \gamma_{ccve}(G). \]
where \( k \) is the number of edge disjoint cycles in \( < V - (\gamma_{ccve}(G) - set) > \).

**Proof.** \( D \) be a \( \gamma_{ccve}(G) - set \). Then \( < V - D > \) has \( p - \gamma_{ccve}(G) \) vertices and atleast \( p - \gamma_{ccve}(G) - 1 + k \) edges. Let \( t \) be the number of edges having one end in \( D \) and another in \( V - D \). Hence,
\[ 2[q - (p - \gamma_{ccve}(G) - 1 + k)] = \sum_{v \in D} \text{deg}(v) + t \geq \gamma_{ccve}(G)\delta(G) + t \geq \gamma_{ccve}(G) + p - \gamma_{ccve}(G) - 1 + k = p + k - 1. \]
This implies,
\[ 2q - 2p + 2\gamma_{ccve}(G) - 2k + 2 \geq p + k - 1 \]
Hence the result follows. \( \square \)

**Theorem 2.9.** Let \( G \) be a \((p,q)\) graph, then
\[ \gamma_{ccve}(G) \leq 2(p + k - 1) - \frac{2q}{\Delta(G)}. \]
where \( k \) is the number of cycles in \( < V - (\gamma_{ccve}(G) - set) > \).

**Proof.** By the construction in the above theorem,
\[ 2[q - (p - \gamma_{ccve}(G) - 1 + k)] = \sum_{v \in D} \text{deg}(v) + t \leq \Delta(G)\gamma_{ccve}(G) + 2(\Delta(G) - 1)(p - \gamma_{ccve}(G) - 1 + k). \]
This implies,
\[ 2q \leq -\Delta(G)\gamma_{ccve}(G) + 2\Delta(G)(p + k - 1) \]
Hence the result follows. \( \square \)

**Theorem 2.10.** Let \( G \) be a \((p,q)\) graph which has \((\omega(G) + 1)\) -regular spanning subgraph, then
\[ \gamma_{ccve}(G) \leq p - \omega(G). \]
where \( \omega(G) \) is the clique number of \( G \).
Proof. Let $S$ be the set of vertices such that $< S >$ is complete and $|S| = \omega(G)$. By the hypothesis it follows that, $V - S$ is a ccved - set of $G$. Hence the result follows.

Note: Since for a $(p, q)$ graph $G$, $(V - S) \bigcup \{v\}_{v \in S}$ is a ccved - set of $G$

$$\gamma_{ccve}(G) \leq |V - S|$$

$$\leq |(V - S)| + |\{v\}|$$

$$\leq p - \omega(G) + 1.$$ 

Here $< S >$ is complete and $|S| = \omega(G)$. □

Theorem 2.11. For a semi complete graph $G$ with $p \geq 3$,

1. $1 \leq \gamma_{ccve}(G) \leq p - 2$.

Proof. Since in a semi complete graph each edge lies on a triangle and also every complete graph with at least three vertices is semi complete, by the above note the inequality follows. □

Note:
(1) For a semi complete graph $G$ having four vertices, $\gamma_{ccve}(G) = p - 3$ if and only if $G$ is a union of two triangles having a common edge.

(2) There is no semi complete graph with three, five vertices having, $\gamma_{ccve}(G) = p - 3$.

Theorem 2.12. $G$ be a semi complete graph with $p > 5$, then $\gamma_{ccve}(G) = p - 3$
if and only if $G$ is isomorphic to

Proof. Assume that $\gamma_{ccve}(G) = p - 3$. This implies that there is a ccved - set $D \subset V$ of cardinality $p - 3$ and $< V - D >$ is isomorphic to

Case 1: $< V - D > = P_3(< v_1v_2v_3 >)$.

Since $G$ is semi complete $v_1v_2, v_2v_3$ lie on two different triangles, say $< v_1v_2v_4 >, < v_2v_3v_5 >$. 
Subcase: $v_4 = v_5$.  
So, both the triangles have a common edge $v_2v_4$ (say). Since $p > 5$, choose a vertex adjacent to $v_4$ say, $v_5$. For, $G$ is semi complete $v_4v_5$ lies on a triangle $< v_4v_5v_6 >$. If $v_6 \in \{v_1, v_2, v_3\}$, then $D - \{v_5\}$ is a ceved - set of cardinality less than $p - 3$, a contradiction to our assumption. If then the result is clear. Suppose not. We can find a vertex say $v_7$ adjacent to either $v_5$ or $v_6$. W.l.g assume that $v_7$ is adjacent to $v_5$. Since $G$ is semi complete $v_4v_7$ is an edge in $G$. If then the result is clear. Suppose not. If there is a vertex $v_8$ in $G$, then by the semi completeness of $G$, $v_4v_8$ is an edge in $G$. This implies that $V - \{v_1, v_2, v_3, v_4\}$ is a ceved- set in $G$ of cardinality $4 (= p - 4)$, a contradiction. Hence $V = \{v_1, v_2, ..., v_7\}$.  
Since none of $v_5, v_6, v_7$ can be adjacent to one of $v_1, v_2, v_3$, the result holds.  

Subcase: $v_4 \neq v_5$.  
Since $p > 5$, choose a vertex $v_6$ from $V - \{v_1, v_2, ..., v_5\}$. If $v_6$ is adjacent to one of $v_1, v_2, v_3$, then $D - \{v_5\}, D - \{v_4, v_6\}, D - \{v_4\}$ is a ceved - set of $G$ respectively, a contradiction to $\gamma_{ccve}(G) = p - 3$. Since $G$ is semi complete, $v_6$ is adjacent with $v_4$ and $v_5$. Then, $D - \{v_4\}$ or $D - \{v_5\}$ is a ceved - set of $G$, a contradiction. So, $v_4 = v_5$.  

Case:2: $< V - D > \neq P_3$.  
Then $< V - D > = K_3$. By the construction as in the case:1, the result follows.  
The converse part is clear.

\[\square\]

Corollary 2.3. $G$ be a semi complete graph. Then $\gamma_{ccve}(G) = p - 3$ if and only if $4 \leq p \leq 7, p \neq 5$. 


Corollary 2.4. For a semi complete graph $G$ with $\gamma_{ccve}(G) = p - 3$, 
$$3 \leq \gamma_{ccve}(G) + \Delta(G) \leq 10.$$ 

Proof. For a semi complete graph $G$, $\Delta(G) \geq 2$ and $\gamma_{ccve}(G) \geq 1$. Also by Theorem 2.12, for a semi complete graph $G$ with $\gamma_{ccve}(G) = p - 3$, $\Delta(G) \leq 6$. Hence the result follows. \qed

Note: The bounds are sharp as the lower and upper bounds are attained in the case of $K_3, W_9$ respectively.

Theorem 2.13. Let $G$ be a semi complete graph with $\gamma_{ccve}(G) > 1$. Then, 
$$\gamma_{ccve}(G) = 1 \text{ if and only if } G \not\in \mathcal{F}.$$ 

Proof. Since for a semi complete graph $\Delta(G) > 2$, by Theorem 2.3 the proof follows. \qed

Theorem 2.14. Let $G$ be a $(p, q)$ graph with $\Delta(G) \geq 3, g(G) \neq 3$, then 
$$\gamma_{ccve}(G) \leq p - \Delta(G).$$ 

Proof. Suppose that $deg(v) = \Delta(G)$ for some $v \in V$. Then, $(V - N[v])$ is a ccved set of $G$. Hence the result. \qed

Theorem 2.15. Let $G$ be a $(p, q)$ graph with $\Delta(G) \geq 3, g(G) > 4$, 
$$\gamma_{ccve}(G) \leq p - k - 3$$ 
where $k$ is the diameter of $G$.

Proof. Let $u$ and $v$ be two vertices with $d(u, v) = k = diam(G)$. Let $<u = v_1v_2...v_{k-1}v_kv_{k+1}v_1> = v$ be a diammetral path in $G$. Since each edge in $G$ is vertex edge dominated by a vertex in $V - \{v_1, v_2, ..., v_{k+1}, u_1, u_{k+1}\}$ (where $u_i, v_i (1 \leq i \leq k + 1), u_i \notin \{v_1, v_2, ..., v_{k+1}\}$) and $<v_1, v_2, ..., v_{k+1}, u_1, u_{k+1}>$ is connected, the former is a ccved set in $G$. Hence, 
$$\gamma_{ccve}(G) \leq |V - \{v_1, v_2, ..., v_{k+1}, u_1, u_{k+1}\}|$$ 
$$\leq p - ((k + 1) + 2)$$ 
$$= p - k - 3.$$ 

We make use of the following result in proving the next result. \qed

Theorem 1.[7] If both $G$ and $\overline{G}$ are connected with $p \geq 6$, then 
$$4 \leq d + \overline{d} \leq p + 1$$ 
where $\overline{d}$ is the diameter of $\overline{G}$.

Corollary 2.5. Suppose both $G, \overline{G}$ are connected with $\delta(G), \delta(\overline{G}) \geq 3$ and $p \geq 6$, then 
$$\gamma_{ccve}(G) + \gamma_{ccve}(\overline{G}) \leq 2p - 10.$$ 

Proof. By Theorem 1.[6], the proof follows. \qed
Theorem 2.16. For a \((p, q)\) graph \(G\) with \(\delta(G) \geq 3, g(G) \geq 7\),
\[
\gamma_{ccve}(G) \leq p - \frac{2}{3}k - 2
\]
where \(k\) is the diameter of \(G\).

Proof. Let \(u\) and \(v\) be two vertices with \(d(u, v) = k = \text{diam}(G)\). Let \(< u = v_1v_2...v_{k-1}v_kv_{k+1} = v >\) be a diametral path in \(G\). Since each edge in \(G\) is vertex edge dominated by a vertex in \(V = \{v_1, v_2, ..., v_{k-1}, u_1, u_4, u_7, ..., u_{k+1}(k = 3m), u_k(k = 3m+1), u_{k-1}(k = 3m+2)\}\) where \(u_i, v_j(1 \leq i \leq k+1), u_i \notin \{v_1, v_2, ..., v_{k+1}\}\) and \(\{v_1, v_2, ..., v_{k+1}, u_1, u_4, u_7, ..., u_{k+1}(k = 3m), u_k(k = 3m+1), u_{k-1}(k = 3m+2)\}\) is connected, the former is a \(ccved\) set in \(G\). Hence,
\[
\gamma_{ccve}(G) \leq |V - \{v_1, v_2, ..., v_{k+1}, u_1, u_4, u_7, ..., u_{k+1}(or)u_k(or)u_{k-1}\}| \leq p - (k + 1) - \left(\frac{k}{3} + 1\right)
= p - \frac{4}{3}k - 2.
\]

Theorem 2.17. If \(D\) is a \(ccved\) set such that no two edges in \(V - D\) are ve - dominated by the same vertex in \(D\), then
\[
\gamma_{ccve}(G) \leq \frac{2p - q - 2}{2}.
\]

Theorem 2.18. For a graph \(G\),
\[
\gamma_{ccve}(G) + \gamma_{cve}(G) \leq p
\]

Proof. Let \(D\) be a \(\gamma_{ccve}(G)\) set. By definition \(V - D\) is connected and each edge in \(G\) is dominated by a vertex in \(V - D\). So, \(V - D\) is a connected ve - dominating set of \(G\). This implies,
\[
\gamma_{cve}(G) \leq |V - D| = p - \gamma_{ccve}(G).
\]
Hence the result follows.

Theorem 2.19. For any graph \(G\), the following conditions are equivalent
1. The set of all pendant vertices form a \(ccved\) set.
2. The set of all pendant edges in \(G\) form an edge dominating set for \(G\).
3. Each non pendant vertex is a support vertex or adjacent to a support vertex.

Proof. Suppose that (1) holds.
Take \(S = \{uv : u or v is a pendant vertex\}\). Let \(e(= v_1v_2) \in E - S\). By our supposition there is atleast one pendant vertex \((say, v_3)\) adjacent to one of the end vertices of \(e\). W.l.g assume that \(v_1v_3 \in E\). Since \(v_3\) is a pendant vertex, \(v_1v_3 \in S\). Hence (2) holds.
Suppose that (2) holds.
Let \( v_1 \) be a non pendant vertex which is neither a support vertex nor adjacent to a support vertex. Then there is a non pendant vertex (say \( v_2 \)) in \( V - \{ \text{support vertices of } G \} \) adjacent to \( v_1 \). This implies \( v_1v_2 \) is not dominated by \( S \), a contradiction to our assumption. Hence (3) holds.

Suppose that (3) holds.
Let \( v_1v_2 \) be an arbitrary non pendant edge in \( G \). By (3), either \( v_1 \) or \( v_2 \) is a support vertex in \( G \). Then, there is a pendant vertex \( v_3 \) such that \( v_1v_3 \) or \( v_2v_3 \) is an edge in \( G \). This implies \( v_1v_2 \) is ve dominated by \( v_3 \). Hence (1) holds.

Corollary 2.6. If a graph \( G \) satisfies any of the conditions mentioned in Theorem 2.19, then
\[
\gamma_{ccve}(G) \leq m.
\]
where \( m \) is the number of pendant vertices in \( G \).

Proof. By hypothesis, it follows that the set of all pendant vertices form a ccved set of \( G \). Hence the result follows.

Furthermore, the bound is sharp as it is attained in the case of \( C_p \circ K_1 \), where \( C_p \circ K_1 \) is the corona of \( C_p \) and \( K_1 \).

Corollary 2.7. \( \gamma_{ccve}(C_p \circ K_1) = p \).

Proof. Since in \( C_p \circ K_1 \), each non pendant vertex is a support vertex, by Theorem 2.19,
\[
\gamma_{ccve}(C_p \circ K_1) \leq p.
\]
Also for any ved set \( D \) with \( |D| < p, < V - D > \) is disconnected. Hence the result follows.

Corollary 2.8. If a graph \( G \) satisfies any of the conditions mentioned in Theorem 2.19, then
\[
\gamma'(G) \leq m.
\]
Furthermore, the bound is sharp as it is attained in the case of \( P_5 \).

Proof. Since a graph satisfies any of the conditions mentioned in Theorem 2.19 has its pendant edges as its edge dominating set, hence the result follows.

Now we give the ccved numbers of some standard graphs.

Theorem 2.20. (1) For any cycle \( C_p \) with \( p \geq 4 \), \( \gamma_{ccve}(G) = p - 3 \).
(2) For any complete graph \( K_p \) (\( p \geq 2 \)), \( \gamma_{ccve}(K_p) = 1 \).
(3) For any wheel graph \( W_p \), \( \gamma_{ccve}(W_p) = 1 \).
(4) For a friendship graph \( F_p \), \( \gamma_{ccve}(F_p) = 2p - 1 \).

Theorem 2.21. For a tree \( T \) having diameter at least four, \( \gamma_{cnve}(T) \leq \gamma_{ccve}(T) \).
Furthermore, equality holds if and only if \( \text{diam}(T) = 4 \).

Proof. Since for a tree having diameter at least four, every ccved-set is a cnved-set the result follows.
Theorem 2.22. For a unicyclic graph $G$ with internal vertices having degree at least three,

$$\gamma_{ccve}(G) \leq p - n,$$

where $n$ is the length of the cycle in $G$.

Furthermore, the bound is sharp as it is attained in the case of $C_n \circ K_1$.

Proof. By the hypothesis it is clear that each vertex in $G$ is either a pendant vertex or a vertex of degree 3. Also the set of all pendant edges in $G$ form an edge dominating set for $G$. Then by Theorem 2.19, the set of all pendant vertices form a ccved set for $G$. Hence the result follows.

Theorem 2.23. For a unicyclic graph $G$, $\gamma_{ccve}(G) = p - n$ if and only if $G$ is isomorphic to one of the following:

Proof. The proof is trivial.

References
