Smart Congruences and Weakly Directly Indecomposable SHADLs

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ABSTRACT. In this paper, we define smart congruence and prove that smart congruences on Semi Heyting Almost Distributive Lattice (SHADL) are determined by filters and also show that smart congruences are congruence permutable. We also define and characterize the weakly directly indecomposable Semi Heyting Almost Distributive Lattice (SHADL) in terms of its Birkhoff centre $C(L)$.

1. Introduction

In [7], Sankappanavar introduced a new equational class $\mathcal{SH}$ of algebras, which he called Semi-Heyting Algebras, as an abstraction of Heyting algebras. This variety includes Heyting algebras and share with them some rather strong properties. For example, the variety of semi-Heyting algebras is arithmetical, semi-Heyting algebras are pseudocomplemented distributive lattices and their congruences are determined by filters. The concept of an Almost Distributive Lattice (ADL) was introduced by U. M. Swamy and Rao G.C. [10] as a common abstraction to most of the existing ring theoretic generalizations of a Boolean algebra on one hand and the class of distributive lattices on the other. Rao G.C., Berhanu Assaye and M. V. Ratna Mani [4], introduced the concept of a Heyting Almost Distributive Lattice (HADL) as a generalization of a Heyting algebra. The concept of a Semi Heyting Almost Distributive Lattice (SHADL) as a generalization of a Semi Heyting algebra was introduced in our earlier paper [5]. In this paper, we define the concept of a smart congruence and prove that the smart congruences on Semi Heyting Almost Distributive Lattice (SHADL) are determined by filters and also we show that smart congruences are permutable. We also define and characterize the weakly directly

indecomposable Semi Heyting Almost Distributive Lattice (SHADL) in terms of
its Birkhoff centre $C(L)$.

2. Preliminaries

In this section we give some important definitions and results that are
frequently used for ready reference.

DEFINITION 2.1. [10] An algebra $(L, \vee, \wedge, 0)$ of type $(2, 2, 0)$ is called ADL if
it satisfies the following axioms: for all $x, y, z \in L$

(1) $x \vee 0 = x$
(2) $0 \wedge x = 0$
(3) $(x \vee y) \wedge z = (x \wedge z) \vee (y \wedge z)$
(4) $x \wedge (y \vee z) = (x \wedge y) \vee (x \wedge z)$
(5) $x \vee (y \wedge z) = (x \vee y) \wedge (x \vee z)$
(6) $(x \vee y) \wedge y = y$

DEFINITION 2.2. [10] Let L be a non-empty set. Fix $x_0 \in L$. For any $x, y \in L$,
define $x \wedge y = y$, $x \vee y = x$ if $x \neq x_0, x_0 \wedge y = x_0$ and $x_0 \vee y = y$. Then $(L, \vee, \wedge, x_0)$
is an ADL and it is called a discrete ADL. Alternately, discrete ADL is defined as
an ADL $(L, \vee, \wedge, 0)$ in which every $x(\neq 0)$ is maximal.

Let $(L, \vee, \wedge, 0)$ be an ADL. For any $x, y \in L$, define $x \leqslant y$ if and only if
$x = x \wedge y$, or equivalently $x \vee y = y$, then $\leqslant$ is a partial ordering on L.

Through out this section L stands for an ADL $(L, \vee, \wedge, 0)$ unless other-
wise specified. In the following theorem some important fundamental properties of
an ADL are given.

THEOREM 2.1. [8] *For any $a, b, c \in L$, we have the following*

(1) $a \vee b = a \Leftrightarrow a \wedge b = b$
(2) $a \vee b = b \Leftrightarrow a \wedge b = a$
(3) $a \wedge b = b \wedge a = a$ *whenever* $a \leqslant b$
(4) $\wedge$ *is associative in* L
(5) $a \wedge b \wedge c = b \wedge a \wedge c$
(6) $(a \vee b) \wedge c = (b \vee a) \wedge c$
(7) $a \wedge b \leqslant b$ *and* $a \leqslant a \vee b$
(8) $a \wedge a = a$ *and* $a \vee a = a$
(9) $a \wedge 0 = 0$ *and* $0 \vee a = a$
(10) *if* $a \leqslant c$ *and* $b \leqslant c$, *then* $a \wedge b = b \wedge a$ *and* $a \vee b = b \vee a$.

DEFINITION 2.3. [8] A non-empty subset F of an ADL L is said to be an filter
of L if it satisfies the following:

(1) $a, b \in F \Rightarrow a \wedge b \in F$
(2) $a \in F, x \in L \Rightarrow x \vee a \in F$

THEOREM 2.2. [8] *The set $F(L)$ of all filters of L forms a distributive lattice
under set inclusion, in which the g.l.d and l.u.b of any filters F and G of L are
given respectively by $F \wedge G = F \bigcap G$ and $F \vee G = \{x \wedge y / \ x \in F \text{ and } y \in G\}$*

DEFINITION 2.4. [10] An equivalence relation θ on an ADL is called a congruence relation on L if $(a \wedge c, b \wedge d) \in \theta$ and $(a \vee c, b \vee d) \in \theta$ for all $(a, b), (c, d) \in \theta$

THEOREM 2.3. $(Con(L), \subseteq)$ *is a lattice in which for any* $\theta_1, \theta_2 \in Con(L)$, *the g.l.b and l.u.b are respectively given by* $\theta_1 \wedge \theta_2 = \theta_1 \cap \theta_2$ *and*
$\theta_1 \vee \theta_2 = \{(x, y) \ / \ $ *there exists a finite sequence of elements*
$x \ = z_0, z_1, ..., z_{n-1} = y \in L$ *such that* $(z_i, z_{i+1}) \in \theta_1 \cup \theta_2$ *for each* $0 \leqslant i \leqslant n - 2\}$

DEFINITION 2.5. [9] Let L be an ADL with maximal elements. Then the set $C(L) = \{a \in L \ / $ there exists $b \in L \ni a \wedge b = 0$ *and* $a \vee b$ *is maximal*$\}$ is called Birkhoff centre of L.

THEOREM 2.4. [9] *Let L be an ADL with maximal elements. Then $C(L)$ is a relatively complemented ADL under operations induced by those of L.*

DEFINITION 2.6. [4] Let $(L, \vee, \wedge, 0, m)$ be an ADL with a maximal element m. Suppose $\rightarrow$ is a binary operation on L satisfying the following conditions for all $x, y, z \in L$.

(1) $x \rightarrow x = m$
(2) $(x \rightarrow y) \wedge y = y$
(3) $x \wedge (x \rightarrow y) = x \wedge y \wedge m$
(4) $x \rightarrow (y \wedge z) = (x \rightarrow y) \wedge (x \rightarrow z)$
(5) $(x \vee y) \rightarrow z = (x \rightarrow z) \wedge (y \rightarrow z)$

Then $(L, \vee, \wedge, \rightarrow, 0, m)$ is called a Heyting Almost Distributive lattice (HADL).

DEFINITION 2.7. [7] An algebra $(L, \vee, \wedge, \rightarrow, 0, 1)$ of type $(2, 2, 2, 0, 0)$ is called a Semi Heyting algebra if it satisfies the following:

(1) $(L, \vee, \wedge, 0, 1)$ is a lattice with $0, 1$
(2) $x \wedge (x \rightarrow y) = x \wedge y$
(3) $x \wedge (y \rightarrow z) = x \wedge (x \wedge y \rightarrow x \wedge z)$
(4) $x \rightarrow x = 1$ for all $x, y, z \in L$

THEOREM 2.5. [6] *Let $(L, \vee, \wedge, \rightarrow, 0, m)$ be an SHADL. For $x \in L$, define $x^* = (x \rightarrow 0) \wedge m$. Then $*$ is a pseudocomplementation on L.*

3. Smart Congruences

We begin with the following definition of SHADL given in [5].

DEFINITION 3.1. [5] Let $(L, \vee, \wedge, 0, m)$ be an ADL with a maximal element m. Suppose there exists a binary operation $\rightarrow$ on L satisfying the following conditions:

(1) $(x \rightarrow x) \wedge m = m$
(2) $x \wedge (x \rightarrow y) = x \wedge y \wedge m$
(3) $x \wedge (y \rightarrow z) = x \wedge (x \wedge y \rightarrow x \wedge z)$
(4) $(x \rightarrow y) \wedge m = x \wedge m \rightarrow y \wedge m$ for all $x, y, z \in L$

Then $(L, \vee, \wedge, \rightarrow, 0, m)$ is a Semi Heyting ADL (SHADL).

In the rest of this section L denotes a SHADL and $F(L)$ denotes the lattice of filters of L.

The following theorem which is taken from [5] will be used frequently in this paper.

THEOREM 3.1. [5] *For any $a, b, c, d, x \in L$ we have the following*

(1) $m \to a = a \wedge m$
(2) $a \wedge b \wedge m \leqslant a \to b$
(3) $(a \to b) \wedge m \leqslant (a \to a \wedge b) \wedge m$
(4) $a \wedge m \leqslant [a \to (b \to a \wedge b)] \wedge m$
(5) $(a \to b) \wedge c = (a \wedge c \to b \wedge c) \wedge c$
(6) $[(a \wedge b) \to (c \wedge d)] \wedge x = [(b \wedge a) \to (d \wedge c)] \wedge x$
(7) $a \wedge m \leqslant [(a \to b) \to b] \wedge m.$

In this Section we define smart congruence and prove that the Smart Congruences on SHADL are determined by Filters.

DEFINITION 3.2. A Congruence θ on an ADL L is called a Smart Congruence if $(x \wedge m, y \wedge m) \in \theta \Rightarrow (x, y) \in \theta.$

We denote the set of all Smart Congruences on L by $Con_0(L)$ and we can clearly observe that $Con_0(L)$ is a sublattice of $Con(L)$ and also $Con_0(L)$ is distributive.

DEFINITION 3.3. Let $F \in F(L)$. Define a binary relation $\theta(F)$ on L by $(x, y) \in \theta(F)$ iff $x \wedge f = y \wedge f$ for some $f \in F$

LEMMA 3.1. $\theta(F) \in Con_0(L)$ and $m/\theta(F) = F.$

PROOF. Assume that L is a $SHADL$.
Clearly $\theta(F)$ is reflexive and symmetric.
Let $x, y, z \in L$ such that $(x, y) \in \theta(F) \Rightarrow x \wedge f_1 = y \wedge f_1$ for some $f_1 \in F$
$(y, z) \in \theta(F) \Rightarrow y \wedge f_2 = z \wedge f_2$ for some $f_2 \in F$
Then $x \wedge f_1 \wedge f_2 = y \wedge f_1 \wedge f_2 = f_1 \wedge y \wedge f_2 = f_1 \wedge z \wedge f_2 = z \wedge f_1 \wedge f_2$ and $f_1 \wedge f_2 \in F$
and hence $(x, z) \in \theta(F)$
Therefore $\theta(F)$ is transitive. Thus $\theta(F)$ is an equivalence relation.
Now, let $(x, y) \in \theta(F), (r, t) \in \theta(F)$. Then $x \wedge f_1 = y \wedge f_1$ and $r \wedge f_2 = t \wedge f_2$ for some $f_1, f_2 \in F$.
Now, $f_1 \wedge f_2 \in F$ and $x \wedge r \wedge f_1 \wedge f_2 = r \wedge x \wedge f_1 \wedge f_2 = r \wedge y \wedge f_1 \wedge f_2$
$= y \wedge f_1 \wedge r \wedge f_2 = y \wedge f_1 \wedge t \wedge f_2 = y \wedge t \wedge f_1 \wedge f_2.$
Thus $(x \wedge r, y \wedge t) \in \theta(F).$
Now, $(x \vee r) \wedge f_1 \wedge f_2 = (x \wedge f_1 \wedge f_2) \vee (r \wedge f_1 \wedge f_2) = (y \wedge f_1 \wedge f_2) \vee (t \wedge f_1 \wedge f_2)$
$= (y \vee t) \wedge f_1 \wedge f_2$ and hence $(x \vee r, y \vee t) \in \theta(F).$
Now, $(x \to r) \wedge f_1 \wedge f_2 = f_1 \wedge f_2 \wedge (x \to r) \wedge f_2 = f_1 \wedge f_2 \wedge (f_1 \wedge f_2 \wedge x \to f_1 \wedge f_2 \wedge r) \wedge f_2$
$= f_1 \wedge f_2 \wedge (x \wedge f_1 \wedge f_2 \to r \wedge f_1 \wedge f_2) \wedge f_2 = f_1 \wedge f_2 \wedge (y \wedge f_1 \wedge f_2 \to t \wedge f_1 \wedge f_2) \wedge f_2$
$= f_1 \wedge f_2 \wedge (f_1 \wedge f_2 \wedge y \to f_1 \wedge f_2 \wedge t) \wedge f_2 = f_1 \wedge f_2 \wedge (y \to t) \wedge f_2 = (y \to t) \wedge f_1 \wedge f_2.$
Thus $(x \to r, y \to t) \in \theta(F).$ Therefore $\theta(F) \in Con(L).$
Now, if $(x \wedge m, y \wedge m) \in \theta(F) \Rightarrow (x, y) \in \theta(F).$ Therefore $\theta(F) \in Con_0(L).$
Now we prove that $m/\theta(F) = F$

Let $x \in m/\theta(F)$. Then $(x, m) \in \theta(F)$. So that $x \wedge f = m \wedge f$ for some $f \in F$.
That is $x \wedge f = f$ and hence $x \vee f = x$. Thus $x \in F$
On the other hand, if $x \in F$, then $x \wedge x = m \wedge x$ and hence $x \in m/\theta(F)$.
Therefore $m/\theta(F) = F$ □

LEMMA 3.2. *If $F \in F(L)$ and $a, b \in L$, then $(a, b) \in \theta(F)$ iff*
$(a \to b) \wedge (b \to a) \wedge m \in F$.

PROOF. Let $(a \wedge m, b \wedge m) \in \theta(F)$. Then
$(a \wedge m \to b \wedge m, b \wedge m \to b \wedge m) \in \theta(F)$
$\Rightarrow ((a \to b) \wedge m, (b \to b) \wedge m) \in \theta(F)$
$\Rightarrow ((a \to b) \wedge m, m) \in \theta(F)$. Similarly, $((b \to a) \wedge m, m) \in \theta(F)$.
Therefore $((a \to b) \wedge (b \to a) \wedge m, m) \in \theta(F)$
$\Rightarrow (a \to b) \wedge (b \to a) \wedge m \in m/\theta(F) = F$.
Thus $(a \to b) \wedge (b \to a) \wedge m \in F$.
Conversely, suppose $(a \to b) \wedge (b \to a) \wedge m \in F$
$\Rightarrow (a \to b) \wedge (b \to a) \wedge m \in m/\theta(F)$
$\Rightarrow ((a \to b) \wedge (b \to a) \wedge m, m) \in \theta(F)$
$\Rightarrow (a \wedge (a \to b) \wedge (b \to a) \wedge m, a \wedge m) \in \theta(F)$
$\Rightarrow (a \wedge b \wedge (b \to a) \wedge m, a \wedge m) \in \theta(F)$
$\Rightarrow (a \wedge b \wedge m, a \wedge m) \in \theta(F)$. Similarly, we get $(a \wedge b \wedge m, b \wedge m) \in \theta(F)$.
Hence $(a \wedge m, b \wedge m) \in \theta(F) \Rightarrow (a, b) \in \theta(F)$. □

THEOREM 3.2. $Con_0(L) \cong F(L)$.

PROOF. Let $\alpha : Con_0(L) \to F(L)$ defined by $\alpha(\theta) = m/\theta$
It is enough if we show that α is onto and $\theta_1 \subseteq \theta_2 \iff \alpha(\theta_1) \subseteq \alpha(\theta_2)$
Let $F \in F(L)$. Then $\theta(F) = \{(x, y) \in L \times L/x \wedge f = y \wedge f$ for some $f \in F\}$.
So that $\theta(F) \in Con_0(L)$ and $\alpha(\theta(F)) = m/\theta(F) = F$. Thus α is onto.
Let $\theta_1, \theta_2 \in Con_0(L)$. Suppose $\alpha(\theta_1) \subseteq \alpha(\theta_2)$ and $(x, y) \in \theta_1$.
Then $(x \wedge m, y \wedge m) \in \theta_1 \Rightarrow (x \wedge m \to y \wedge m, y \wedge m \to y \wedge m) \in \theta_1$
$\Rightarrow ((x \to y) \wedge m, (y \to y) \wedge m) \in \theta_1$
$\Rightarrow ((x \to y) \wedge m, m) \in \theta_1$
$\Rightarrow (x \to y) \wedge m \in m/\theta_1 \subseteq m/\theta_2$
$\Rightarrow ((x \to y) \wedge m, m) \in \theta_2$
$\Rightarrow (x \wedge (x \to y) \wedge m, x \wedge m) \in \theta_2$
$\Rightarrow (x \wedge y \wedge m, x \wedge m) \in \theta_2$.
Similarly, $(x \wedge y \wedge m, y \wedge m) \in \theta_2$.
Thus $(x \wedge m, y \wedge m) \in \theta_2$. Since $\theta_2 \in Con_0(L)$, we get $(x, y) \in \theta_2$.
Thus $\theta_1 \subseteq \theta_2$. It can be routinely verified that $\theta_1 \subseteq \theta_2 \Rightarrow \alpha(\theta_1) \subseteq \alpha(\theta_2)$.
Therefore $Con_0(L) \cong F(L.)$ □

Before going to the next theorem we need the following.

LEMMA 3.3. *Let L be a SHADL and $b, c \in L$. Then*
$[(c \to c) \to b] \wedge [(b \to c) \to c] \wedge (c \vee b) \wedge m = b \wedge m$.

PROOF. Let $b, c \in L$. Then $[(c \to c) \to b] \wedge [(b \to c) \to c] \wedge (c \vee b) \wedge m$
$= [m \to b] \wedge [(b \to c) \to c] \wedge (c \vee b) \wedge m = b \wedge m \wedge [(b \to c) \to c] \wedge (c \vee b) \wedge m$
$= b \wedge (c \vee b) \wedge m = b \wedge m$ $\qquad \square$

THEOREM 3.3. $Con_0(L)$ is a permutable sublattice of $Con(L)$.

PROOF. Let $\theta_1, \theta_2 \in Con_0(L)$ and $(a, b) \in \theta_1 \; o \; \theta_2$. Then there exists $c \in L$
such that $(a, c) \in \theta_1$ and $(c, b) \in \theta_2$.
Since $(a, c) \in \theta_1$, we get $(c, a) \in \theta_1$. From this we get that the paries
$((c \to c) \to b, (a \to c) \to b), ((b \to c) \to c, (b \to c) \to a)$ and
$((c \vee b) \wedge m, (a \vee b) \wedge m)$ belong to θ_1.
Hence $(((c \to c) \to b) \wedge ((b \to c) \to c) \wedge (c \vee b) \wedge m, ((a \to c) \to b)$
$\wedge ((b \to c) \to a) \wedge (a \vee b) \wedge m) \in \theta_1$.
Thus by above Lemma, we get that
$(b \wedge m, ((a \to c) \to b) \wedge ((b \to c) \to a) \wedge (a \vee b) \wedge m) \in \theta_1$.
That is $(b, ((a \to c) \to b) \wedge ((b \to c) \to a) \wedge (a \vee b)) \in \theta_1$.
Similarly, since $(c, b) \in \theta_2$, we get
$(((a \to c) \to b) \wedge ((b \to c) \to a) \wedge (a \vee b), a) \in \theta_2$.
Hence $(b, a) \in \theta_1 \; o \; \theta_2$ or $(a, b) \in \theta_2 \; o \; \theta_1$.
Therefore $\theta_1 \; o \; \theta_2 \subseteq \theta_2 \; o \; \theta_1$. By, symmetry we also get that $\theta_2 \; o \; \theta_1 \subseteq \theta_1 \; o \; \theta_2$.
Thus $\theta_1 \; o \; \theta_2 = \theta_2 \; o \; \theta_1$. Therefore $Con_0(L)$ is congruence permutable. $\qquad \square$

In the following theorem we give a set of conditions for $Con_0(L)$ to be congruence permutable when L is an ADL.

THEOREM 3.4. Let $L = (L, \vee, \wedge, m)$ be an ADL with a maximal element and $\to$ be a binary operation such that the following conditions hold:

  (1) $m \to x = x \wedge m$
  (2) $x \wedge m \leqslant ((x \to y) \to y) \wedge m$
  (3) $(x \to y) \wedge m = x \wedge m \to y \wedge m$.

Then $Con_0(L)$ is permutable.

PROOF. From (2), we have $x \wedge [(x \to y) \to y] \wedge m = x \wedge m$.
Taking m in place of x, we get $m \wedge [(m \to y) \to y] \wedge m = m$
$\Rightarrow [(m \to y) \to y] \wedge m = m$
$\Rightarrow [y \wedge m \to y] \wedge m = m$
$\Rightarrow (y \to y) \wedge m = m$.
Hence, from the above theorem, it follows that $Con_0(L)$ is congruence permutable.
$\qquad \square$

4. Weakly Directly Indecomposable SHADLs

In this section we define and characterize weakly directly indecomposable Semi Heyting ADLs. Throughout this section L stands for a SHADL. We denote the center of L by $C(L)$. Recall that $C(L)$ is a relatively complemented ADL. We begin with the following definition.

DEFINITION 4.1. Let $T(L) = \{a \in L / a = 0$ or a is a maximal element of $L\}$. Then $T(L)$ is a sub ADL of $C(L)$.

Let L be a SHADL and for $a \in L$, write $L_a = ([\ 0, a \wedge m], \vee, \wedge, \to^a 0, a \wedge m)$ where $c \to^a d = (c \to d) \wedge a \wedge m$ for $c, d \in [0, a \wedge m]$. Then we have proved in [6] that L_a is a Semi Heyting algebra .

LEMMA 4.1. *Let L be an SHADL and $a \in L$. If $L = L_a$ then a is a maximal element in L.*

Now, we prove the following.

LEMMA 4.2. *Let $a \in L$. Then the function $f_a : L \to L_a$ defined by $f_a(x) = x \wedge a \wedge m$ is a homomorphism onto L_a.*

PROOF. Define $f_a : L \to L_a$ by $f_a(x) = x \wedge a \wedge m$. Clearly we can verify that $f_a(x \wedge y) = f_a(x) \wedge f_a(y), f_a(x \vee y) = f_a(x) \vee f_a(y)$, for $x, y \in L$.
Now, consider $f_a(x \to y) = (x \to y) \wedge a \wedge m = (x \wedge a \wedge m \to y \wedge a \wedge m) \wedge a \wedge m$
$= x \wedge a \wedge m \to^a y \wedge a \wedge m = f_a(x) \to^a f_a(y)$.
Therefore f_a is a homomorphism and clearly f_a is onto. $\square$

DEFINITION 4.2. Let L and L' be two SHADLs. A homomorphism $f : L \to L'$ is said to be weak isomorphism if f is onto and $f(x) = f(y) \Rightarrow x \wedge m = y \wedge m$

DEFINITION 4.3. An algebra $\mathcal{A}$ is said to be weakly directly indecomposable if it is not weak isomorphic to the direct product of two non-trivial algebras of same type as $\mathcal{A}$.

LEMMA 4.3. *Let L and L' be two SHADLs. If $\phi : L \to L'$ is a weak isomorphism, then t is maximal in L iff $\phi(t)$ is maximal in L'.*

PROOF. Suppose t is maximal in L. Let $y \in L'$, then $y = \phi(x)$ for some $x \in L$.
Then, $\phi(t) \wedge y = \phi(t) \wedge \phi(x) = \phi(t \wedge x) = \phi(x) = y$
Therefore $\phi(t)$ is maximal in L'.
Conversely, suppose that $\phi(t)$ is maximal in L'.
Let $x \in L$, then $\phi(x) \in L'$. Now, $\phi(t) \wedge \phi(x) = \phi(x)$
$\Rightarrow \phi(t \wedge x) = \phi(x) \Rightarrow t \wedge x \wedge m = x \wedge m \Rightarrow t \wedge x \wedge m \wedge x = x \wedge m \wedge x \Rightarrow t \wedge x = x$
Therefore t is maximal in L $\square$

THEOREM 4.1. *Let L be an SHADL. Then the following are equivalent:*
  (1) *L is weakly directly indecomposable.*
  (2) *$C(L) = T(L)$*
  (3) *$(a \vee a^*) \wedge m < m$ for all elements $a \in L - T(L)$. Where $a^* = (a \to 0) \wedge m$.*

PROOF. $(1) \Rightarrow (2)$:
Suppose L is weakly directly indecomposable.
Let $a \in C(L)$. Then there exists $b \in L$ such that $a \wedge b = 0$, $a \vee b$ is a maximal element of L. Define $h : L \to L_a \times L_b$ by $h(x) = (f_a(x), f_b(x))$. By Lemma 4.2, we get that h is a homomorphism. Now, let $(x, y) \in L_a \times L_b$.
Then $x \leqslant a \wedge m$. So that $x \wedge b \leqslant a \wedge m \wedge b = a \wedge b = 0$. Thus $x \wedge b = 0$. Similarly $y \wedge a = 0$.
Now $x \vee y \in L$ and $h(x \vee y) = (f_a(x \vee y), f_b(x \vee y))$
$= ((x \vee y) \wedge a \wedge m, (x \vee y) \wedge b \wedge m)$

$= ((x \wedge a \wedge m) \vee (y \wedge a \wedge m), (x \wedge b \wedge m) \vee (y \wedge b \wedge m))$

$= (x \wedge a \wedge m, y \wedge b \wedge m)$

$= (x, y).$

Therefore h is a surjective homomorphism.

Finally, let $x, y \in L$ and $h(x) = h(y)$. Then $(f_a(x), f_b(x)) = (f_a(y), f_b(y))$.

So that $x \wedge a \wedge m = y \wedge a \wedge m$ and $x \wedge b \wedge m = y \wedge b \wedge m$.

Since $a \vee b$ is a maximal element of L, we get that

$x \wedge m = (a \vee b) \wedge x \wedge m$

$= (a \wedge x \wedge m) \vee (b \wedge x \wedge m)$

$= (x \wedge a \wedge m) \vee (x \wedge b \wedge m)$

$= (y \wedge a \wedge m) \vee (y \wedge b \wedge m)$

$= (a \wedge y \wedge m) \vee (b \wedge y \wedge m)$

$= (a \vee b) \wedge y \wedge m$

$= y \wedge m.$

Therefore h is a weak isomorphism. Since L is weakly directly indecomposable, we get that either $L_a = L$ or $L_b = L$. Hence, we get that a is a maximal element of L or b is a maximal element of L. Suppose b is a maximal element of L. Then $a \wedge b = 0 \Rightarrow b \wedge a = 0 \Rightarrow a = 0.$

Thus $a = 0$ or a is a maximal element of L. Thus $a \in T(L)$.

Hence $C(L) = T(L)$.

$(2) \Rightarrow (3)$:

Suppose $C(L) = T(L)$. Suppose $a \in L - T(L)$. Clearly $a \wedge a^* = 0$. Suppose $(a \vee a^*) \wedge m = m$. Then, for any $x \in L$,

$(a \vee a^*) \wedge x = (a \vee a^*) \wedge m \wedge x$

$= m \wedge x$

$= x$ and

hence $a \vee a^*$ is a maximal element of L. So that $a \in C(L) = T(L)$. This is a contradiction. Therefore $(a \vee a^*) \wedge m < m$.

Finally, we prove $(3) \Rightarrow (1)$:

Assume (3). Suppose L is not weakly directly indecomposable. Then there exist two non-trivial SHADLs L_1 and L_2 and a weak isomorphism $\phi : L \to L_1 \times L_2$. Let m_1 and m_2 denote the maximal elements of L_1 and L_2. Choose $a, b \in L$ such that $\phi(a) = (0, m_2)$ and $\phi(b) = (m_1, 0)$. Now

$\phi(a \wedge b) = \phi(a) \wedge \phi(b)$

$= (0, m_2) \wedge (m_1, 0)$

$= (0, 0)$ and hence $a \wedge b = 0$. Again,

$\phi(a \vee b) = \phi(a) \vee \phi(b)$

$= (0, m_2) \vee (m_1, 0)$

$= (m_1, m_2)$, a maximal element in $L_1 \times L_2$.

Therefore $a \vee b$ is a maximal element in L by Lemma 4.3. Thus $a \in C(L)$.

If $a = 0$, then $\phi(a) = (0, 0)$. Thus $m_2 = 0$. A contradiction since L_2 is nontrivial.

If a is a maximal element of L, then $b = a \wedge b = 0$. Thus $\phi(b) = (0, 0)$. Hence $m_1 = 0$. Which is again a contradiction since L_1 is nontrivial.

Therefore $a \in L - T(L)$.

Now, $a \wedge b = 0 \Rightarrow b \wedge m \leqslant a^* \wedge m$

$\Rightarrow (a \wedge m) \vee (b \wedge m) \leqslant (a \wedge m) \vee (a^* \wedge m)$
$\Rightarrow (a \vee b) \wedge m \leqslant (a \vee a^*) \wedge m$
$\Rightarrow m \leqslant (a \vee a^*) \wedge m$
$\Rightarrow (a \vee a^*) \wedge m = m.$

This is a contradiction since $a \in L - T(L)$. Therefore L is weakly directly inde-composable. $\qquad\square$

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