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Some properties of L_p^k-convex sequences

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Abstract. We have defined a new class of convex sequences, referred to as L_p^k-convex sequences. These sequences represent generalization of a class of k-convex sequences, $k \geqslant 2$. Some important properties of L_p^k-convex sequences are proved.

1. Introduction

Let $(a_n), n \in N$, be a sequence of real numbers, $a_n \in R$, and Δ^k a difference operator of order $k, k \in N_o$, defined by

$$\Delta^k a_n = \Delta(\Delta^{k-1} a_n) = \Delta^{k-1} a_{n+1} - \Delta^{k-1} a_n, \qquad \Delta^0 a_n = a_n$$

The above sequence is convex (concave) of order k if and only if $\Delta^k a_n \geqslant 0$, ($\Delta^k a_n \leqslant 0$), for each $n \in N$. Classes of these sequences have wide applications in solving problems in various mathematical disciplines (see for example [**2, 3, 6, 7, 8, 11, 12, 13**]). There are a number of generalizations of a notion of convexity. Hereby we will adduce some which are important for our work.

Let p be a positive real number, $p \in R^+$, and $(a_n), n \in N$ a sequence of real numbers.This sequence is p-monotone nondecreasing (non-increasing) (see for example [**1, 4, 5, 10**]), if, for each $n, n \in N$, the following inequality

$$L_p(a_n) = a_{n+1} - pa_n \geqslant 0 \qquad (L_p(a_n) = a_{n+1} - pa_n \leqslant 0). \tag{1.1}$$

holds.

A notion of convex sequences was extended in [**9**] by the following definition.

Definition 1.1. Let k be a fixed natural number, $k \geqslant 2$. A sequence $(a_n), n \in N$, is called k-convex, if for all $n \in N$, holds

$$a_{n+1} \leqslant a_n + \frac{a_{n+k} - a_n}{k}, \quad \text{and } a_{n+k-1} \leqslant a_n + \frac{(k-1)(a_{n+k} - a_n)}{k}. \tag{1.2}$$

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In this paper we will define a class of L_p^k-convex sequences which represents a generalization of the class of k-convex sequences. Then we will prove two fundamental properties of L_p^k-convex sequences.

2. L_p^k - convex sequences

Let p be a positive real number, $p \in R^+$, and (Wn) and (V_n), sequences defined by

$$W_n = \begin{cases} \frac{p^n-1}{p-1}, & p \neq 1 \\ n, & p = 1 \end{cases} \qquad \text{and} \qquad V_n = \frac{W_n}{p^{n-1}},$$

for each $n \in N$. We define a class of L_p^k-convex sequences as follows.

Definition 2.1. Let $k, k \geqslant 2$, be a fixed natural number. A sequence of real numbers $(a_n), n \in N$, is L_p^k-convex if following inequalities

$$W_k a_{n+1} \leqslant a_{n+k} + pW_{k-1}a_n \qquad \text{and} \qquad W_k a_{n+k-1} \leqslant W_{k-1}a_{n+k} + p^{k-1}a_n \tag{2.1}$$

are valid for each $n \in N$.

According to (2.1) it is not difficult to conclude that if sequence (a_n) is L_p^k-convex, then for each $n, n \in N$, the following inequality is valid

$$a_{n+k} - a_{n+k-1} - p^{k-1}a_{n+1} + p^{k-1}a_n \geqslant 0. \tag{2.2}$$

It is easy to verify that each L_p^k-convex sequence satisfies the inequality (2.2), but not vice versa.

Note that for $p = 1$ Definition 2.1 becomes Definition 1.1.

The following results can be easily proved for sequences (W_n) and $(a_n), n \in N$.

Lemma 2.1. *The following equalities are valid for elements of sequence* (W_n), $n \in N$

$$\begin{aligned} W_{n+1} &= (W_{n+k} - W_{k-1})p^{1-k} = 1 + pW_n \\ W_{n+k-1} &= (W_{n+k} - 1)p^{-1} = W_{k-1} + p^{k-1}W_n. \end{aligned}$$

Lemma 2.2. *If a real sequence* $(a_n), n \in N$, *is p-monotone nondecreasing (nonincreasing), then the inequality*

$$a_{n+k} \geqslant p^k a_n, \qquad (a_{n+k} \leqslant p^k a_n),$$

holds for each $n \in N$.

Now, based on the results of Lemma 2.1 and 2.2 we will prove two crucial properties of L_p^k-convex sequences.

Theorem 2.1. *Let* $(a_n), n \in N$, *be a sequence of real numbers that is p-monotone nondecreasing and* L_p^k*-convex. Then, the sequence* $(V_n a_n), n \in N$, *is* L_p^k*-convex as well.*

Proof. We will prove that under given conditions for the sequence $(a_n), n \in N$, the sequence $(V_n a_n), n \in N$, satisfies the inequality (2.1). Since sequence $(a_n), n \in N$, is p-monotone nondecreasing and L_p^k-convex, the inequality

$$\begin{aligned} p^{k-1}W_k W_{n+1} a_{n+1} &\leqslant p^{k-1} W_{n+1} a_{n+k} + p^k W_{k-1} W_{n+1} a_n = \\ &= (W_{n+k} - W_{k-1}) a_{n+k} + p^k W_{k-1}(1 + pW_n) a_n = \\ &= W_{n+k} a_{n+k} + p^{k+1} W_{k-1} W_n a_n + W_{k-1}(p^k a_n - a_{n+k}) \leqslant \\ &\leqslant W_{n+k} a_{n+k} + p^{k+1} W_{k-1} W_n a_n \end{aligned}$$

holds for each $n \in N$. By multiplying the last inequality with p^{1-k-n} we obtain

$$W_k V_{n+1} a_{n+1} \leqslant V_{n+k} a_{n+k} + pW_{k-1} V_n a_n,$$

which means that elements of the sequence $(V_n a_n), n \in N$, satisfy the first inequality in (2.1).

Similarly, from the inequality

$$\begin{aligned} W_k W_{n+k-1} a_{n+k-1} &\leqslant W_{k-1} W_{n+k-1} a_{n+k-1} + p^{k-1} W_{n+k-1} a_n = \\ &= W_{k-1} p^{-1} (W_{n+k} - 1) a_{n+k} + p^{k-1}(W_{k-1} + p^{k-1} W_n) a_n = \\ &= W_{k-1} p^{-1} W_{n+k} a_{n+k} + p^{2k-2} W_n a_n + p^{-1} W_{k-1}(p^k a_n - a_{n+k}) \leqslant \\ &\leqslant W_{k-1} p^{-1} W_{n+k} a_{n+k} + p^{2k-2} W_n a_n, \end{aligned}$$

we obtain inequality

$$W_k V_{n+k-1} a_{n+k-1} \leqslant W_{k-1} V_{n+k} a_{n+k} + p^{k-1} V_n a_n,$$

meaning that the sequence $(V_n a_n), n \in N$, satisfies the second inequality in (2.1), also.

□

THEOREM 2.2. *Let $(a_n), n \in N$, be L_p^k-convex sequence. If $(\frac{a_n}{V_n}), n \in N$, is p-monotone non-increasing, then it is L_p^k-convex as well.*

Proof. We will conduct the proof by contradiction. Suppose that for some fixed $n, n \in N$, holds the inequality

$$W_k \frac{a_{n+1}}{V_{n+1}} \geqslant \frac{a_{n+k}}{V_{n+k}} + pW_{k-1} \frac{a_n}{V_n}.$$

If so, then the following would be valid

$$\begin{aligned} W_k a_{n+1} &\geqslant \frac{V_{n+1}}{V_{n+k}} a_{n+k} + pW_{k-1} \frac{V_{n+1}}{V_n} a_n = \\ &= a_{n+k} + pW_{k-1} a_n + \frac{V_{k-1}}{p^{n+k-1}} \left(p^k \frac{a_n}{V_n} - \frac{a_{n+k}}{V_{n+1}} \right) \geqslant \\ &\geqslant a_{n+k} + pW_{k-1} a_n. \end{aligned}$$

This means that the sequence $(a_n), n \in N$, is not L_p^k-convex, which is in contradiction with the assumption of Theorem 2.2.

□

Let us note that when $p = 1$ results given by Theorems 2.1 and 2.2 reduce to the one proved in [**9**].

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