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Edge Scattering Number of Gear Graphs

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ABSTRACT. The edge scattering number of a noncomplete connected graph G is defined to be $es(G) = max\{\omega(G-S)-|S| : S \subseteq E(G),\ \omega(G-S) > 1\}$ where $\omega(G-S)$ denote the number of components in $G-S$. A set $S \subseteq E(G)$, is said to be the es-set of G, if $es(G) = \omega(G-S) - |S|$. In this paper contains results on bounds for the edge scattering number. Moreover we give some results about the edge scattering number of graphs obtained from graph operations *between gear graphs and* K_2 *complete graphs.*

1. Introduction

In a communication network, the vulnerability measures the resistance of the network to disruption of operation after the failure of certain stations or communication links. To measure the vulnerability we have some parameters which are connectivity and edge-connectivity [**16**], integrity and edge-integrity [**3**], toughness and edge-toughness [**6, 12**], tenacity and edge-tenacity [**7, 13**], scattering number [**8**] and edge scattering number [**8, 1**].

Terminology and notation not defined in this paper can be found in [**5**]. Let G be a finite simple graph with vertex set *V(G)* and edge set *E(G)*.

If the network does get disconnected, then remaining components should continue to function with reduced capacity. We would prefer a network which would disconnect in such a way that its capacity is almost seem as before. That is, we have the fundamental question: "How difficult is it to reconstruct the network?." This question is analyzed by considering the number of components of the remaining graph. Therefore, we are concerned with the edge scattering number of a graph as a measure of graph vulnerability.

DEFINITION 1.1. [**1**] The edge scattering number of a noncomplete connected graph G is defined to be

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$$es(G) = max\{\omega(G-S) - |S| : S \subseteq E(G),\ \omega(G-S) > 1\}$$

where $\omega(G-S)$ denote the number of components in $G-S$. A set $S \subseteq E(G)$, is said to be the *es*-set of G, if

$$es(G) = \omega(G-S) - |S|.$$

Next we give some lower and upper bounds for the edge scattering number in terms of well known graph parameters.

THEOREM 1.1. [**1**] *Let G be a connected graph. Then,*

$$es(G) \leqslant 1.$$

THEOREM 1.2. [**1**] *In the graph G, n and m denote the number of vertices and the number of edges, respectively. Let G be a connected graph. Then,*

$$es(G) \geqslant n - m.$$

THEOREM 1.3. [**1**] *Let G be a graph. If G is λ-edge-connected then,*

$$es(G) \geqslant 2 - \lambda.$$

THEOREM 1.4. [**1**] *Let G be a connected graph and $\delta(G)$ be the minimum degree of G. Then,*

$$es(G) \geqslant 2 - \delta(G).$$

Geared systems are used in dynamic modelling. These are graph theoretic models that are obtained by using gear graphs. Similarly the complement of a gear graph, the cartesian product of gear graphs and the sequential join of gear graphs can be used to design a gear network.

Consequently these considerations motivated us to investigate the vulnerability of gear graphs by using the edge scattering number. Now we give the following definition.

DEFINITION 1.2. [**3**] The *gear graph* is a wheel graph with a vertex added between each pair adjacent graph vertices of the outer cycle. The gear graph G_n has $2n+1$ vertices and $3n$ edges.

In Section 2 we compute the edge scattering number of a gear graph. Also we give some results about the edge scattering number of graphs obtained from graph operations *between gear graphs and K_2 complete graphs.*

2. Gear Graphs and Graph Operations

In this section we first calculate the edge scattering number of a gear graph.

THEOREM 2.1. *The edge scattering number of the gear graph G_n $(n \geqslant 3)$ is* 0.

Proof. Let S be an edge cut set of G_n. If $|S| = r$, then we have at most r components. From the definition of edge scattering number we have,

$$\omega(G_n - S) - |S| \leqslant r - r = 0$$

and when we take the maximum of both sides we have,

$$es(G_n) \leqslant 0. \tag{2.1}$$

On the other hand it is easy see that $\delta(G_n) = 2$. Then, by Theorem 1.4, we get

$$es(G_n) \geqslant 2 - \delta(G_n) = 2 - 2 = 0. \tag{2.2}$$

By (2.1) and (2.2) we have,

$$es(G_n) = 0.$$

The proof is completed.

Theorem 2.2. *Let $\overline{G_n}$ be a complement graph of a gear graph G_n $(n \geqslant 3)$. Then,*

$$es(\overline{G_n}) = 2 - n.$$

Proof. The graph $\overline{G}_n$ has two complete subgraphs, namely K_{n1} and K_{n2}. Each vertices of K_{n1} is joined to the vertices of K_{n2} with $(n-2)$ edges. Let S be an edge cut set of $\overline{G_n}$ and $|S| = r$. Then we have two cases:
Case 1: Suppose that if $1 \leqslant r < n$ then,

$$\omega(\overline{G_n} - S) = 1.$$

This is not satisfying for the definition of edge scattering number. Namely, it should have,

$$\omega(\overline{G_n} - S) > 1$$

Case 2: If $n \leqslant r < E(G)$, then we have at most $\lfloor \frac{r}{n} \rfloor + 1$ components. Hence

$$\omega(\overline{G}_n - S) - |S| \leqslant \lfloor \tfrac{r}{n} \rfloor + 1 - r$$

and when we take the maximum of both sides we have,

$$es(\overline{G}_n) \leqslant max\{\lfloor \tfrac{r}{n} \rfloor + 1 - r\}.$$

The function $f(r) = \lfloor \frac{r}{n} \rfloor + 1 - r$ takes its maximum value at $r = n$ and we get

$$es(\overline{G}_n) \leqslant 2 - n.$$

It can be easily seen that there is an edge cut set S^* of G, such that $|S^*| = n$ and $\omega(\overline{G}_n - S) = 2$. Therefore

$$es(\overline{G}_n) = 2 - n.$$

The proof is completed.

Definition 2.1. [**5**] The Cartesian product $G_1 \times G_2$ of graphs G_1 and G_2 has $V(G_1) \times V(G_2)$ as its vertex set and (u_1, u_2) is adjacent to (v_1, v_2) if either $u_1 = v_1$ and u_2 is adjacent to v_2 or $u_2 = v_2$ and u_1 is adjacent to v_1.

Theorem 2.3. *Let G_n $(n \geqslant 3)$ be a gear graph. Then,*

$$es(K_2 \times G_n) = -1.$$

Proof. The graph $K_2 \times G_n$ has $4n+2$ vertices and has two subgraphs, namely G_{n1} and G_{n2} . Gear graph contains vertices set of whell graph. Let S be an edge cut set of graph $K_2 \times G_n$ and $|S| = r$. Since $\lambda(K_2 \times G_n)$ edge-connected for $K_2 \times G_n$ and $\lambda(K_2 \times G_n) = 3$. By Theorem 1.3 we have

$$es(K_2 \times G_n) \geqslant 2 - 3 = -1. \tag{2.3}$$

If $r < 3$ then $\omega(K_2 \times G_n - S) = 1$ it is a contradiction.
If $r \geqslant 3$ then

$$\omega((K_2 \times G_n) - S) \leqslant \lfloor \tfrac{r+1}{3} \rfloor + 1.$$

Thus

$$\omega((K_2 \times G_n) - S) - |S| \leqslant \lfloor \tfrac{r+1}{3} \rfloor + 1 - r$$

and when we take the maximum of both sides we have

$$es(K_2 \times G_n) \leqslant max\{\lfloor \tfrac{r+1}{3} \rfloor + 1 - r\}.$$

The function $f(r) = \lfloor \frac{r+1}{3} \rfloor + 1 - r$ takes its maximum value at $r = 3$ and we get

$$es(K_2 \times G_n) \leqslant -1. \tag{2.4}$$

By (2.3) and (2.4) we get

$$es(K_2 \times G_n) = -1.$$

The proof is completed.

THEOREM 2.4. *Let $m \geqslant 3$ and $n \geqslant 3$ be positive integers. Then,*

$$es(G_m \times G_n) = -3.$$

Proof. Let $\lambda(G_m \times G_n)$ be edge-connected for $G_m \times G_n$. Then, we know that

$$\lambda(G_m \times G_n) = 5.$$

By Theorem 1.3 and we get,

$$es(G_m \times G_n) \geqslant 2 - \lambda(G_m \times G_n) = 2 - 5 = -3. \tag{2.5}$$

If $r < 5$ then $\omega(G_m \times G_n - S) = 1$ it is a contradiction.
On the other hand let S be an edge cut set of $G_m \times G_n$ and $|S| = r$. If $r \geqslant 5$ then,

$$\omega((G_m \times G_n) - S) \leqslant \lfloor \tfrac{r}{5} \rfloor + 1$$

Therefore

$$\omega((G_m \times G_n) - S) - |S| \leqslant \lfloor \tfrac{r}{5} \rfloor + 1 - r$$

and when we take the maximum of both sides we have,

$$es(G_m \times G_n) \leqslant max\{\lfloor \tfrac{r}{5} \rfloor + 1 - r\}$$

The function $f(r) = \lfloor \frac{r}{5} \rfloor + 1 - r$ takes its maximum value at $r = 5$ and we get

$$es(G_m \times G_n) \leqslant -3 \tag{2.6}$$

From (2.5) and (2.6) we have

$$es(G_m \times G_n) = -3.$$

We complete the proof.

Definition 2.2. [**2**] Let G_1 and G_2 be two graphs. The union $G = G_1 \cup G_2$ has $V(G) = V(G_1) \cup V(G_2)$ and $E(G) = E(G_1) \cup E(G_2)$. The join is denoted $V(G_1) + V(G_2)$ and consist of $V(G_1) \cup V(G_2)$ and all edges joining $V(G_1)$ with $V(G_2)$. For three or more disjoint graphs $G_1, G_2, \ldots, G_n$, the sequential join $G_1 + G_2 + \ldots + G_n$ is $(G_1 + G_2) \cup (G_2 + G_3) \cup \ldots \cup (G_{n-1} + G_n)$.

Theorem 2.5. *Let $3 \leqslant m \leqslant n$ be positive integers. Then,*

$$es(G_m + G_n) = -2m - 1.$$

Proof. Let $\delta(G_m + G_n)$ be the minimum degree of $G_m + G_n$. Then,

$$\delta(G_m + G_n) = 2m + 3$$

By Theorem 1.4 and we have,

$$es(G_m + G_n) \geqslant 2 - \delta(G_m + G_n) = 2 - (2m + 3) = -2m - 1. \tag{2.7}$$

On the other hand let S be an edge cut set of $G_m + G_n$ and $|S| = r$. If $r \geqslant 2m + 3$ then we have

$$\omega((G_m + G_n) - S) \leqslant \lfloor \tfrac{r}{2m+1+2} \rfloor + 1.$$

Thus

$$\omega((G_m + G_n) - S) - |S| \leqslant \lfloor \tfrac{r}{2m+3} \rfloor + 1 - r$$

and when we take the maximum of both sides we have,

$$es(G_m + G_n) \leqslant max\{\lfloor \tfrac{r}{2m+3} \rfloor + 1 - r\}.$$

The function $f(r) = \lfloor \frac{r}{2m+3} \rfloor + 1 - r$ takes its maximum value at $r = 2m + 3$ and we get

$$es(G_m + G_n) \leqslant -2m - 1 \tag{2.8}$$

By (2.7) and (2.8) we have,

$$es(G_m + G_n) = -2m - 1$$

The proof is completed.

Proposition 2.1. *One can easily show that $es(G_3 + G_4) = -7$.*

Theorem 2.6. *Let $n \geqslant 5$ be a positive integer. Then,*

$$es(G_3 + G_4 + \ldots + G_n) = -9.$$

Proof. Let S be an edge cut set of graph $G_3 + G_4 + \ldots + G_n$ and set $|S| = r$. It is easy see that $\lambda(G_3 + G_4 + \ldots + G_n) = 11$. By Theorem 1.3 and we have two cases:

$$es(G_3 + G_4 + \ldots + G_n) \geqslant 2 - \lambda(G_3 + G_4 + \ldots + G_n) = -9. \tag{2.9}$$

Case 1: If $r < 11$ then $\omega(G_3 + G_4 + \ldots + G_n) = 1$. It is a contradiction.

Case 2: If $r \geqslant 11$ then,

$$\omega((G_3 + G_4 + \ldots + G_n) - S) \leqslant \lfloor \tfrac{r}{11} \rfloor + 1.$$

So

$$\omega((G_3 + G_4 + \ldots + G_n) - S) - |S| \leqslant \lfloor \tfrac{r}{11} \rfloor + 1 - r$$

when we take the maximum of both sides we have,

$$es(G_3 + G_4 + \ldots + G_n) \leqslant max\{\lfloor \frac{r}{11} \rfloor + 1 - r\}$$

The function $f(r) = \lfloor \frac{r}{11} \rfloor + 1 - r$ takes its maximum value at $r = 11$ and we get

$$es(G_3 + G_4 + \ldots + G_n) \leqslant -9 \tag{2.10}$$

By (2.9) and (2.10) we have

$$es(G_3 + G_4 + \ldots + G_n) = -9.$$

We complete the proof.

3. Conclusion

A network has often as considerable an impact on network's performance as the edges or vertices themselves. Performance measures for the networks are essential to guide the designer in choosing an appropriate topology. In order to measure the performance we are interested the following performance metrics:

1. The number of elements that are not functioning,

2. The number of the components of the remaining network,

Many graph-theoretical parameters have been used in the past to describe the stability of communication networks. We can say that the disruption is more successful if the disconnected network contains more components. In order to reconstruct a disrupted network easily, the number of connected components, formed after the edges deleted, should be possibly small.

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