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Closed and Dense Elements in Semi Heyting Almost Distributive Lattices

G.C. Rao, M.V.Ratnamani, and Berhanu Assaye

Abstract. In this paper, we define the concept of a closed element and dense element in a Semi Heyting Almost Distributive Lattice (SHADL) L and derive some properties of closed elements and dense elements of L. We also observe that every SHADL is a pseudocomplemented ADL and that the set $L^* = \{x^*/x \in L\}$ of all closed elements of an SHADL L, forms a Boolean algebra with the operation $\underline{\vee}$ defined as $x \underline{\vee} y = (x^* \wedge y^*)^*$ for every $x, y \in L^*$ where, $x^* = (x \to 0) \wedge m$.

1. Introduction

The concept of an Almost Distributive Lattice (ADL) was introduced by U. M. Swamy and G. C. Rao [10] as a common abstraction to most of the existing ring theoretic generalizations of a Boolean algebra on one hand and the class of distributive lattices on the other. The concept of Heyting Almost Distributive Lattice (HADL) was introduced as a generalization of a Heyting algebra and many fundamental properties of HADLs were derived in our earlier paper [4]. Later, closed elements and dense elements in Heyting Almost Distributive Lattices (HADL) were studied by G. C. Rao and Berhanu Assaye in [5] and [6] respectively. The concept of a Semi Heyting Almost Distributive Lattice (SHADL) as a generalization of a Semi Heyting algebra was introduced in our earlier paper [7]. In this paper we study some properties of closed elements and dense elements of a SHADL. We also observe that every SHADL is a pseudocomplemented ADL and the set $L^* = \{x^*/x \in L\}$ of all closed elements of an SHADL L forms a Boolean algebra.

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2. Preliminaries

In this section we give some important definitions and results that are frequently used for ready reference.

Definition 2.1. [10] An algebra $(L, \vee, \wedge, 0)$ of type $(2, 2, 0)$ is called ADL if it satisfies the following axioms: for all $x, y, z \in L$

(1) $x \vee 0 = x$
(2) $0 \wedge x = 0$
(3) $(x \vee y) \wedge z = (x \wedge z) \vee (y \wedge z)$
(4) $x \wedge (y \vee z) = (x \wedge y) \vee (x \wedge z)$
(5) $x \vee (y \wedge z) = (x \vee y) \wedge (x \vee z)$
(6) $(x \vee y) \wedge y = y$

Definition 2.2. [10] Let L be a non-empty set. Fix $x_0 \in L$. For any $x, y \in L$, define $x \wedge y = y, x \vee y = x$ if $x \neq x_0, x_0 \wedge y = x_0$ and $x_0 \vee y = y$. Then $(L, \vee, \wedge, x_0)$ is an ADL and it is called a discrete ADL. Alternately, discrete ADL is defined as an ADL $(L, \vee, \wedge, 0)$ in which every $x(\neq 0)$ is maximal.

If $(L, \vee, \wedge, 0)$ is an ADL. For any $x, y \in L$, define $x \leqslant y$ if and only if $x = x \wedge y$, or equivalently $x \vee y = y$, then $\leqslant$ is a partial ordering on L.

Through out this section L stands for an ADL $(L, \vee, \wedge, 0)$ unless otherwise specified. In the following theorem some important fundamental properties of an ADL are given.

Theorem 2.1. [9] *For any $a, b, c \in L$, we have the following*

(1) $a \vee b = a \Leftrightarrow a \wedge b = b$
(2) $a \vee b = b \Leftrightarrow a \wedge b = a$
(3) $a \wedge b = b \wedge a = a$ *whenever* $a \leqslant b$
(4) $\wedge$ *is associative in* L
(5) $a \wedge b \wedge c = b \wedge a \wedge c$
(6) $(a \vee b) \wedge c = (b \vee a) \wedge c$
(7) $a \wedge b \leqslant b$ *and* $a \leqslant a \vee b$
(8) $a \wedge a = a$ *and* $a \vee a = a$
(9) $a \wedge 0 = 0$ *and* $0 \vee a = a$
(10) *if* $a \leqslant c$ *and* $b \leqslant c$*, then* $a \wedge b = b \wedge a$ *and* $a \vee b = b \vee a$*.*

Definition 2.3. [11] Let L be an ADL. A unary operation * on L is called a pseudocomplementation on L if, for any $x, y \in L$, the following conditions hold:

(1) $x \wedge y = 0 \Leftrightarrow x^* \wedge y = y$
(2) $(x \vee y)^* = x^* \wedge y^*$
(3) $x \wedge x^* = 0$

Definition 2.4. [4] Let $(L, \vee, \wedge, 0, m)$ be an ADL with a maximal element m. Suppose $\rightarrow$ is a binary operation on L satisfying the following conditions for all $x, y, z \in L$.

(1) $x \rightarrow x = m$

(2) $(x \to y) \wedge y = y$
(3) $x \wedge (x \to y) = x \wedge y \wedge m$
(4) $x \to (y \wedge z) = (x \to y) \wedge (x \to z)$
(5) $(x \vee y) \to z = (x \to z) \wedge (y \to z)$

Then $(L, \vee, \wedge, \to, 0, m)$ is called a Heyting Almost Distributive Lattice (HADL).

Definition 2.5. [5] Let $(L, \vee, \wedge, \to, 0, m)$ be a HADL. Define for any $x \in L$. $x^* = (x \to 0)$ and $L^* = \{x^* / \ x \in L\}$. Then an element of L^* is called a closed element of L. Also, for any $x, y \in L^*$. We define $x \underline{\vee} y = (x^* \wedge y^*)^*$.

Definition 2.6. [6] Let $(L, \vee, \wedge, \to, 0, m)$ be a HADL. Define $D_L = \{x \in L / \ x^* = 0\}$. Then an element of D_L is called a dense element of L.

Definition 2.7. [8] An algebra $(L, \vee, \wedge, \to, 0, 1)$ of type $(2, 2, 2, 0, 0)$ is called a Semi Heyting algebra if it satisfies the following:

(1) $(L, \vee, \wedge, 0, 1)$ is a lattice with 0, 1
(2) $x \wedge (x \to y) = x \wedge y$
(3) $x \wedge (y \to z) = x \wedge (x \wedge y \to x \wedge z)$
(4) $x \to x = 1$ for all $x, y, z \in L$

3. Closed and Dense Elements in Semi Heyting Almost Distributive Lattices

We begin with the following definition of SHADL given in [7].

Definition 3.1. [7] Let $(L, \vee, \wedge, 0, m)$ be an ADL with a maximal element m. Suppose there exists a binary operation $\to$ on L satisfying the following conditions:

(1) $(x \to x) \wedge m = m$
(2) $x \wedge (x \to y) = x \wedge y \wedge m$
(3) $x \wedge (y \to z) = x \wedge (x \wedge y \to x \wedge z)$
(4) $(x \to y) \wedge m = x \wedge m \to y \wedge m$ for all $x, y, z \in L$

Then $(L, \vee, \wedge, \to, 0, m)$ is a Semi Heyting ADL (SHADL).

The following theorem which is taken from [7] will be used frequently in this paper. Through out this section L denotes an SHADL.

Theorem 3.1. [7] *For any $a, b, c, d, x \in L$ we have the following*

(1) $m \to a = a \wedge m$
(2) $a \wedge b \wedge m \leqslant a \to b$
(3) $(a \to b) \wedge m \leqslant (a \to a \wedge b) \wedge m$
(4) $a \wedge m \leqslant [a \to (b \to a \wedge b)] \wedge m$
(5) $(a \to b) \wedge c = (a \wedge c \to b \wedge c) \wedge c$
(6) $[(a \wedge b) \to (c \wedge d)] \wedge x = [(b \wedge a) \to (d \wedge c)] \wedge x$
(7) $a \leqslant b$ *and* $a \leqslant c \Rightarrow a \wedge m \leqslant (b \to c) \wedge m.$

In this section we introduce the concepts of closed elements and dense elements in an SHADL analogous to those given in HADL.

In the following, we give the definitions of a closed element and dense element in an SHADL.

DEFINITION 3.2. If $(L, \vee, \wedge, \rightarrow, 0, m)$ is an SHADL and $x \in L$, then we write $x^* = (x \rightarrow 0) \wedge m$. If $x^* = 0$, then x is called a dense element of L and if $x = y^*$ for some $y \in L$, then x is called a closed element of L. We denote the set of closed (dense) elements of L by $L^*(D_L)$.

In the following lemma we prove the fundamental properties of closed elements and dense elements of L

LEMMA 3.1. *Let L be an SHADL and $a, b, c \in L$. Then*

(1) $a \wedge (a \rightarrow b)^* = a \wedge b^*$.
(2) $a \wedge b = 0 \Leftrightarrow a \wedge m \leqslant b^*$.
(3) $m^* = 0$.
(4) $a^* = m \Leftrightarrow a = 0$.
(5) $a \wedge b^* = 0 \Rightarrow a \wedge m \leqslant b^{**}$.
(6) $a \leqslant b \Rightarrow b^* \leqslant a^*, a^{**} \leqslant b^{**}$.
(7) $a \wedge b^* = a \wedge (a \wedge b)^*$.
(8) $(a \wedge b)^* = (b \wedge a)^*$. *In particular,* $(a \wedge m)^* = a^*$.
(9) $a \wedge a^{**} = a \wedge m$ *and* $a^{**} \wedge a = a$.
(10) $a^* = a^{***}$.
(11) $a \in L^*$ *iff* $a = a^{**}$.
(12) $a \in D_L$ *iff* $a^{**} = m$.
(13) $(a \vee b)^* = a^* \wedge b^*$.
(14) $b \wedge a = a \Rightarrow a \wedge b^* = 0$.
(15) $a^* \wedge b^* = b^* \wedge a^*$.
(16) $a^* \vee b^* = b^* \vee a^*$.
(17) $a \wedge b = 0 \Rightarrow a^{**} \wedge b = 0$.
(18) $(a \wedge b)^{**} \leqslant a^{**}$.
(19) $a^{**} \wedge b^{**} = (a \wedge b)^{**}$.
(20) $a^{**} \wedge (a \rightarrow b)^{**} = a^{**} \wedge b^{**}$.
(21) *If a is dense, then* $(a \rightarrow b)^* = b^*$.
(22) *If a and b are dense elements in L, then $a \rightarrow b$ is also dense.*
(23) *If* $a \wedge b = 0$, *then* $a^* \wedge b = b$.
(24) $(0 \rightarrow m) \wedge m = 0$ *if and only if* $(0 \rightarrow a) \wedge m \leqslant a^*$ *for all* $a \in L$.
In particular, $(0 \rightarrow m) \wedge m = 0$ *if and only if* $(0 \rightarrow a) \wedge m = 0$ *for all dense elements a of L.*
(25) $a^* \leqslant (0 \rightarrow a) \wedge m$. *In particular* $a^* \leqslant (0 \rightarrow a^{**}) \wedge m$.
(26) $a \wedge m \leqslant (0 \rightarrow a^*) \wedge m$.
(27) $(a \rightarrow a^*) \wedge m \leqslant a^* \leqslant (a^{**} \rightarrow a) \wedge m$.
(28) *If* $a^* \leqslant 0 \rightarrow a^*$, *then* $(a \rightarrow a^*) \wedge m = a^*$.
(29) $a \wedge m \leqslant (a^{**} \rightarrow a) \wedge m$
(30) $a \wedge m \leqslant (a \rightarrow a^{**}) \wedge m$.
(31) $(a^{**} \rightarrow a^*) \wedge m \leqslant a^* \leqslant (a \rightarrow a^{**}) \wedge m$.
(32) $(a \vee a^*) \wedge m \leqslant (a \rightarrow a^{**}) \wedge m$. *Hence* $(a \rightarrow a^{**}) \in D(L)$.

(33) $c \leqslant a \Rightarrow c \wedge (a \rightarrow b^*) = c \wedge b^*$.
(34) $a \wedge m \leqslant (0 \rightarrow a^{**}) \wedge m$ *if and only if* $a \wedge m \leqslant (0 \rightarrow a) \wedge m$.
(35) $b \wedge (a \rightarrow b^*) \wedge m = b \wedge a^*$.
(36) *If* a *is dense or if* $a^* \leqslant b^*$, *then* $(a \rightarrow b^*) \wedge m \leqslant b^*$.
(37) *If* $a \wedge b = 0$, *then* $(a \rightarrow b) \wedge m \leqslant a^*$.
(38) $(a^* \vee b^*)^{**} = (a \wedge b)^* = a^* \underline{\vee} b^*$ *where* $a \underline{\vee} b = (a^* \wedge b^*)^*$.
(39) $a \in L^* \Rightarrow a^* \underline{\vee} a = m$.

Proof. (1) $a \wedge (a \rightarrow b)^* = a \wedge [(a \rightarrow b) \rightarrow 0] \wedge m$
$= a \wedge [a \wedge (a \rightarrow b) \rightarrow 0] \wedge m$
$= a \wedge [a \wedge b \wedge m \rightarrow 0] \wedge m$
$= a \wedge (b \wedge m \rightarrow 0) \wedge m$
$= a \wedge (b \rightarrow 0) \wedge m$
$= a \wedge b^*$.

(2) If $a \wedge b = 0$, then $a \wedge b^* = a \wedge (b \rightarrow 0) \wedge m$
$= a \wedge (a \wedge b \rightarrow 0) \wedge m$
$= a \wedge (0 \rightarrow 0) \wedge m$
$= a \wedge m$.
Thus $a \wedge m \leqslant b^*$.
Conversely, $a \wedge m \leqslant b^* \Rightarrow a \wedge b^* = a \wedge m$
$\Rightarrow a \wedge b = a \wedge m \wedge b = a \wedge b^* \wedge b = 0$.

(3) $m^* = (m \rightarrow 0) \wedge m = m \wedge (m \rightarrow 0) \wedge m = m \wedge 0 \wedge m = 0$.

(4) If $a^* = m$, then $a = m \wedge a = a^* \wedge a = 0$.
Conversely, assume that $a = 0$, then $a^* = (a \rightarrow 0) \wedge m$
$= (0 \rightarrow 0) \wedge m = m$.

(5) If $a \wedge b^* = 0$ then $a \wedge b^{**} = a \wedge m$ (by (2) above)
and hence, $a \wedge m \leqslant b^{**}$.

(6) Suppose $a \leqslant b$. Then $a \wedge b^* \leqslant b \wedge b^* = 0$. Thus $b^* \leqslant a^*$ (by (2) above) and hence $a^{**} \leqslant b^{**}$.

(7) $a \wedge b^* = a \wedge (b \rightarrow 0) \wedge m = a \wedge (a \wedge b \rightarrow 0) \wedge m = a \wedge (a \wedge b)^*$.

(8) $(a \wedge b)^* = ((a \wedge b) \rightarrow 0) \wedge m = ((b \wedge a) \rightarrow 0) \wedge m = (b \wedge a)^*$.

(9) $a \wedge a^{**} = a \wedge (a^* \rightarrow 0) \wedge m = a \wedge (a \wedge a^* \rightarrow 0) \wedge m = a \wedge m$.
Now, $a^{**} \wedge a = a \wedge a^{**} \wedge a = a \wedge m \wedge a = a$.

(10) By (2) above, we get that $a^* \wedge a^{**} = 0 \Rightarrow a^* \leqslant a^{***}$.
Also, $a \wedge m \leqslant a^{**} \Rightarrow a^{***} \leqslant (a \wedge m)^* = a^*$ (by (8))
Therefore, $a^* = a^{***}$.

(11) Follows from (10) above.

(12) $a \in D_L \Rightarrow a^* = 0 \Rightarrow a^{**} = 0^* = m$.
Conversely, if $a^{**} = m \Rightarrow a^* = a^{***} = m^* = 0 \Rightarrow a \in D_L$.

(13) $(a \vee b) \wedge (a^* \wedge b^*) = (a \wedge a^* \wedge b^*) \vee (b \wedge a^* \wedge b^*) = 0$.
$\Rightarrow a^* \wedge b^* \leqslant (a \vee b)^*$
Also, $a \wedge m \leqslant (a \vee b) \wedge m$ and $b \wedge m \leqslant (a \vee b) \wedge m$
$\Rightarrow [(a \vee b) \wedge m]^* \leqslant (a \wedge m)^*$ and $[(a \vee b) \wedge m]^* \leqslant (b \wedge m)^*$
$\Rightarrow (a \vee b)^* \leqslant a^*$ and $(a \vee b)^* \leqslant b^*$

Therefore, $(a \vee b)^* \leqslant a^* \wedge b^*$.
Hence, $(a \vee b)^* = a^* \wedge b^*$.

(14) Suppose $b \wedge a = a$, then $a \wedge b^* = b \wedge a \wedge b^* = a \wedge b \wedge b^* = 0$.

(15) $a^* \wedge b^* = a^* \wedge b^* \wedge m = b^* \wedge a^* \wedge m = b^* \wedge a^*$.

(16) $a^* \vee b^* = (a^* \wedge m) \vee (b^* \wedge m) = (a^* \vee b^*) \wedge m = (b^* \vee a^*) \wedge m = b^* \vee a^*$.

(17) $a \wedge b = 0 \Rightarrow b \wedge m \leqslant a^* \Rightarrow a^{**} \leqslant b^* \Rightarrow a^{**} \wedge b \leqslant b^* \wedge b = 0$.

(18) $(a \wedge b)^{**} = (b \wedge a)^{**} \leqslant a^{**}$ from (6).

(19) From (18) we get $(a \wedge b)^{**} \leqslant a^{**}$ and $(a \wedge b)^{**} \leqslant b^{**}$
and hence, $(a \wedge b)^{**} \leqslant a^{**} \wedge b^{**}$.
Now, by (17) above, $a \wedge b \wedge (a \wedge b)^* = 0 \Rightarrow a^{**} \wedge b \wedge (a \wedge b)^* = 0$
$\Rightarrow b \wedge a^{**} \wedge (a \wedge b)^* = 0$
$\Rightarrow b^{**} \wedge a^{**} \wedge (a \wedge b)^* = 0$
$\Rightarrow a^{**} \wedge b^{**} \wedge (a \wedge b)^* = 0$
$\Rightarrow a^{**} \wedge b^{**} \leqslant (a \wedge b)^{**}$
Therefore, $(a \wedge b)^{**} = a^{**} \wedge b^{**}$.

(20) $a^{**} \wedge (a \to b)^{**} = [a \wedge (a \to b)]^{**} = (a \wedge b \wedge m)^{**} = (a \wedge b)^{**} = a^{**} \wedge b^{**}$.

(21) a is dense $\Rightarrow a^* = 0 \Rightarrow a^{**} = 0^* = m$.
Thus $(a \to b)^{**} = m \wedge (a \to b)^{**} = a^{**} \wedge (a \to b)^{**}$
$= a^{**} \wedge b^{**}$
$= m \wedge b^{**}$
$= b^{**}$ and hence
$(a \to b)^* = (a \to b)^{***} = b^{***} = b^*$.

(22) Suppose a and b are dense elements of L. Then by (20) we get
$(a \to b)^{**} = a^{**} \wedge (a \to b)^{**} = a^{**} \wedge b^{**} = m$
Therefore, $a \to b$ is also a dense element of L.

(23) Suppose $a \wedge b = 0$. Then, $a^* \wedge b = (a \to 0) \wedge m \wedge b$
$= (a \to 0) \wedge b$
$= (b \wedge a \to 0) \wedge b$
$= (0 \to 0) \wedge b$
$= m \wedge b$
$= b$.

(24) $(0 \to m) \wedge m = 0 \Rightarrow a \wedge (0 \to m) \wedge m = 0$
$\Rightarrow a \wedge (0 \to (a \wedge m)) \wedge m = 0$
$\Rightarrow a \wedge (0 \to a) \wedge m = 0$
$\Rightarrow (0 \to a) \wedge m \leqslant a^*$.
Conversely, assume that $(0 \to a) \wedge m \leqslant a^*$, for all $a \in L$.
When $a = m$, we get $(0 \to m) \wedge m \leqslant m^* = 0$.
If a is dense element of L then $a^* = 0$ and hence the result follows.

(25) $a^* \wedge (0 \to a) \wedge m = a^* \wedge (0 \to a^* \wedge a) \wedge m$
$= a^* \wedge (0 \to 0) \wedge m$
$= a^* \wedge m = a^*$.
Therefore, $a^* \leqslant (0 \to a) \wedge m$.
On replacing a by a^{**} in this, we get the rest.

(26) $a \wedge m \wedge (0 \to a^*) \wedge m = a \wedge (0 \to a^*) \wedge m$
$= a \wedge (0 \to a \wedge a^*) \wedge m$

$= a \wedge (0 \to 0) \wedge m = a \wedge m.$
Therefore, $a \wedge m \leqslant (0 \to a^*) \wedge m.$

(27) From (3) of Theorem 3.1, we get $(a \to a^*) \wedge m \leqslant a^*.$
Now, $a^* \wedge (a^{**} \to a) \wedge m = a^* \wedge ((a^* \wedge a^{**}) \to (a^* \wedge a)) \wedge m$
$= a^* \wedge (0 \to 0) \wedge m$
$= a^* \wedge m = a^*.$

(28) Suppose $a^* \leqslant (0 \to a^*)$. Then
$a^* \wedge (a \to a^*) \wedge m = a^* \wedge (a^* \wedge a \to a^*) \wedge m$
$= a^* \wedge (0 \to a^*) \wedge m$
$= a^* \wedge m = a^*.$
Therefore, $a^* \leqslant (a \to a^*) \wedge m$. and hence by (27) above, we get
$(a \to a^*) \wedge m = a^*.$

(29) $a \wedge m \wedge (a^{**} \to a) \wedge m = a \wedge (a \wedge a^{**} \to a) \wedge m = a \wedge (a \wedge m \to a) \wedge m =$
$a \wedge (a \to a) \wedge m = a \wedge m$ and hence, $a \wedge m \leqslant (a^{**} \to a) \wedge m.$

(30) Follows from (4) of Theorem 3.1, by taking $b = a^*$.

(31) Since $a \wedge (a^{**} \to a^*) \wedge m \leqslant a^{**} \wedge (a^{**} \to a^*) \wedge m = a^{**} \wedge a^* \wedge m = 0$
$\Rightarrow (a^{**} \to a^*) \wedge m \leqslant a^*$. Now,
$a^* \wedge (a \to a^{**}) \wedge m = a^* \wedge (a^* \wedge a \to a^* \wedge a^{**}) \wedge m$
$= a^* \wedge (0 \to 0) \wedge m$
$= a^* \wedge m = a^*.$
Therefore, $a^* \leqslant (a \to a^{**}) \wedge m.$

(32) Consider $(a \vee a^*) \wedge (a \to a^{**}) \wedge m = [a \wedge (a \to a^{**})] \vee [a^* \wedge (a \to a^{**})] \wedge m$
$= (a \wedge m) \vee (a^* \wedge m)$
$= (a \vee a^*) \wedge m.$
Therefore, $(a \vee a^*) \wedge m \leqslant (a \to a^{**}) \wedge m.$
Since $a \vee a^* \in D_L$, we get $a \to a^{**} \in D_L$

(33) $c \wedge (a \to b^*) = c \wedge (c \wedge a \to c \wedge b^*) = c \wedge (c \to c \wedge b^*) = c \wedge (c \to b^*) = c \wedge b^*.$

(34) $a \wedge (0 \to a^{**}) \wedge m = a \wedge (0 \to a \wedge a^{**}) \wedge m = a \wedge (0 \to a) \wedge m$ and hence we get $a \wedge m \leqslant (0 \to a^{**}) \wedge m$ iff $a \wedge m \leqslant (0 \to a) \wedge m.$

(35) $b \wedge (a \to b^*) \wedge m = b \wedge (b \wedge a \to b \wedge b^*) \wedge m$
$= b \wedge (b \wedge a \to 0) \wedge m$
$= b \wedge (a \to 0) \wedge m = b \wedge a^*.$

(36) By 35 above, $b \wedge (a \to b^*) \wedge m = b \wedge a^*$. If a is dense then $b \wedge a^* = 0$ or if $a^* \leqslant b^*$, then $b \wedge a^* = 0$. Thus $b \wedge (a \to b^*) \wedge m = 0$.
Hence $(a \to b^*) \wedge m \leqslant b^*.$

(37) Since $a \wedge (a \to b) = a \wedge b \wedge m = 0$, we get $(a \to b) \wedge m \leqslant a^*.$

(38) $(a^* \vee b^*)^{**} = [(a^* \vee b^*)^*]^* = [a^{**} \wedge b^{**}]^* = a^* \underline{\vee} b^*.$

(39) $a^* \underline{\vee} a = (a^{**} \wedge a^*)^* = 0^* = m.$

□

THEOREM 3.2. *Let L be an SHADL and let $a, b \in L$ with $a \wedge m \leqslant b \wedge m$. For $c, d \in [a \wedge m, b \wedge m]$, define $c \to^{ab} d = (c \to d) \wedge b \wedge m$. Then the algebra $L_0 = ([a \wedge m, b \wedge m], \vee, \wedge, \to^{ab}, a \wedge m, b \wedge m)$ is a Semi Heyting algebra. Further more, if L is a HADL. Then L_0 is also a Heyting algebra.*

PROOF. Let $c, d, e \in [a \wedge m, b \wedge m]$.
First we note that $c \rightarrow^{ab} d \in [a \wedge m, b \wedge m]$
Since $a \wedge m \leqslant c \wedge m, a \wedge m \leqslant d \wedge m$, we get $a \wedge m \leqslant (c \rightarrow d) \wedge m$ and hence $c \rightarrow^{ab} d \in [a \wedge m, b \wedge m]$.
$c \wedge (c \rightarrow^{ab} d) = c \wedge (c \rightarrow d) \wedge b \wedge m$
$= c \wedge d \wedge b \wedge m$
$= c \wedge d$.
$e \wedge (c \rightarrow^{ab} d) = e \wedge (c \rightarrow d) \wedge b \wedge m$
$= e \wedge (e \wedge c \rightarrow e \wedge d) \wedge b \wedge m$
$= e \wedge (e \wedge c \rightarrow^{ab} e \wedge d)$. Finally,
$(c \rightarrow^{ab} c) = (c \rightarrow c) \wedge b \wedge m$
$= m \wedge b \wedge m$
$= b \wedge m$.
Hence L_0 is a semi Heyting algebra.
Now, suppose L is a HADL. Then
$c \wedge d \leqslant c \Rightarrow (c \rightarrow c) \leqslant c \wedge d \rightarrow c$
$\Rightarrow m \leqslant c \wedge d \rightarrow c$
$\Rightarrow c \wedge d \rightarrow c = m$
Therefore $c \wedge d \rightarrow^{ab} c = (c \wedge d \rightarrow c) \wedge b \wedge m = b \wedge m$.
Hence L_0 is a Heyting algebra. □

THEOREM 3.3. *Let $(L, \vee, \wedge, \rightarrow, 0, m)$ be an SHADL. For $x \in L$, define $x^* = (x \rightarrow 0) \wedge m$. Then $*$ is a pseudocomplementation on L.*

PROOF. Clearly, $a \wedge b = 0$ iff $a^* \wedge b = b, a \wedge a^* = 0$ and from (13) of lemma 3.4, we get that $*$ is a pseudocomplementation on L. □

THEOREM 3.4. *Let L be an SHADL and $a, b \in L$ such that $a \wedge m \leqslant b \wedge m$. For $c \in [a \wedge m, b \wedge m]$, define $c^{*ab} = (c \rightarrow a) \wedge b \wedge m$. Then the algebra $([a \wedge m, b \wedge m], \vee, \wedge, ^{*ab}, a \wedge m, b \wedge m)$ is a pseudocomplemented lattice.*

PROOF. It is enough to verify that $x \wedge y = a \wedge m \Leftrightarrow x \leqslant y^{*ab}$ for all $x, y \in [a \wedge m, b \wedge m]$
Let $c \in [a \wedge m, b \wedge m]$, Since $a \wedge m \leqslant c \wedge m$ we have $a \wedge m \leqslant (c \rightarrow a) \wedge m$
$\Rightarrow a \wedge b \wedge m \leqslant (c \rightarrow a) \wedge b \wedge m \Rightarrow a \wedge m \leqslant (c \rightarrow a) \wedge b \wedge m \leqslant b \wedge m$.
Therefore, $c^{*ab} \in [a \wedge m, b \wedge m]$.
Let $x, y \in [a \wedge m, b \wedge m]$
Assume that $x \wedge y = a \wedge m$. Then $y^{*ab} = (y \rightarrow a) \wedge b \wedge m$
$x \wedge y^{*ab} = x \wedge (y \rightarrow a) \wedge b \wedge m = x \wedge (y \rightarrow a) \wedge m = x \wedge (x \wedge y \rightarrow a) \wedge m$
$= x \wedge (a \rightarrow a) \wedge m = x$.
Therefore, $x \leqslant y^{*ab}$
Conversely, Suppose $x \leqslant y^{*ab} \Rightarrow x = x \wedge (y \rightarrow a) \wedge b \wedge m$.
Now $y \wedge x = y \wedge x \wedge (y \rightarrow a) \wedge b \wedge m = x \wedge y \wedge a \wedge b \wedge m = a \wedge m$.

Therefore, $([a \wedge m, b \wedge m], \vee, \wedge, ^{*ab}, a \wedge m, b \wedge m)$ is a pseudocomplemented lattice. □

Corollary 3.1. *Let L be an SHADL. Then the algebra* $([0,m], \vee, \wedge, ^*, 0, m)$, *where* $c^* = (c \to 0) \wedge m$ *for* $c \in [0,m]$, *is a pseudocomplemented lattice.*

Corollary 3.2. *Let L be an SHADL. Then the following are equivalent.*

(1) *L is a lattice.*
(2) *L is a Semi Heyting algebra*
(3) *L is a pseudocomplemented lattice*
(4) *L is a distributive lattice*
(5) *L is a modular lattice*

The proof of the following theorem can be verified routinely.

Theorem 3.5. *Let* $(L, \vee, \wedge, \to, 0, m)$ *be an SHADL. Then* $(L^*, \underline{\vee}, \wedge, *, 0, m)$ *is a Boolean algebra. Where* $x \underline{\vee} y = (x^* \wedge y^*)^*$ *for any* $x, y \in L^*$.

We know that a Boolean algebra is a Heyting algebra in which $a \to b = a^* \vee b$. On the other hand, in a SHADL, we have the following.

Theorem 3.6. *Let L be an SHADL. Then, for* $a, b \in L, (a \to b)^{**} \leqslant a^* \underline{\vee}\, b^{**}$.

Proof. $(a \to b)^{**} = (a^{**} \underline{\vee}\, a^*) \wedge (a \to b)^{**}$
$= [a^{**} \wedge (a \to b)^{**}] \underline{\vee}\, [a^* \wedge (a \to b)^{**}]$
$= [a \wedge (a \to b)]^{**} \underline{\vee}\, [a^* \wedge (a \to b)^{**}]$
$= [a^{**} \wedge b^{**}] \underline{\vee}\, [a^* \wedge (a \to b)^{**}]$
$= [(a^{**} \wedge b^{**}) \underline{\vee}\, a^*] \wedge [(a^{**} \wedge b^{**}) \underline{\vee}\, (a \to b)^{**}]$
$\leqslant [a^* \underline{\vee}\, (a^{**} \wedge b^{**})]$
$= (a^* \underline{\vee}\, a^{**}) \wedge (a^* \underline{\vee}\, b^{**})$
$= m \wedge (a^* \underline{\vee}\, b^{**})$
$= a^* \underline{\vee}\, b^{**}$.
Therefore $(a \to b)^{**} \leqslant a^* \underline{\vee}\, b^{**}$. □

In the following theorems we derive some important properties of SHADL involving the operation $*$.

Theorem 3.7. *Let L be an SHADL and $a, b \in L$. Then* $(a \vee a^*) \wedge (a \to b) \wedge m \leqslant (a^* \vee b) \wedge m$.

Proof. $(a \vee a^*) \wedge (a \to b) \wedge m = [[a \wedge (a \to b)] \vee [a^* \wedge (a \to b)]] \wedge m$
$= [(a \wedge b \wedge m) \vee (a^* \wedge (a \to b))] \wedge m$
$= [(a \wedge b \wedge m) \vee a^*] \wedge [(a \wedge b \wedge m) \vee (a \to b)] \wedge m$
$\leqslant [a^* \vee (a \wedge b \wedge m)]$
$= (a^* \vee a) \wedge (a^* \vee (b \wedge m))$
$\leqslant (a^* \vee b) \wedge m$. □

Theorem 3.8. *Let L be an SHADL and $a, b, c \in L$. Then*

(1) $a^{**} \wedge (a \to b)^* = a^{**} \wedge b^*$.
(2) $a^{**} \wedge (b \to c)^* = a^{**} \wedge (a \wedge b \to a \wedge c)^*$.
(3) $b^* \wedge (a \to b) \wedge m = b^* \wedge a^*$.

PROOF. (1) $a^*\underline{\vee}(a \to b)^* = (a^{**} \wedge (a \to b)^{**})^* = (a \wedge (a \to b))^*$
$= (a \wedge b \wedge m)^*$
$= (a \wedge b)^* = a^*\underline{\vee}b^*$.
Therefore, $a^{**} \wedge (a \to b)^* = a^{**} \wedge b^*$.
(2) $a^*\underline{\vee}(b \to c)^* = [a \wedge (b \to c)]^* = [a \wedge (a \wedge b \to a \wedge c)]^* = a^*\underline{\vee}(a \wedge b \to a \wedge c)^*$.
Therefore, $a^{**} \wedge (b \to c)^* = a^{**} \wedge (a \wedge b \to a \wedge c)^*$.
(3) $b^* \wedge (a \to b) \wedge m = b^* \wedge (b^* \wedge a \to b^* \wedge b) \wedge m = b^* \wedge (b^* \wedge a \to 0) \wedge m$
$= b^* \wedge (a \to 0) \wedge m = b^* \wedge a^*$.
□

THEOREM 3.9. *Let $(L, \vee, \wedge, \to, 0, m)$ be an SHADL. Then for any element $x \in L$ there exists $d \in D_L$ such that $x = x^{**} \wedge d$.*

PROOF. Let $d = (x \vee x^*)$, then $d^* = (x \vee x^*)^* = x^* \wedge x^{**} = 0$.
Therefore $d \in D_L$. Now,
$x^{**} \wedge d = x^{**} \wedge (x \vee x^*) = [\ (x^{**} \wedge x) \vee (x^{**} \wedge x^*)\ ] = x \vee 0 = x$
□

COROLLARY 3.3. *Let $(L, \vee, \wedge, \to, 0, m)$ be an SHADL and $x, y \in L$ such that $x^{**} = y^{**}$. Then there exists $d \in D_L$ such that $x \wedge d = y \wedge d$.*

PROOF. Let $x, y \in L$, by above theorem there exists $d_1, d_2 \in D_L$ such that $x = x^{**} \wedge d_1,\ y = y^{**} \wedge d_2$.
Let $d = d_1 \wedge d_2$, then d is a dense element of L. Now, consider
$x \wedge d \wedge m = x^{**} \wedge d_1 \wedge d_2 \wedge m = y^{**} \wedge d_1 \wedge d_2 \wedge m = y \wedge d \wedge m$ and hence $x \wedge d = y \wedge d$. □

COROLLARY 3.4. *Let $(L, \vee, \wedge, \to, 0, m)$ be an SHADL and x be an element of L. Then x is dense if and only if there is an element y of L such that $x \wedge m = y^{**} \to y$.*

PROOF. Suppose x is a dense element of L. Then
$x^{**} \to x = m \to x = x \wedge m$.
Conversely, assume that $x \wedge m = y^{**} \to y$ for some $y \in L$. First we show that $y^{**} \to y$ is a dense element.
We know that $y^{**} \wedge (y^{**} \to y) = y^{**} \wedge y \wedge m = y \wedge m$.
Now, $y^{**} = (y \wedge m)^{**} = [y^{**} \wedge (y^{**} \to y)]^{**} = y^{**} \wedge (y^{**} \to y)^{**}$
$\Rightarrow y^{**} \leqslant (y^{**} \to y)^{**} \Rightarrow (y^{**} \to y)^* \leqslant y^*$. Also, by Lemma 3.4 (27),
$y^* \leqslant (y^{**} \to y) \wedge m \Rightarrow (y^{**} \to y)^* \leqslant y^{**}$
Therefore $(y^{**} \to y)^* = 0$ and hence $y^{**} \to y$ is a dense element.
Thus $x \wedge m$ is a dense element of L and hence x is a dense element of L. □

If $(L, \vee, \wedge, \to, 0, m)$ and $(L', \vee, \wedge, \to, 0', m')$ are two SHADLs. Then a mapping $\alpha : L \to L'$ is said to be a homomorphism of L into L' if for any $x, y \in L$ the following hold.

(1) $\alpha(x \wedge y) = \alpha(x) \wedge \alpha(y)$
(2) $\alpha(x \vee y) = \alpha(x) \vee \alpha(y)$
(3) $\alpha(x \to y) = \alpha(x) \to \alpha(y)$
(4) $\alpha(0) = 0'$

Further if $\alpha : L \to L'$ is a homomorphism, then $\{x \in L/\alpha(x) = m'\}$ is called the kernel of α and is denoted by $ker\alpha$.

Finally, we conclude this paper with the following.

THEOREM 3.10. *Let $(L, \vee, \wedge, \to, 0, m)$ be an SHADL and $\alpha : L \to L^*$ be defined by $\alpha(x) = x^{**}$ for all $x \in L$ and suppose $x, y \in L$. Then*

(1) *α is isotone.*
(2) $\alpha(x \wedge y) = \alpha(x) \wedge \alpha(y)$
(3) $\alpha(x \vee y) = \alpha(x)\underline{\vee}\alpha(y)$
(4) $ker(\alpha) = D_L$

PROOF. Let $x, y \in L$

(1) Assume $x \leqslant y \Rightarrow x^{**} \leqslant y^{**} \Rightarrow \alpha(x) \leqslant \alpha(y)$
(2) $\alpha(x \wedge y) = (x \wedge y)^{**} = x^{**} \wedge y^{**} = \alpha(x) \wedge \alpha(y)$
(3) $\alpha(x \vee y) = (x \vee y)^{**} = (x^* \wedge y^*)^* = x^{**}\underline{\vee}\ y^{**} = \alpha(x)\underline{\vee}\alpha(y)$
(4) Let $x \in ker(\alpha) \Rightarrow \alpha(x) = m \Rightarrow x^{**} = m \Rightarrow x^* = 0 \Rightarrow x \in D_L$.
Conversely, assume that $x \in D_L \Rightarrow x^* = 0 \Rightarrow x^{**} = m$
$\Rightarrow \alpha(x) = m \Rightarrow x \in ker(\alpha)$.
Hence $ker(\alpha) = D_L$.

□

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Department of Mathematics, Andhra University, Visakhapatnam, A.P, INDIA
E-mail address: gcraomaths@yahoo.co.in

AITAM Enggenering College, Tekkali, Srikakulam, Andhra Pradesh, INDIA
E-mail address: vvratnamani@gmail.com

Department of Mathematics, Bahir Dar University, Bahir Dar, Ethiopia.