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FURTHER RESULTS ON NON-LINEAR DIFFERENTIAL POLYNOMIALS SHARING A SMALL FUNCTION WITH FINITE WEIGHT

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Abstract. We investigate the uniqueness of meromorphic functions when certain types of non-linear differential polynomial as introduced in [19] sharing a small function with finite weight. The results of the paper improve, extend, unify and generalize a number of recent results.

1. Introduction, Definitions and Results

In this paper by meromorphic functions we will always mean meromorphic functions in the complex plane.

Let f and g be two non-constant meromorphic functions and let a be a finite complex number. We say that f and g share a CM, provided that $f-a$ and $g-a$ have the same zeros with the same multiplicities. Similarly, we say that f and g share a IM, provided that $f-a$ and $g-a$ have the same zeros ignoring multiplicities. In addition we say that f and g share ∞ CM, if $1/f$ and $1/g$ share 0 CM, and we say that f and g share ∞ IM, if $1/f$ and $1/g$ share 0 IM.

Let m be a positive integer or infinity and $a \in \mathbb{C} \cup \{\infty\}$. We denote by $E_{m)}(a; f)$ the set of all a-points of f with multiplicities not exceeding m, where an a-point is counted according to its multiplicity. Also we denote by $\overline{E}_{m)}(a; f)$ the set of distinct a-points of $f(z)$ with multiplicities not greater than m. If α is a small function we define that $E_{m)}(\alpha, f) = E_{m)}(\alpha, g)$ ($\overline{E}_{m)}(\alpha, f) = \overline{E}_{m)}(\alpha, g)$) which means $E_{m)}(0, f-\alpha) = E_{m)}(0, g-\alpha)$ ($\overline{E}_{m)}(0, f-\alpha) = \overline{E}_{m)}(0, g-\alpha)$).

We adopt the standard notations of value distribution theory (see [6]). We denote by $T(r)$ the maximum of $T(r, f)$ and $T(r, g)$. The notation $S(r)$ denotes any quantity satisfying $S(r) = o(T(r))$ as $r \longrightarrow \infty$, outside of a possible exceptional

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set of finite linear measure.
Throughout this paper, we need the following definition.

$$\Theta(a,f)=1-\limsup_{r\longrightarrow\infty}\frac{\overline{N}(r,a;f)}{T(r,f)},$$

where a is a value in the extended complex plane.

In 1997, Yang and Hua [20] made some important breakthrough by obtaining some relationship between two meromorphic functions related to value sharing of some specific type of non-linear differential polynomials namely differential monomials. Below we are stating their result.

Theorem A. [20] *Let f and g be two non-constant meromorphic functions, $n \geq 11$ be a positive integer and $a \in \mathbb{C}-\{0\}$. If $f^n f'$ and $g^n g'$ share a CM, then either $f(z)=c_1e^{cz}$, $g(z)=c_2e^{-cz}$, where c_1, c_2 and c are three constants satisfying $(c_1c_2)^{n+1}c^2=-1$ or $f\equiv tg$ for a constant t such that $t^{n+1}=1$.*

Further progresses in the direction of the above theorem are remarkable. Specially the introduction of the new notion of scaling between CM and IM, known as weighted sharing of values by I. Lahiri {[7]-[8]} in 2001 influences the investigations to a large extent. The veracity of the statement can easily be verified if one goes through the references {see [1]-[4], [10]-[14], [16]}.

Below we are giving the definition of weighted sharing.

Definition 1. [7, 8] *Let k be a nonnegative integer or infinity. For $a\in\mathbb{C}\cup\{\infty\}$ we denote by $E_k(a;f)$ the set of all a-points of f, where an a-point of multiplicity m is counted m times if $m\leq k$ and $k+1$ times if $m>k$. If $E_k(a;f)=E_k(a;g)$, we say that f, g share the value a with weight k.*

The definition implies that if f, g share a value a with weight k then z_0 is an a-point of f with multiplicity m $(\leq k)$ if and only if it is an a-point of g with multiplicity m $(\leq k)$ and z_0 is an a-point of f with multiplicity m $(>k)$ if and only if it is an a-point of g with multiplicity n $(>k)$, where m is not necessarily equal to n.

We write f, g share (a,k) to mean that f, g share the value a with weight k. Clearly if f, g share (a,k), then f, g share (a,p) for any integer p, $0\leq p<k$. Also we note that f, g share a value a IM or CM if and only if f, g share $(a,0)$ or (a,∞) respectively. If α is a small function we define that f and g share (α,l) which means f and g share α with weight l if $f-\alpha$ and $g-\alpha$ share $(0,l)$.

In 2004, Lin and Yi [15] further improved the result of Fang and Hong [5] in the following manner.

Theorem B. [15] *Let f and g be two non-constant meromorphic functions satisfying $\Theta(\infty,f)>\frac{2}{(n+1)}$, $n(\geq 12)$ an integer. If $f^n(f-1)f^{'}$ and $g^n(g-1)g^{'}$ share $(1,\infty)$, then $f\equiv g$.*

Theorem C. [15] *Let f and g be two non-constant meromorphic functions and $n(\geq 13)$ be an integer. If $f^n(f-1)^2f^{'}$ and $g^n(g-1)^2g^{'}$ share $(1,\infty)$, then $f\equiv g$.*

In 2005, Xiong, Lin and Mori [19] considered the following slightly different type of non-linear differential polynomial than that was considered earlier. Suppose h is a non-constant meromorphic function and $\psi_1(h)=h^{n+1}(g^m+a)+\alpha$, where a is a constant and $\alpha\not\equiv 0,\infty$ is a small function. Xiong, Lin and Mori [19] proved the following theorem.

Theorem D. [19] *Let f and g be two transcendental meromorphic functions. Let m, n, k are positive integers such that $(k-1)n > 14+3m+k(10+m)$ and $E_{k)}(0,\psi_1'(f)) = E_{k)}(0,\psi_1'(g))$, then*
(i) if $m \geq 2$, then $f(z) = g(z)$;
(ii) if $m = 1$, then either $f(z) \equiv= g(z)$, or f and g satisfy the algebraic equation $R(f,g) \equiv 0$, where $R(\omega_1,\omega_2) = (n+1)(\omega_1^{n+2} - \omega_2^{n+2}) - (n+2)(\omega_1^{n+1} - \omega_2^{n+1})$.

In 2007, Shen-Li [18] improved and supplemented *Theorem D*. In 2008, C. Meng [17], improved and supplemented *Theorem D* by the notion of weighted sharing. Meng obtained the following results.

Theorem E. [17] *Let f and g be two transcendental meromorphic functions. Let $\psi_1'(f)$ and $\psi_1'(g)$ share $(0,l)$. If*

(i) $l = 2$ *and* $n > m+10$*; or if*
(ii) $l = 1$ *and* $n > \frac{3m}{2} + 12$*; or if*
(iii) $l = 0$ *and* $n > 4m+22$,

then the conclusion of Theorem D holds.

Throughout the paper we define two non-zero polynomials $P_1(z)$ and $P(z)$ as follows:

$$P_1(z) = \frac{a_m}{n+m+1}z^m + \frac{a_{m-1}}{n+m}z^{m-1} + \ldots + \frac{a_0}{n+1}, \tag{1.1}$$

and

$$P(z) = a_m z^m + a_{m-1}z^{m-1} + \ldots + a_1 z + a_0, \tag{1.2}$$

where $m \geq 1$ is an integer and a_0, a_1,..., a_m are complex constants.

If $P(z)$ is non-constant and $a_m \neq 0$, $a_0 \neq 0$, let t be the number of distinct roots of the equation $P(z) = 0$. We define s by

$$s = \frac{4m}{t} - (m-1). \tag{1.3}$$

Clearly $t \leq m$.

Next we recall the following result of Zhang-Chen-Lin [22] as it has some relevance with the above discussion.

Theorem F. [22] *Let f and g be two non-constant meromorphic functions. Let n and m be two positive integers such that $n > \max\{m+10, 3m+3\}$ and $P(z)$ be such that $a_0(\neq 0)$, a_1, ... , $a_m(\neq 0)$ are complex constants. If $f^nP(f)f'$ and $g^nP(g)g'$ share $(1,\infty)$ then either $f(z) = tg(z)$ for a constant t such that $t^d = 1$, where $d = gcd(n+m+1, ..., n+m+1-i, ..., n+1)$, $a_{m-i} \neq 0$ for some $i \in \{0,1,2,...,m\}$ or f and g satisfy the algebraic equation $R(f,g) \equiv 0$, where $R(\omega_1,\omega_2) = \omega_1^{n+1}(\frac{a_m\omega_1^m}{n+m+1} + \frac{a_{m-1}\omega_1^{m-1}}{n+m} + \ldots + \frac{a_0}{n+1}) - \omega_2^{n+1}(\frac{a_m\omega_2^m}{n+m+1} + \frac{a_{m-1}\omega_2^{m-1}}{n+m} + \ldots + \frac{a_0}{n+1})$.*

In 2011, Zhang-Xu [23] generalised *Theorem F* for small functions. In this paper we will obtain a single result which will unify, improve, extend and generalize all the results stated so far.

Let m^* be a non non-negative integer defined as follows:

$$m^* = \begin{cases} m, & \text{if } a_m \neq 0 \\ 0, & \text{if } a_0 \neq 0 \text{ and } a_i = 0, 1 \leq i \leq m \end{cases}.$$

For a non-constant meromorphic function h we define $\psi(h)$ as

$$\psi(h)=[h^{n+1}\{\frac{a_m}{n+m+1}h^m+\frac{a_{m-1}}{n+m}h^{m-1}+\ldots+\frac{a_0}{n+1}\}]+\alpha$$

Theorem 1. *Let f and g be two non-constant meromorphic functions, and $\alpha(z)(\not\equiv 0,\infty)$ be a small function with respect to f and g. Also we suppose that $\psi^{'}(f)$ and $\psi^{'}(g)$ share $(0,2)$, where $n>\max\{m^*+10,s\}$, is an integer. Then the following conclusions hold.*

- (I) *When $a_m\neq 0$, $a_0\neq 0$ and at least one of $a_{m-i}, i=1,2,\ldots,m-1\neq 0$ then one of the following two conditions holds:*
  - (I1) *$f(z)\equiv tg(z)$ for a constant t such that $t^d=1$, where $d=gcd(n+m+1,n+m,\ldots,n+m+1-i,\ldots,n+1)$, $a_{m-i}\neq 0$ for some $i=0,1,2,\ldots,m$;*
  - (I2) *f and g satisfy the algebraic equation $R(f,g)\equiv 0$, where $R(\omega_1,\omega_2)=\omega_1^{n+1}(a_m\omega_1^m+a_{m-1}\omega_1^{m-1}+\ldots+a_0)-\omega_2^{n+1}(a_m\omega_2^m+a_{m-1}\omega_2^{m-1}+\ldots+a_0)$*
- (II) *When $a_m\neq 0$, $a_0\neq 0$ and all of a_{m-i}'s, $i=1,2,\ldots,m-1$ are zero then*
  - (II1) *if $m=1$, $\Theta(\infty,f)+\Theta(\infty,g)>4/(n+1)$; or*
  - (II2) *if $m\geq 2$*
    we have for some constant t, satisfying $t^d\equiv 1$, $f\equiv tg$, where $d=gcd(m,n+1)$.
- (III) *When $\mid a_m\mid+\mid a_0\mid\neq 0$, but $\mid a_m\mid.\mid a_0\mid=0$ and all of a_{m-i}'s, $i=1,2,\ldots,m-1$ are zero then one of the following two conditions holds:*
  - (III1) *$f(z)\equiv tg(z)$ where t is a constant satisfying $t^{n+m^*+1}=1$.*
  - (III2) *$a_{m^*}^2[f^{n+m^*+1}]'[g^{n+m^*+1}]'\equiv\alpha^2$. In particular when $\alpha(z)=d=$ constant, we get $f(z)=c_1e^{cz}$, $g(z)=c_2e^{-cz}$, where c_1, c_2 and c are constants satisfying $a_{m^*}^2(c_1c_2)^{n+m^*+1}((n+m^*+1)c)^2=-d^2$.*

Theorem 2. *Let f and g be two non-constant meromorphic functions, and $\alpha(z)(\not\equiv 0,\infty)$ be a small function with respect to f and g. Also we suppose that $\psi^{'}(f)$ and $\psi^{'}(g)$ share $(0,l)$, where $n>\max\{m^*+10,s\}$, is an integer. Then the following conclusions hold. If*

- (a) *$l=1$ and $n>\max\{\frac{3m^*}{2}+12,s\}$;*
- (b) *$l=0$ and $n>\max\{4m^*+22,s\}$,*

then the conclusions of Theorem 1 *holds.*

We now explain following definitions and notations which are used in the paper.

Definition 2. [12] *Let p be a positive integer and $a\in\mathbb{C}\cup\{\infty\}$.*

- (i) *$N(r,a;f\mid\geq p)$ $(\overline{N}(r,a;f\mid\geq p))$denotes the counting function (reduced counting function) of those a-points of f whose multiplicities are not less than p.*
- (ii) *$N(r,a;f\mid\leq p)$ $(\overline{N}(r,a;f\mid\leq p))$denotes the counting function (reduced counting function) of those a-points of f whose multiplicities are not greater than p.*

Definition 3. {*11, cf.*[21]} *For $a \in \mathbb{C} \cup \{\infty\}$ and a positive integer p we denote by $N_p(r,a;f)$ the sum $\overline{N}(r,a;f) + \overline{N}(r,a;f \mid\geq 2) + \ldots \overline{N}(r,a;f \mid\geq p)$. Clearly $N_1(r,a;f) = \overline{N}(r,a;f)$.*

2. Lemmas

Let F and G be two non-constant meromorphic functions defined in $\mathbb{C}$. We denote by H the function as follows:

$$H = \left(\frac{F''}{F'} - \frac{2F'}{F-1}\right) - \left(\frac{G''}{G'} - \frac{2G'}{G-1}\right). \tag{2.1}$$

Lemma 1. [12] *Let f be a non-constant meromorphic function and let $a_n(z)(\not\equiv 0)$, $a_{n-1}(z)$, ... , $a_0(z)$ be meromorphic functions such that $T(r, a_i(z)) = S(r,f)$ for $i = 0,1,2,...,n$. Then*

$$T(r, a_n f^n + a_{n-1} f^{n-1} + ... + a_1 f + a_0) = nT(r,f) + S(r,f).$$

Lemma 2. [24] *Let f be a non-constant meromorphic function, and p, k be positive integers. Then*

$$N_p\left(r,0;f^{(k)}\right) \leq T\left(r,f^{(k)}\right) - T(r,f) + N_{p+k}(r,0;f) + S(r,f), \tag{2.2}$$

$$N_p\left(r,0;f^{(k)}\right) \leq k\overline{N}(r,\infty;f) + N_{p+k}(r,0;f) + S(r,f). \tag{2.3}$$

Lemma 3. [9] *If $N(r,0;f^{(k)} \mid f \neq 0)$ denotes the counting function of those zeros of $f^{(k)}$ which are not the zeros of f, where a zero of $f^{(k)}$ is counted according to its multiplicity then*

$$N(r,0;f^{(k)} \mid f \neq 0) \leq k\overline{N}(r,\infty;f) + N(r,0;f \mid< k) + k\overline{N}(r,0;f \mid\geq k) + S(r,f).$$

Lemma 4. [20] *Let f and g be two non-constant meromorphic functions, $n \geq 6$ be an integer. If $f^n f' g^n g' = 1$ then $f(z) = c_1 e^{cz}$, $g(z) = c_2 e^{-cz}$ where c, c_1 and c_2 are constants satisfying $(c_1c_2)^{n+1}c^2 = -1$.*

Lemma 5. *Let f, g be two non-constant meromorphic functions and n be a positive integer such that $n > 6$. If $a_{m^*}^2 (f^{n+m^*+1})'(g^{n+m^*+1})' \equiv d^2$, then $f = c_1 e^{cz}$, $g = c_2 e^{-cz}$, where c_1, c_2 and c are constants such that $a_{m^*}^2 (c_1c_2)^{n+m^*+1}((n+m^*+1)c)^2 = -d^2$.*

Proof. From the given condition we can write

$$f^{n+m^*} f' g^{n+m^*} g' \equiv \left(\frac{d}{a_{m^*}(n+m^*+1)^2}\right)^2 = k^2, \tag{2.4}$$

where $k = d/(a_{m^*}(n+m^*+1)^2)$. We put $f_1 = \frac{f}{k^{\frac{1}{n+m^*+1}}}$, $g_1 = \frac{g}{k^{\frac{1}{n+m^*+1}}}$. Then (2.4) reduces to

$$f_1^{n+m^*} f_1' g_1^{n+m^*} g_1' = 1.$$

Using *Lemma 4* we have $f = c_1 e^{cz}$, $g = c_2 e^{-cz}$, where c_1, c_2 and c are constants such that $a_{m^*}^2 (c_1c_2)^{n+m^*+1}((n+m^*+1)c)^2 = -d^2$. □

Lemma 6. [8] *If F, G share $(1,2)$ then one of the following cases holds.*

$$(i)\ max\{\,T(r,F),T(r,G)\} \leq N_2(r,0;F)+N_2(r,0;G)+N_2(r,\infty;F)+N_2(r,\infty;G) +S(r,F)+S(r,G)$$

(ii) $F\equiv G$
(iii) $FG\equiv 1$.

Lemma 7. [1] *Let F, G be two non-constant meromorphic functions such that they share $(1,1)$ and $H\not\equiv 0$. Then*

$$\begin{aligned} T(r,F) &\leq N_2(r,0;F)+N_2(r,\infty;F)+N_2(r,0;G)+N_2(r,\infty;G)\\ &\quad +\frac{1}{2}\overline{N}(r,0;F)+\frac{1}{2}\overline{N}(r,\infty;F)+S(r,F)+S(r,G). \end{aligned}$$

Lemma 8. [1] *Let F, G be two non-constant meromorphic functions such that they share $(1,0)$ and $H\not\equiv 0$. Then*

$$\begin{aligned} T(r,F) &\leq N_2(r,0;F)+N_2(r,\infty;F)+N_2(r,0;G)+N_2(r,\infty;G)+2\overline{N}(r,0;F)\\ &\quad +2\overline{N}(r,\infty;F)+\overline{N}(r,0;G)+\overline{N}(r,\infty;G)+S(r,F)+S(r,G). \end{aligned}$$

Lemma 9. *Let f, g be two non-constant meromorphic functions and $F=\frac{[f^{n+1}P_1(f)]'}{-\alpha}$, $G=\frac{[g^{n+1}P_1(g)]'}{-\alpha}$, where $\alpha(z)(\not\equiv 0,\infty)$ be a small function with respect to f and g, n is a positive integer such that $n>m^*+5$. If $H\equiv 0$ then either $[f^{n+1}P_1(f)]'\equiv[g^{n+1}P_1(g)]'$ or $[f^{n+1}P_1(f)]'[g^{n+1}P_1(g)]'\equiv\alpha^2$.*

Proof. Since $H\equiv 0$, on integration we get

$$\frac{1}{F-1}\equiv\frac{bG+a-b}{G-1}, \tag{2.5}$$

where a, b are constants and $a\neq 0$. We now consider the following cases.
Case 1. Let $b\neq 0$ and $a\neq b$.
If $b=-1$, then from (2.5) we have

$$F\equiv\frac{-a}{G-a-1}.$$

Therefore

$$\overline{N}(r,a+1;G)=\overline{N}(r,\infty;F)=\overline{N}(r,\infty;f).$$

So in view of *Lemma 2* and the second fundamental theorem we get

$$\begin{aligned} & (n+m^*+1)\,T(r,g)\\ \leq\ & T(r,G)+N_2(r,0;g^{n+1}P_1(g))-\overline{N}(r,0;G)\\ \leq\ & \overline{N}(r,\infty;G)+\overline{N}(r,0;G)+\overline{N}(r,a+1;G)+N_2(r,0;g^{n+1}P(g))\\ -\ & N(r,0;G)+S(r,g)\\ \leq\ & \overline{N}(r,\infty;g)+N_2(r,0;g^{n+1}P(g))+\overline{N}(r,\infty;f)+S(r,g)\\ \leq\ & T(r,f)+(m^*+3)\,T(r,g)+S(r,f)+S(r,g). \end{aligned}$$

Without loss of generality, we suppose that there exists a set I with infinite measure such that $T(r,f)\leq T(r,g)$ for $r\in I$.
So for $r\in I$ we have

$$(n-3)\,T(r,g)\leq S(r,g),$$

which is a contradiction.
If $b \neq -1$, from (2.5) we obtain that

$$F - (1 + \frac{1}{b}) \equiv \frac{-a}{b^2[G + \frac{a-b}{b}]}.$$

So

$$\overline{N}(r, \frac{(b-a)}{b}; G) = \overline{N}(r, \infty; F) = \overline{N}(r, \infty; f).$$

Using *Lemma 2* and by the same argument as used in the case when $b = -1$ we can get a contradiction.
Case 2. Let $b \neq 0$ and $a = b$.
If $b = -1$, then from (2.5) we have

$$FG \equiv \alpha^2,$$

that is

$$[f^{n+1}P_1(f)]^{'}[g^{n+1}P_1(g)]^{'} \equiv \alpha^2.$$

If $b \neq -1$, from (2.5) we have

$$\frac{1}{F} \equiv \frac{bG}{(1+b)G - 1}.$$

Therefore

$$\overline{N}(r, \frac{1}{1+b}; G) = \overline{N}(r, 0; F).$$

So in view of *Lemma 2* and the second fundamental theorem we get

$$\begin{aligned}
& (n + m^* + 1)\, T(r, g) \\
\leq\ & T(r, G) + N_2(r, 0; g^{n+1}P_1(g)) - \overline{N}(r, 0; G) + S(r, g) \\
\leq\ & \overline{N}(r, \infty; G) + \overline{N}(r, 0; G) + \overline{N}(r, \frac{1}{1+b}; G) + N_2(r, 0; g^{n+1}P_1(g)) \\
-\ & N(r, 0; G) + S(r, g) \\
\leq\ & (m^* + 3)T(r, g) + \overline{N}(r, 0; F) + S(r, g) \\
\leq\ & \overline{N}(r, \infty; f) + 2\overline{N}(r, 0; f) + (m^* + 3)T(r, g) \\
\leq\ & (m^* + 3)\{T(r, g) + T(r, f)\} + S(r, f) + S(r, g).
\end{aligned}$$

So for $r \in I$ we have

$$\{n - m^* - 5\}\, T(r, g) \leq S(r, g),$$

which is a contradiction since $n > m^* + 5$.
Case 3. Let $b = 0$. From (2.5) we obtain

$$F \equiv \frac{G + a - 1}{a}. \tag{2.6}$$

If $a \neq 1$ then from (2.6) we obtain

$$\overline{N}(r, 1 - a; G) = \overline{N}(r, 0; F).$$

We can similarly deduce a contradiction as in *Case 2*. Therefore $a = 1$ and from (2.6) we obtain

$$F \equiv G.$$

i.e.,

$$[f^{n+1}P_1(f)]^{'} \equiv [g^{n+1}P_1(g)]^{'}.$$

□

Lemma 10. *Let f and g be two non-constant meromorphic functions and $\alpha(z)(\not\equiv 0,\infty)$ be a small function of f and g. Let n be a positive integer such that $n>s$, where s be defined by (1.3). Suppose that $P(z)\neq a_iz^i$, for $i=1,2,\ldots,m$ be a non-constant polynomial. Then*

$$f^nP(f)f^{'}g^nP(g)g^{'}\not\equiv\alpha^2,$$

Proof. First suppose that

$$f^nP(f)f^{'}g^nP(g)g^{'}\equiv\alpha^2(z). \tag{2.7}$$

Let d_i be the distinct zeros of $P(z)=0$ with multiplicity p_i, where $i=1,2,\ldots,t$, $1\le t\le m$ and $\sum\limits_{i=1}^{t}p_i=m$.
Now by the second fundamental theorem for f and g we get respectively

$$t_2T(r,f)\le\overline{N}(r,0;f)+\overline{N}(r,\infty;f)+\sum_{i=1}^{t}\overline{N}(r,d_i;f)-\overline{N}_0(r,0;f^{'})+S(r,f), \tag{2.8}$$

and

$$tT(r,g)\le\overline{N}(r,0;g)+\overline{N}(r,\infty;g)+\sum_{i=1}^{t}\overline{N}(r,d_i;g)-\overline{N}(r,0;g^{'})+S(r,g), \tag{2.9}$$

where $\overline{N}(r,0;f^{'})$ denotes the reduced counting function of those zeros of $f^{'}$ which are not the zeros f and $f-d_i$, $i=1,2,\ldots,t$ and $\overline{N}(r,0;g^{'})$ can be similarly defined. Let z_0 be a zero of f with multiplicity p but $\alpha(z_0)\neq 0,\infty$. Clearly z_0 must be a pole of g with multiplicity q. Then from (2.7) we get $np+p-1=nq+mq+q+1$. This gives

$$mq+2=(n+1)(p-q). \tag{2.10}$$

From (2.10) we get $p-q\ge 1$ and so $q\ge\frac{n-1}{m}$. Now $np+p-1=nq+mq+q+1$ gives
$p\ge\frac{n+m-1}{m}$. Thus we have

$$\overline{N}(r,0;f)\le\frac{m}{n+m-1}\,N(r,0;f)\le\frac{m}{n+m-1}\,T(r,f). \tag{2.11}$$

Let $z_1(\alpha(z_1)\neq 0,\infty)$ be a zero of $f-d_i$ with multiplicity q_i, $i=1,2,\ldots,t$. Then z_1 must be a pole of g with multiplicity $r(\ge 1)$. So from (2.7) we get $q_ip_i+q_i-1=(n+m+1)r+1\ge n+m+2$. This gives $q_i\ge\frac{n+m+2}{p_i+1}$ for $i=1,2,\ldots,t$ and so we get

$$\overline{N}(r,d_i;f)\le\frac{p_i+1}{n+m+3}\,N(r,d_i;f)\le\frac{p_i+1}{n+m+3}\,T(r,f).$$

Clearly

$$\sum_{i=1}^{t}\overline{N}(r,d_i;f)\le\frac{m+t}{n+m+3}\,T(r,f). \tag{2.12}$$

Similarly we have

$$\overline{N}(r,0;g)\le\frac{m}{n+m-1}\,T(r,g), \tag{2.13}$$

and

$$\sum_{i=1}^{t} \overline{N}(r,d_i;g) \leq \frac{m+t}{n+m+3}\, T(r,g). \tag{2.14}$$

Also it is clear from (2.13) and (2.14) that

$$\begin{aligned} & \overline{N}(r,\infty;f) \qquad (2.15)\\ \leq\ & \overline{N}(r,0;g) + \sum_{i=1}^{t}\overline{N}(r,d_i;g) + \overline{N}_0(r,0;g^{'}) + S(r,f) + S(r,g)\\ \leq\ & \left(\frac{m}{n+m-1} + \frac{m+t}{n+m+3}\right) T(r,g) + \overline{N}_0(r,0;g^{'}) + S(r,f) + S(r,g). \end{aligned}$$

Then by (2.8), (2.11), (2.12) and (2.15) we get

$$\begin{aligned} & t\,T(r,f) \qquad (2.16)\\ \leq\ & \left(\frac{m}{n+m-1} + \frac{m+t}{n+m+3}\right)\{T(r,f)+T(r,g)\} + \overline{N}_0(r,0;g^{'})\\ & -\overline{N}_0(r,0;f^{'}) + S(r,f) + S(r,g). \end{aligned}$$

Similarly we have

$$\begin{aligned} & t\,T(r,g) \qquad (2.17)\\ \leq\ & \left(\frac{m}{n+m-1} + \frac{m+t}{n+m+3}\right)\{T(r,f)+T(r,g)\} + \overline{N}_0(r,0;f^{'})\\ & -\overline{N}_0(r,0;g^{'}) + S(r,f) + S(r,g). \end{aligned}$$

So from (2.16) and (2.17) we get

$$\begin{aligned} & t\{T(r,f)+T(r,g)\}\\ \leq 2\ & \left(\frac{m}{n+m-1} + \frac{m+t}{n+m+3}\right)\ \{T(r,f)+T(r,g)\} + S(r,f) + S(r,g), \end{aligned}$$

i.e.,

$$\left(t - \frac{2m}{n+m-1} - \frac{2(m+t)}{n+m+3}\right)\{T(r,f)+T(r,g)\} \leq S(r,f) + S(r,g). \tag{2.18}$$

Since

$$\begin{aligned} & \left(t - \frac{2m}{n+m-1} - \frac{2(m+t)}{n+m+3}\right)\\ =\ & \frac{(n+m-1)^2 t + 2(n+m-1)(t-2m) - 8m}{(n+m-1)(n+m+3)}, \end{aligned}$$

we note that when $n+m-1 > \frac{4m}{t}$, i.e., when $n > \frac{4m}{t} - (m-1) = s$, then clearly $t - \frac{2m}{n+m-1} - \frac{2(m+t)}{n+m+3} > 0$ and so (2.18) leads to a contradiction. This completes the proof. □

Lemma 11. *Let f and g be two non-constant meromorphic (entire) functions and $n(\geq 2)$, $m(\geq 1)$ be two distinct integers satisfying $n+m \geq d+6$ ($n+m \geq d+2$). Then for two constants λ, μ, with $|\lambda|+|\mu| \neq 0$,*

$$f^{n+1}\,(\mu f^m + \lambda) \equiv g^{n+1}\,(\mu g^m + \lambda)$$

implies the following.
(i) if $\lambda\mu \neq 0$ *and*
(a) $m = 1$, $\Theta(\infty, f) + \Theta(\infty, g) > 4/n + 1$*;*
or (b) $m \geq 2$ *and for some constant* t*, satisfying* $t^d \equiv 1$*,*
we have $f \equiv tg$*, where* $d = (m, n + 1)$*.*
(ii) if $\lambda\mu = 0$*, then* $f = tg$*, where* t *is a constant satisfying* $t^{n+m^*+1} = 1$*.*

Proof. Let $m = 1$. In this case noting that $d = 1 = (n + 2, n + 1)$, proceeding in the same way as done in *Lemma 6* of [10] we can show when $\Theta(\infty, f) + \Theta(\infty, g) > 4/(n + 1)$, we have $f \equiv g$.

Next suppose $m \geq 2$. Let $f \not\equiv tg$ for a constant t satisfying $t^d = 1$. We put $h = \frac{f}{g}$. Then $h^d \not\equiv 1$, i.e., $(h - v_0)(h - v_1)\dots(h - v_{d-1}) \not\equiv 0$, where $v_k = exp\left(\frac{2k\pi i}{d}\right)$, $k = 0, 1, 2, \dots, d-1$. First suppose that h is constant. Now from the given condition we have

$$\mu g^m(h^{n+m+1} - 1) \equiv -\lambda(h^{n+1} - 1).$$

Since $gcd(n + 1, m) = d$, it follows that $gcd(n + m + 1, n + 1) = d$. Eliminating d common factors namely $h - v_k$, $k = 0, 1, \dots, d - 1$ from both sides we are left with

$$ag^m(h - \alpha_1)(h - \alpha_2)\dots(h - \alpha_{n+m+1-d}) \equiv (h - \beta_1)(h - \beta_2)\dots(h - \beta_{n+1-d}),$$

where α_i and β_j are those zeros of $h^{n+m+1} - 1$ and $h^{n+1} - 1$ which are not the zeros of $h^d - 1$, $i = 1, 2, \dots, n + m + 1 - d$ and $j = 1, 2, \dots, n + 1 - d$. Also we note that none of the α_i's coincides with β_j's. So if $h = \alpha_i$ or β_j, then we have either $(h - \beta_1)(h - \beta_2)\dots(h - \beta_{n-d}) \equiv 0$ or $g \equiv 0$ and in both case we get contradictions. So we assume neither $h^{n+m+1} \equiv 1$ nor $h^{n+1} \equiv 1$. Hence we may write

$$g^m = -\frac{\lambda}{\mu}\ \frac{h^{n+1} - 1}{h^{n+m+1} - 1}. \tag{2.19}$$

It follows from above that g is a constant, which is impossible. So h is non-constant. We observe that since a non-constant meromorphic function can not have more than two Picard exceptional values h can take at least $n + m - d - 1$ values among $u_j = exp\left(\frac{2j\pi i}{n+m+1}\right)$, where $j = 0, 1, 2, \dots, n+m$. Since f^m has no simple pole $h-u_j$ has no simple zero for at least $n + m - d - 1$ values of u_j, for $j = 0, 1, 2, \dots, n + m$ and for these values of j we have $\Theta(u_j; h) \geq \frac{1}{2}$, which leads to a contradiction.

Therefore $h^d \equiv 1$. i.e., $f \equiv tg$ for a constant t satisfying $t^d = 1$, where $d = gcd(n + 1, m)$. **Subcase 2.2:** Let $\lambda\mu = 0$ but $|\lambda| + |\mu| \neq 0$. Then from the given condition we get $f^{n+m^*+1} \equiv g^{n+m^*+1}$ and so $f \equiv tg$, where t is a constant satisfying $t^{n+m^*+1} = 1$. □

3. Proofs of the Theorems

Proof of Theorem 1. Since

$$\psi(f) = [f^{n+1}\{\frac{a_m}{n+m+1}f^m + \frac{a_{m-1}}{n+m}f^{m-1} + \dots + \frac{a_0}{n+1}\}] + \alpha = f^{n+1}P_1(f) + \alpha$$

and

$$\psi(g) = [g^{n+1}\{\frac{a_m}{n+m+1}g^m + \frac{a_{m-1}}{n+m}g^{m-1} + \dots + \frac{a_0}{n+1}\}] + \alpha = g^{n+1}P_1(g) + \alpha.$$

We have

$$\psi'(f) = f^n[a_m f^m + a_{m-1}f^{m-1} + \dots + a_0]f' + \alpha' = f^nP(f)f' + \alpha'$$

and

$$\psi^{'}(g)=g^{n}[a_{m}g^{m}+a_{m-1}g^{m-1}+\ldots+a_{0}]g^{'}+\alpha^{'}=g^{n}P(g)g^{'}+\alpha^{'}.$$

Let

$$F_{1}=f^{n+1}\left\{\frac{a_{m}}{n+m+1}f^{m}+\frac{a_{m-1}}{n+m}f^{m-1}+\ldots+\frac{a_{0}}{n+1}\right\}=f^{n+1}P_{1}(f),$$

$$G_{1}=g^{n+1}\left\{\frac{a_{m}}{n+m+1}g^{m}+\frac{a_{m-1}}{n+m}g^{m-1}+\ldots+\frac{a_{0}}{n+1}\right\}=g^{n+1}P_{1}(g),$$

$$F=\frac{f^{n}[a_{m}f^{m}+a_{m-1}f^{m-1}+\ldots+a_{0}]f^{'}}{-\alpha^{'}}=\frac{f^{n}P(f)f^{'}}{-\alpha^{'}}=\frac{(f^{n+1}P_{1}(f))^{'}}{-\alpha^{'}}$$

and

$$G=\frac{g^{n}[a_{m}g^{m}+a_{m-1}g^{m-1}+\ldots+a_{0}]g^{'}}{-\alpha^{'}}=\frac{g^{n}P(g)g^{'}}{-\alpha^{'}}=\frac{(g^{n+1}P_{1}(g))^{'}}{-\alpha^{'}}.$$

Then it follows that F and G share $(1,2)$, except the zeros and poles of $\alpha^{'}$. Also $F_{1}^{'}=-\alpha^{'}F$ and $G_{1}^{'}=-\alpha^{'}G$. First suppose that case (i) of *Lemma 6* holds. Then from *Lemmas 1* and *2* we get

$$\begin{aligned}
&(n+m^{*}+1)T(r,f) \qquad (3.1)\\
\leq\;& T(r,F_{1}^{'})+N_{3}(r,0;F_{1})-N_{2}(r,0;F_{1}^{'})+S(r,f)\\
\leq\;& T(r,F)+N_{3}(r,0;F_{1})-N_{2}(r,0;F)+S(r,f)\\
\leq\;& 2\overline{N}(r,\infty,f)+2\overline{N}(r,\infty;g)+N_{2}(r,0;G)+N_{3}(r,0;F_{1})+S(r,f)+S(r,g)\\
\leq\;& 2\overline{N}(r,\infty;f)+3\overline{N}(r,\infty;g)+N_{3}(r,0;f^{n+1}P_{1}(f))+N_{3}(r,0;g^{n+1}P_{1}(g))\\
&+S(r,f)+S(r,g)\\
\leq\;& 2\overline{N}(r,\infty;f)+3\overline{N}(r,\infty;g)+3\{\overline{N}(r,0;f)+\overline{N}(r,0;g)\}+m^{*}\{T(r,f)+T(r,g)\}\\
&+S(r,f)+S(r,g)\\
\leq\;& (2m^{*}+11)T(r)+S(r).
\end{aligned}$$

In a similar way we can obtain

$$(n+m^{*}+1)\;T(r,g)\leq(2m^{*}+11)\;T(r)+S(r). \qquad (3.2)$$

Combining (3.1) and (3.2) we see that

$$(n-m^{*}-10)\;T(r)\leq S(r). \qquad (3.3)$$

Clearly (3.3) leads to a contradiction.

Again from *Lemma 10* we see that when $P(z)$ is a non-constant polynomial with $a_{m}\neq 0$ and $a_{0}\neq 0$, case (iii) of *Lemma 6* does not hold . If $\mid a_{m}\mid+\mid a_{0}\mid\neq 0$ but $\mid a_{m}\mid.\mid a_{0}\mid=0$ and all a_{m-i}, $i=1,2,\ldots,m-1$ are zero from case (iii) of *Lemma 6* we get

$$a_{m^{*}}^{2}(f^{n+m^{*}+1})^{'}(g^{n+m^{*}+1})^{'}\equiv\alpha^{2}.$$

In particular, if $\alpha=d=$ *constant*, the conclusion of the theorem follows form *Lemma 5.*

So we must have $F\equiv G$. i.e., $(f^{n+1}P_{1}(f))^{'}\equiv(g^{n+1}P_{1}(g))^{'}$. Integrating, we obtain

$$f^{n+1}P_{1}(f)\equiv g^{n+1}P_{1}(g)+c.$$

If possible suppose $c \neq 0$.
Now using the second fundamental theorem we get

$$\begin{aligned} & (n+m^*+1)T(r,f) \\ \leq\ & \overline{N}(r,0;f^{n+1}P_1(f))+\overline{N}(r,\infty;f^{n+1}P_1(f))+\overline{N}(r,c_0;f^{n+1}P_1(f)) \\ \leq\ & \overline{N}(r,0;f)+m^*T(r,f)+\overline{N}(r,\infty;f)+\overline{N}(r,0;g^{n+1}P_1(g)) \\ \leq\ & (m^*+2)\,T(r,f)+\overline{N}(r,0;g)+m^*T(r,g)+S(r,f) \\ \leq\ & (m^*+2)\,T(r,f)+(m^*+1)\,T(r,g)+S(r,f)+S(r,g) \\ \leq\ & \{2m^*+3\}\,T(r)+S(r). \end{aligned}$$

Similarly we get

$$(n+m^*+1)\,T(r,g) \leq \{2m^*+3\}\,T(r)+S(r).$$

Combining these we get

$$(n-m^*-2)\,T(r) \leq S(r),$$

which is a contradiction since $n > m^*+2$.
Therefore $c = 0$ and so

$$f^{n+1}P_1(f) \equiv g^{n+1}P_1(g).$$

i.e.,

$$\frac{a_m}{n+m+1}f^{n+m+1}+\frac{a_{m-1}}{n+m}f^{n+m}+\ldots+\frac{a_0}{n+1}f^{n+1} \equiv \frac{a_m}{n+m+1}g^{n+m+1}+\frac{a_{m-1}}{n+m}g^{n+m}+\ldots+\frac{a_0}{n+1}g^{n+1}.$$

If $a_{m-i} = 0$, for $i = 1,2,\ldots m-1$, then since $P(z)$ is a non-zero polynomial, it follows that $\mid a_m \mid + \mid a_0 \mid \neq 0$. If $\mid a_m \mid . \mid a_0 \mid \neq 0$, the conclusion of the theorem follows from *Lemma 11 (i)* otherwise it follows from *Lemma 11 (ii)*. Let at least one of $a_{m-i} \neq 0$, for $i = 1,2,\ldots m-1$. Suppose $h = \frac{f}{g}$. If h is a constant, by putting $f = hg$ in the above expression we get

$$\frac{a_m}{n+m+1}g^m(h^{n+m+1}-1)+\frac{a_{m-1}}{n+m}g^{m-1}(h^{n+m}-1)$$

$+\ldots+$

$$\frac{a_1}{n+2}g(h^{n+2}-1)+\frac{a_0}{n+1}(h^{n+1}-1) \equiv 0,$$

which implies that $h^d = 1$, where $d = gcd(n+m+1,\ldots,n+m+1-i,\ldots,n+1)$, $a_{m-i} \neq 0$ for some $i \in \{0,1,\ldots,m\}$. Thus $f \equiv tg$ for a constant t such that $t^d = 1$,where $d = gcd(n+m+1,\ldots,n+m+1-i,\ldots,n+1)$, $a_{m-i} \neq 0$ for some $i \in \{0,1,\ldots,m\}$.
If h is not constant then f and g satisfy the algebraic equation $R(f,g) \equiv 0$, where $R(\omega_1,\omega_2) = \omega_1^{n+1}(\frac{a_m\omega_1^m}{n+m+1}+\frac{a_{m-1}\omega_1^{m-1}}{n+m}+\ldots+\frac{a_0}{n+1}) - \omega_2^{n+1}(\frac{a_m\omega_2^m}{n+m+1}+\frac{a_{m-1}\omega_2^{m-1}}{n+m}+\ldots+\frac{a_0}{n+1})$. □

Proof of Theorem 2. In this case by the same argument as used in *Theorem 1* we have F and G share $(1,1)$, except the zeros and poles of α'.
First suppose $H \not\equiv 0$. While $l = 1$, from *Lemmas 7*, *1* and 2 we get from second

fundamental theorem that either $FG \equiv \alpha^2$ or $F \equiv G$. The rest of the proof follows from *Theorem 1*. So we omit the details.

$$\begin{aligned}
& (n+m^*+1)T(r,f) && (3.4)\\
\leq\ & \frac{5}{2}\overline{N}(r,\infty,f)+2\overline{N}(r,\infty;g)+N_2(r,0;G)+N_3(r,0;F_1)+\frac{1}{2}\overline{N}(r,0;F)\\
& +S(r,f)+S(r,g)\\
\leq\ & 3\overline{N}(r,\infty;f)+3\overline{N}(r,\infty;g)+N_3(r,0;f^{n+1}P_1(f))+\frac{1}{2}N_2(r,0;f^{n+1}P_1(f))\\
& +N_3(r,0;g^{n+1}P_1(g))+S(r,f)+S(r,g)\\
\leq\ & 3\overline{N}(r,\infty;f)+3\overline{N}(r,\infty;g)+4\overline{N}(r,0;f)+3\overline{N}(r,0;g)+\frac{3m^*}{2}T(r,f)+m^*T(r,g)\\
& +S(r,f)+S(r,g)\\
\leq\ & (\frac{5m^*}{2}+13)T(r)+S(r).
\end{aligned}$$

In a similar way we can obtain

$$(n+m^*+1)T(r,g) \quad \leq \quad (\frac{5m^*}{2}+13)T(r)+S(r). \tag{3.5}$$

Combining (3.4) and (3.5) we see that

$$(n+m^*+1)\ T(r) \leq \left(\frac{5m^*}{2}+13\right)\ T(r)+S(r),$$

i.e

$$\left(n-\frac{3m^*}{2}-12\right)\ T(r) \leq S(r). \tag{3.6}$$

Since $n > \frac{3m^*}{2}+12$, (3.6) leads to a contradiction.
While $l=0$, from *Lemmas 8*, *1* and 2 we get from second fundamental theorem that

$$\begin{aligned}
& (n+m^*+1)T(r,f) && (3.7)\\
\leq\ & 4\overline{N}(r,\infty,f)+3\overline{N}(r,\infty;g)+2\overline{N}(r,0;F)+N_3(r,0;f^{n+1}P_1(f))\\
& +N_2(r,0;G)+\overline{N}(r,0;G)+S(r,f)+S(r,g)\\
\leq\ & 6\overline{N}(r,\infty;f)+5\overline{N}(r,\infty;g)+7\overline{N}(r,0;f)+5\overline{N}(r,0;g)+3m^*T(r,f)\\
& +2m^*T(r,g)\}+S(r,f)+S(r,g)\\
\leq\ & (5m^*+23)\ T(r)+S(r).
\end{aligned}$$

In a similar way we can obtain

$$(n+m^*+1)\ T(r,g) \leq (5m^*+23)\ T(r)+S(r). \tag{3.8}$$

Combining (3.7) and (3.8) we see that

$$(n+m^*+1)\ T(r) \leq (5m^*+23)\ T(r)+S(r)),$$

i.e.,

$$(n-4m^*-22)\ T(r) \leq S(r). \tag{3.9}$$

Since $n > 4m^*+22$, (3.9) leads to a contradiction. Next suppose $H \equiv 0$. Then by *Lemma 9* and following the same procedure as adopted in the proof of *Theorem 1* we can easily deduce the conclusions of the theorem. So we omit the details. □

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