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On decompositions of generalized continuity

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ABSTRACT. In this paper μ-locally closed sets (introduced by Ekici [**6**]), μ_t-sets, μ_B-sets have been studied. Using these concepts we have obtained the notions for decomposition of continuity and contra continuity in generalized topological spaces.

1. Introduction

In the past few years, different forms of open sets have been studied. Recently a significant contribution to the theory of generalized open sets was extended by A. Császár. Especially, the author have defined some basic operators on generalized topological spaces. On the other hand the notion of decompositions of continuity on topological spaces was first introduced by Tong [**9**]. Recently, decompositions of continuity on topological spaces with a GT on it was studied by Roy and Sen [**8**]. Owing to the fact that corresponding definitions have many features in common, it is quite natural to conjecture that they can be deduced from suitable more general definitions.

We recall some notions defined in [**2**]. Let X be a non-empty set, $expX$ denotes the power set of X. We call a class $\mu \subsetneqq expX$ a generalized topology [**2**], (briefly, GT) if $\varnothing \in \mu$ and union of elements of μ belongs to μ. A set X, with a GT μ on it is said to be a generalized topological space (briefly, GTS) and is denoted by (X, μ).

For a GTS (X, μ), the elements of μ are called μ-open sets and the complement of μ-open sets are called μ-closed sets. For $A \subsetneqq X$, we denote by $c_\mu(A)$ the intersection of all μ-closed sets containing A, i.e., the smallest μ-closed set containing A; and by $i_\mu(A)$ the union of all μ-open sets contained in A, i.e., the largest μ-open set contained in A (see [**2, 3**]).

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It is easy to observe that i_μ and c_μ are idempotent and monotonic, where $\gamma : expX \to expX$ is said to be idempotent iff $A \subseteq B \subseteq X$ implies $\gamma(\gamma(A)) = \gamma(A)$ and monotonic iff $\gamma(A) \subseteq \gamma(B)$. It is also well known from [**3, 4**] that if μ is a GT on X and $A \subseteq X$, $x \in X$, then $x \in c_\mu(A)$ iff $x \in M \in \mu$ implies $M \cap A \neq \varnothing$ and $c_\mu(X \setminus A) = X \setminus i_\mu(A)$.

2. μ-locally closed sets and μ_t-sets

DEFINITION 2.1. [**6**] A subset A of a GTS (X, μ) is said to be μ-locally closed if $A = U \cap F$ where U is μ-open and F is μ-closed in X.

REMARK 2.1. In a GTS (X, μ), every μ-open set is μ-locally closed and if $X \in \mu$ then every μ-closed set is μ-locally closed.

EXAMPLE 2.1. (a) Let $X = \{a, b, c\}$, $\mu = \{\varnothing, \{a\}, \{a, b\}\}$. Then (X, μ) is a GTS. It is easy to check that $\{b, c\}$ is a μ-closed set but not μ-locally closed.
(b) Let $X = \{a, b, c\}$, $\mu = \{\varnothing, \{b\}, \{a, b\}, \{a, c\}, X\}$. Then (X, μ) is a GTS. It can be checked that $\{b\}$ and $\{c\}$ are two μ-locally closed sets but their union is not μ-locally closed.
(c) Let $X = \{a, b, c\}$, $\mu = \{\varnothing, \{a, b\}, \{a, c\}, X\}$. Then (X, μ) is a GTS such that $\{a, b\}$ and $\{a, c\}$ are two μ-locally closed subsets of X but their intersection is not μ-locally closed.

THEOREM 2.1. *If A is a μ-locally closed set in a GTS (X, μ), then there exists a μ-closed set K in X such that $A \cap K = \varnothing$.*

PROOF. Let A be a μ-locally closed subset of X. Then $A = U \cap F$, where U is μ-open and F is μ-closed. Let $K = F \cap (X \setminus U)$. Then K is a μ-closed subset of X such that $A \cap K = \varnothing$. □

THEOREM 2.2. *For a subset A of a GTS (X, μ), the followings are equivalent:*
(i) A is μ-locally closed;
(ii) $A = U \cap c_\mu(A)$ for some μ-open set U;
(iii) $c_\mu(A) \setminus A$ is μ-closed;
(iv) $A \cup (X \setminus c_\mu(A))$ is μ-open;
(v) $A \subseteq i_\mu(A \cup (X \setminus c_\mu(A)))$.

PROOF. (i) $\Rightarrow$ (ii) : Let A be a μ-locally closed subset of X. Then $A = U \cap F$, where U is μ-open and F is μ-closed. Then $A \subseteq F$ implies that $c_\mu(A) \subseteq F$. So $A = U \cap F \supseteq U \cap c_\mu(A)$. Again, $A \subseteq U$ and $A \subseteq c_\mu(A)$ implies that $A \subseteq U \cap c_\mu(A)$. Thus $A = U \cap c_\mu(A)$.
(ii) $\Rightarrow$ (iii) : $c_\mu(A) \setminus A = c_\mu(A) \setminus [U \cap c_\mu(A)]$ for some μ-open set U. Then $c_\mu(A) \cap [X \setminus (U \cap c_\mu(A))] = c_\mu(A) \cap [(X \setminus U) \cup (X \setminus c_\mu(A))] = [c_\mu(A) \cap (X \setminus U)] \cup [c_\mu(A) \cap (X \setminus c_\mu(A))] = c_\mu(A) \cap (X \setminus U)$ which is a μ-closed set.
(iii) $\Rightarrow$ (iv) : Since $c_\mu(A) \setminus A$ is μ-closed, then $X \setminus (c_\mu(A) \setminus A)$ is μ-open and $X \setminus (c_\mu(A) \setminus A) = X \setminus (c_\mu(A) \cap (X \setminus A)) = A \cup (X \setminus c_\mu(A))$.
(iv) $\Rightarrow$ (v) : $A \subseteq [A \cup (X \setminus c_\mu(A))] = i_\mu[A \cup (X \setminus c_\mu(A))]$.
(v) $\Rightarrow$ (i) : $A \subseteq i_\mu[A \cup (X \setminus c_\mu(A))]$. Thus $A = i_\mu[A \cup (X \setminus c_\mu(A))] \cap c_\mu(A)$ where $i_\mu[A \cup (X \setminus c_\mu(A))]$ is μ-open and $c_\mu(A)$ is μ-closed. □

THEOREM 2.3. *Let (X, μ) be a GTS. If $A \subsetneqq B \subsetneqq X$ and B is μ-locally closed, then there exists a μ-locally closed set C such that $A \subsetneqq C \subsetneqq B$.*

PROOF. As B is μ-locally closed by Theorem 2.2, $B = U \cap c_\mu(B)$ where U is μ-open. Then $A \subsetneqq B \subsetneqq U$. So $A \subsetneqq U \cap c_\mu(A) = C$ (say). Then C is μ-locally closed and $A \subsetneqq C \subsetneqq B$. □

DEFINITION 2.2. Let (X, μ) be a GTS. Then a subset A of X is called μ-dense [**5**] if $c_\mu(A) = X$. The space (X, μ) is called μ-submaximal [**6**] if every μ-dense subset is μ-open in X.

THEOREM 2.4. [**6**] *A GTS (X, μ) is μ-submaximal if and only if every subset of X is μ-locally closed.*

DEFINITION 2.3. Let (X, μ) be a GTS. Then a subset A of X is called a μ-generalized closed set (in short, μg-closed set) [**7**] iff $c_\mu(A) \subseteq U$ whenever $A \subseteq U$ where U is μ-open in X. The complement of a μg-closed set is called a μg-open set.

THEOREM 2.5. *Let (X, μ) be a GTS. If A is μg-closed and μ-locally closed, then it is μ-closed. The converse is also true if $X \in \mu$.*

PROOF. Suppose that A is μg-closed and μ-locally closed. Thus $A = U \cap F$, where $U \in \mu$ and F is μ-closed. So $A \subsetneqq U$ and $A \subsetneqq F$. So by hypothesis $c_\mu(A) \subsetneqq U$ and $c_\mu(A) \subsetneqq c_\mu(F) = F$. Thus $c_\mu(A) \subsetneqq U \cap F = A$. Thus A is μ-closed.

Conversely, suppose that A is a μ-closed set in X. Let $A \subsetneqq U$ where U is μ-open in X. Then $c_\mu(A) = A \subsetneqq U$. Thus A is μg-closed. Since A is μ-closed it is μ-locally closed (by Remark 2.1). □

EXAMPLE 2.2. (a) Let $X = \{a, b, c, d\}$, $\mu = \{X, \varnothing, \{d\}, \{a, b\}, \{a, b, d\}\}$. Then (X, μ) is a GTS. It can be easily verified that $\{a, b\}$ is a μ-locally closed set but not a μg-closed set. **(b)** Let $X = \{a, b, c\}$, $\mu = \{\varnothing, X, \{a, b\}, \{a, c\}\}$. Then (X, μ) is a GTS. It can be easily verified that $\{b, c\}$ is not a μ-locally closed set but it is a μg-closed set.

DEFINITION 2.4. Let (X, μ) be a GTS. Then a subset A of X is said to be a
(i) μ_t-set if $i_\mu(A) = i_\mu(c_\mu(A))$;
(ii) μ_B-set if $A = U \cap V$, $U \in \mu$, V is a μ_t-set;
(iii) μ-semiopen [**3**] if $A \subsetneqq c_\mu(i_\mu(A))$;
(iv) μ-preopen [**3**] if $A \subsetneqq i_\mu(c_\mu(A))$.

A is called μ-semiclosed [**3**] if its complement is μ-semiopen.

PROPOSITION 2.1. *Let (X, μ) be a GTS. Then*
(i) A is a μ_t-set if and only if it is a μ-semiclosed set.
(ii) If A is a μ-closed set then it is a μ_t-set.
(iii) If $X \in \mu$, then A is a μ_t-set implies that it is also a μ_B-set.
(iv) Every μ-locally closed set is a μ_B-set.

Proof. (i) Let A be a μ_t-set. Then $i_\mu(A) = i_\mu(c_\mu(A))$. Therefore $i_\mu(c_\mu(A)) = i_\mu(A) \subsetneqq A$. Hence A is a μ-semiclosed set. Conversely, if A is μ-semiclosed, then $i_\mu(c_\mu(A)) \subsetneqq A$. Thus $i_\mu(c_\mu(A)) \subsetneqq i_\mu(A)$. Also $A \subsetneqq c_\mu(A)$. Therefore $i_\mu(A) = i_\mu(c_\mu(A))$.
(ii) Let A be a μ-closed set. Then $A = c_\mu(A)$. Thus $i_\mu(A) = i_\mu(c_\mu(A))$. Therefore A is a μ_t-set.
(iii) Let A be a μ_t-set. Then $A = X \cap A$. The rest follows from the definition of a μ_B-set.
(iv) Let A be a μ-locally closed subset of X. Then $A = U \cap F$, where U is μ-open in X and F is μ-closed. Then by (ii), F is a μ_t-set and hence A is a μ_B-set. □

Example 2.3. Let $X = \{a, b, c\}$ and $\mu = \{\varnothing, \{a\}, \{a, b\}, \{b, c\}, X\}$. Then μ is a GT on X. We here observe that $\{a\}$ and $\{c\}$ are two μ_t-sets but their union is not so. It can also be verified that $\{a, b\}$ is a μ_B-set which is not a μ_t-set.

Example 2.4. Let $X = \{a, b, c, d\}$ and $\mu = \{\varnothing, \{a\}, \{a, b\}, \{a, c\}, \{a, b, c\}\}$. Then μ is a GT on X. It can be checked that $\{c, d\}$ is a μ_t-set which is not μ-locally closed and $\{a\}$ is a μ-locally closed set which is not a μ_t-set. Note that $\{b, d\}$ is a μ_t-set but not a μ_B-set.

Lemma 2.1. [**6**] *For a GTS (X, μ) and $S, T \subsetneqq X$*
(i) $i_\mu(S \cap T) \subsetneqq i_\mu(S) \cap i_\mu(T)$.
(ii) $c_\mu(S) \cup c_\mu(T) \subsetneqq c_\mu(S \cup T)$.

Remark 2.2. [**6**] For a GTS (X, μ) and $S, T \subsetneqq X$ $i_\mu(S) \cap i_\mu(T) \subsetneqq i_\mu(S \cap T)$ is not true in general as shown in [**6**].

Proposition 2.2. *Let (X, μ) be a GTS. If A is μ-open then A is μ-preopen and a μ_B-set. The converse is true if μ is closed under finite intersection.*

Proof. Let A be μ-open. By [**3**], it is μ-preopen. Also $A = A \cap X$ where A is μ-open and X is a μ_t-set.

Conversely, since A is a μ_B-set, $A = U \cap V$, where U is μ-open in X and V is a μ_t-set. As A is μ-preopen by Lemma 2.1, $A \subsetneqq i_\mu(c_\mu(A)) \subsetneqq i_\mu(c_\mu(U \cap V)) \subsetneqq i_\mu(c_\mu(U) \cap c_\mu(V)) = i_\mu(c_\mu(U)) \cap i_\mu(V)$. Hence $A = U \cap V = (U \cap V) \cap U \subsetneqq (i_\mu(c_\mu(U)) \cap i_\mu(V)) \cap U = (i_\mu(c_\mu(U)) \cap U) \cap i_\mu(V) = U \cap i_\mu(V)$. Thus A is a μ-open set. □

Example 2.5. Consider Example 2.3. It can be shown that $\{a, c\}$ is a μ-preopen set but not a μ_B-set. Also in Example 2.4, $\{b\}$ is a μ_B-set but not μ-preopen.

3. Decompositions of μ-continuity

Definition 3.1. A function $f : (X, \mu) \to (Y, \lambda)$ is said to be μg-continuous (resp. μ-contra lc-continuous) if $f^{-1}(F)$ is μg-closed (resp. μ-locally closed) for each λ-closed set F of (Y, λ).

DEFINITION 3.2. A function $f : (X, \mu) \to (Y, \lambda)$ is said to be (μ, λ)-continuous [**2**] if for each $x \in X$ and for each λ-open set V of Y containing $f(x)$, there exists $U \in \mu$ containing x such that $f(U) \subseteqq V$.

THEOREM 3.1. [**2**] *For a function $f : (X, \mu) \to (Y, \lambda)$ the followings are equivalent :*
(i) f is (μ, λ)-continuous;
(ii) for every λ-open set V of Y, $f^{-1}(V)$ is μ-open in X;
(iii) for every λ-closed set F of Y, $f^{-1}(F)$ is μ-closed in X.

It follows from Theorem 2.5,

THEOREM 3.2. *If a function $f : (X, \mu) \to (Y, \lambda)$ is μg-continuous and μ-contra lc-continuous, then it is (μ, λ)-continuous. The converse is true if $X \in \mu$.*

EXAMPLE 3.1. (a) Let $X = \{a, b, c\}$, $\mu = \{\varnothing, \{a\}, \{a, b\}, \{b, c\}, X\}$, $\lambda = \{\varnothing, \{b\}, \{b, c\}, X\}$. It can be checked that the identity function $f : (X, \mu) \to (X, \lambda)$ is μg-continuous but not μ-contra lc-continuous.
(b) Let $X = \{a, b, c\}$, $\mu = \{\varnothing, \{a\}, \{c\}, \{a, b\}, \{a, c\}, X\}$, $\lambda = \{\varnothing, \{b, c\}, \{a, c\}, X\}$. It can be checked that the identity function $f : (X, \mu) \to (X, \lambda)$ is μ-contra lc-continuous but not μg-continuous.

DEFINITION 3.3. A function $f : (X, \mu) \to (Y, \lambda)$ is said to be contra (μ, λ)-continuous [**1**] if $f^{-1}(F)$ is μ-open in X for each λ-closed set F of (Y, λ).

THEOREM 3.3. *A contra (μ, λ)-continuous function $f : (X, \mu) \to (Y, \lambda)$ is (μ, λ)-continuous if and only if it is μg-continuous.*

PROOF. Let f be contra (μ, λ)-continuous and μg-continuous. Let F be a λ-closed set in Y. Then by contra (μ, λ) continuity of f, $f^{-1}(F)$ is μ-open in X. Thus $f^{-1}(F)$ is μ-locally closed in X. Since f is μg-continuous, $f^{-1}(F)$ is μg-closed. Thus by Theorem 2.5, $f^{-1}(F)$ is μ-closed, showing f to be (μ, λ)-continuous.

Converse part is obvious as every μ-closed set is μg-closed. □

DEFINITION 3.4. A mapping $f : (X, \mu) \to (Y, \lambda)$ is said to be contra μg-continuous (resp. μ-lc-continuous) if $f^{-1}(V)$ is μg-closed (resp. μ-locally closed) in X for each λ-open set V of Y.

THEOREM 3.4. *If a mapping $f : (X, \mu) \to (Y, \lambda)$ is μ-lc-continuous and contra μg-continuous, then it is contra (μ, λ)-continuous. The converse is true if $X \in \mu$.*

PROOF. Follows from Theorem 2.5. □

EXAMPLE 3.2. (a) Let $X = \{a, b, c\}$, $\mu = \{\varnothing, X, \{a, b\}, \{a, c\}\}$, $\lambda = \{\varnothing, X, \{b\}, \{b, c\}\}$. It can be easily verified that the identity mapping $f : (X, \mu) \to (X, \lambda)$ defined by is contra μg-continuous but not μ-lc-continuous.
(b) Let $X = \{a, b, c, d\}$, $\mu = \{X, \varnothing, \{d\}, \{a, b\}, \{a, b, d\}\}$ and $\lambda = \{\varnothing, \{a, b\}, \{a, c\}, \{a, b, c\}, X\}$. Then the identity mapping $f : (X, \mu) \to (X, \lambda)$ is not contra μg-continuous but μ-lc-continuous.

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