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Construction solutions differential operator type Sturm-Liouvulle border task with linear delay

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Abstract. In this paper we construct the solution and the characteristic function of the boundary task generated a linear differential equation with delay, then establishes an important relation between the potential q on the segment $[\xi_1, \pi]$ and the so-called transitional function $\tilde{q}$ on the segment $[-\pi, \pi]$. The obtained result opens the possibility of a solution of the inverse task.

1. Introduction

This work is dedicated to establishing a relation between the potential and the so-called transitional function $\tilde{q}$ spectral assignment of the following type

$$-y^{''}(x) + q(x)\, y(x - \tau(x)) = \lambda y(x) \tag{1.1}$$

$$y(x - \tau(x)) \equiv 0 \;\; x < \tau(x) \tag{1.2}$$

$$y'(0) - hy(0) = 0 \tag{1.3}$$

$$y'(\pi) + Hy(\pi) = 0 \quad h, H \in \overline{R} \tag{1.4}$$

Suppose that $q \in L_2[0, \pi]$ and q satisfies Dirihleove conditions. Moreover, let $\tau(x) = \alpha x + \beta, \alpha, \beta \in R^+$ and $\alpha < 1$. Function $\gamma(x) = x - \tau(x) = (1 - \alpha)x - \beta$ is strictly increasing, because $\gamma'(x) = 1 - \alpha > 0$. Its inverse function is

$$\gamma^{-1}(y) = \frac{1}{1-\alpha} y + \frac{\beta}{1-\alpha} \quad \text{to applie} \quad \gamma^{-1}(0) = \frac{\beta}{1-\alpha} = \xi_1.$$

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The condition (1.2) is met at intervals $[0,\, \xi_1)$. Limit ourselves to the case $h = H = \infty$. Then from (1.1) and (1.3) we obtain an integral equation

$$(1.5) \qquad y\,(x,z) = \sin\, zx + \frac{1}{z}\int_0^x q\,(t_1) \sin\, z(x{-}t_1) y\,(t_1 - \tau\,(t_1)\,, z)\, dt_1, z^2 = \lambda$$

Solution task (1.1 - 1.3) is dedicated to the monograph [**3**].

2. Solution task (1.1, 1.2, 1.3)

For $x \in [0, \xi_1]$ from (1.5) we obtain

$$y(x,z) = \sin zx$$

At intervals $(\xi_1,\, \pi]$ equation (1.5) becomes

$$(2.1) \qquad y\,(x,z) = \sin\, zx + \frac{1}{z}\int_{\xi_1}^x q\,(t_1) \sin\, z(x{-}t_1) y\,(t_1 - \tau\,(t_1)\,, z)\, dt_1, z^2 = \lambda$$

LEMMA 2.1. *There* $k_0 \in N_0$ *such that valid*

$$\xi_{k_0} < \pi \leqslant \xi_{k_0+1};\ \ \xi_l = \gamma^{-l}\,(0)\,.$$

PROOF. If $\frac{\beta}{1-\alpha} \geqslant \pi$ then $k_0 = 0$ so there is nothing to prove. Suppose that $\xi_1 = \gamma^{-1}\,(0) = \frac{\beta}{1-\alpha} < \pi$. Construct a series of

$$\xi_l = \gamma^{-l}\,(0) = \gamma^{-1}\left(\gamma^{-l+1}\,(0)\right), l = 2,3,\ldots For\ l = 2\ we\ obtain$$

$$\xi_2 = \gamma^{-l}\,(\xi_1) = \gamma^{-1}\left(\frac{\beta}{1-\alpha}\right) = \frac{\beta}{(1-\alpha)^2} + \frac{\beta}{1-\alpha}$$

Following the procedure of construction members a number xi_l we get

$$xi_l = \sum_{k=1}^{l} \frac{\beta}{(1-\alpha)^k} = \frac{\beta}{1-\alpha}\left[1 + \frac{1}{1-\alpha} + \cdots + \frac{\beta}{(1-\alpha)^{l-1}}\right].$$

Because of $0 < \alpha < 1$ is valid $\frac{1}{1-\alpha} > 1$ and geometric series

$$\sum_{k=0}^{\infty} \frac{1}{(1-\alpha)^k}\ diverges\ toward + \infty.$$

Therefore there is the slightest $k_0 \in N_0$ such that valid $\xi_{k_0+1} \geqslant \pi$ while $\xi_{k_0} < \pi$. □

Result of 1:

$$(*) \qquad (\xi_1,\, \pi] = \bigcup_{l=2}^{k_0} (\xi_{l-1},\, \xi_l] \cup (\xi_{k_0},\, \pi]$$

On the basis of (*) is the solution of integral equation (2.1) is constructed using variable step at a distance $(\xi_{l-1},\, \xi_l]$

Let us introduce the following functions

$$a_{s^2}(x,z)=\int_{\xi_1}^{x} q(t_1)\sin z(x-t_1)\sin z\gamma(t_1)\,dt_1$$

$$a_{s^{l+1}}(x,z)=\int_{\xi_l}^{x} q(t_1)\sin z(x-t_1)a_{s^l}(\gamma(t_1),z)\,dt_1,\, l=\overline{1,k_0}$$

LEMMA 2.2. *Solution* $y_l(x,z)$ *at a distance* $(\xi_{l-1},\xi_l]$ $l=\overline{1,k_{0-1}}$ *is given with*

$$y_l(x,z)=\sin zx+\sum_{k=1}^{l}\frac{1}{z^k}a_{s^{k+1}}(x,z) \tag{2.2}$$

The proof is carried out using mathematical induction. For illustration writ solution for $l=2$.

$$y_2(x,z)=\sin zx+\frac{1}{z}\int_{\xi_1}^{x} q(t_1)\sin z(x-t_1)\sin z\gamma(t_1)\,dt_1+$$

$$+\frac{1}{z^2}\int_{\xi_2}^{x} q(t_1)\sin z(x-t_1)\int_{\xi_1}^{\gamma(t_1)} q(t_2)\sin z(\gamma(t_1)-t_2)\sin z\gamma(t_2)\,dt_2dt_1$$

At the final interval $(\xi_{k_0},\pi]\subset(\xi_{k_0},\xi_{k_0+1}]$ solution has the form

$$y_{k_0}(x,z)=\sin zx+\frac{1}{z}a_{s^2}(x,z)+\sum_{k=2}^{k_0}\frac{1}{z^k}a_{s^{k+1}}(x,z) \tag{2.3}$$

Note 1: In [13] has observed the case of a linear delay with a view to construction of asymptotic eigenvalues.

3. Characteristic functions

Putting $x=\pi$ and using the boundary condition $y(\pi,z)=0$ we obtain the characteristic function $F(z)=y(\pi,z)$, $z\in C$, the operator L.

Thus,

$$F(z)=\sin\pi z+\frac{1}{z}a_{s^2}(\pi,z)+\sum_{k=2}^{k_0}\frac{1}{z^k}a_{s^{k+1}}(\pi,z) \tag{3.1}$$

The function F is the entire function of exponential type with apparent singularity at zero.

In fact sheet, $\lim_{z\to 0}F(z)=0$.

$$\sin z(\pi-t_1)\sin z(t_1-\tau(t_1))=\frac{1}{2}[\cos z(\pi-2t_1+\tau(t_1)-\cos z(\pi-\tau(t_1))],$$

we have,

$$a_{s^2}(\pi,z)=\frac{1}{z}\int_{\xi_1}^{\pi} q(t_1)\cos z(\pi-2t_1+\tau(t_1))dt_1-\frac{1}{z}\int_{\xi_1}^{\pi} q(t_1)\cos z(\pi-\tau(t_1))dt_1=$$

$$\frac{1}{2}\int\limits_{\xi_1}^{\pi} q(t_1)\cos\, z(\pi+(\alpha-2)t_1+\beta)dt_1 - \frac{1}{2}\int\limits_{\xi_1}^{\pi} q(t_1)\cos\, z(\pi-\alpha t_1-\beta)dt_1$$

We put

$$\theta_1 = \pi + 2\beta + (\alpha-2)t_1,\ dt_1 = \frac{d\theta}{\alpha-2}$$

Respectively $\theta_1^* = \pi - \alpha t_1 - \beta dt_1 = -\frac{d\theta}{\alpha} t_1 \in [\xi_1, \pi]$

Then apply

$$a_{s^2}(\pi,z) = \frac{1}{2}\int\limits_{-\gamma(\pi)}^{\pi-\xi_1} \frac{1}{2-\alpha} q(\frac{\pi+\beta-\theta}{2-\alpha})\cos\, z\theta d\theta - \frac{1}{2}\int\limits_{\gamma(\pi)}^{\pi-\xi_1} \frac{1}{\alpha} q(\frac{\pi-\beta-\theta}{\alpha})\cos\, z\theta d\theta$$

Defines the so-called transitional function $\tilde{q}$ as follows

$$\tilde{q} = \begin{cases} 0, & \theta \in [-\pi, -\gamma(\pi)) \cup (\pi-\xi_1, \pi] \\ \frac{1}{2-\alpha} q(\frac{\pi+\beta-\theta}{2-\alpha}), & \theta \in [-\gamma(\pi), \gamma(\pi)] \\ \frac{1}{2-\alpha} q(\frac{\pi+\beta-\theta}{2-\alpha}) - \frac{1}{\alpha} q(\frac{\pi-\beta-\theta}{\alpha}), & \theta \in (\gamma(\pi), \pi-\xi_1] \end{cases} \tag{3.2}$$

From (3.2) we see that the known potential q on the segment $[\xi_1, \pi]$ and the known delay $\tau(x) = \alpha x + \beta,\ \xi_1 = \frac{\beta}{1-\alpha}$ unambiguously defines the transition function $\tilde{q}$ the segment $[-\pi, \pi]$.

In fact, given the delay means knowing the function $\gamma(x)$, and are known and numbers $-\gamma(\pi)$ i $\gamma(\pi)$.

Ask the reverse question: whether a given transition function $\tilde{q}$ well-defined potential q ?

Parameters $\xi_1, \pm\gamma(\pi), \alpha$ i β we believe to be famous. For the purpose of answering this question, we perform a partition segments $[\xi_1, \pi]$ i $[-\gamma(\pi), \pi-\xi_1]$ in accordance with the nature of the mapping exercise that function $\theta_1 = \pi + \beta - (2-\alpha)t_1$ odnosno $t_1 = \frac{\pi+\beta-\theta_1}{2-\alpha}$.

Function $\theta_1 : [\xi_1, \pi] \to [-\gamma(\pi), \pi-\xi_1]$ is strictly decreasing. At the same time function $\theta_1^* = \pi - \beta - \alpha t_1 : [\xi_1, \pi] \to [\gamma(\pi), \pi-\xi_1]$ is also strictly decreasing. We put $\delta_1 = \frac{2\beta+\alpha\pi}{2-\alpha}$

Then we have $\theta_1([\delta_1, \pi]) = [-\gamma(\pi), \gamma(\pi)]$

$$\theta_1^*([\delta_1, \pi]) = [\gamma(\pi), \eta_1],\quad \eta_1 = \theta_1^*(\delta_1) = \pi - \beta - \frac{\alpha^2\pi + 2\alpha\beta}{2-\alpha}$$

Let $\delta_0 = \pi,\ \eta_0 = \gamma(\pi)$. Next $\theta_1(\delta_2) = \eta_1,\ \theta_1^*(\ \delta_2) =\ \eta_2$

In the general case

$$\theta_1(\delta_k) = \eta_{k-1},\quad \theta_1^*(\delta_k) =\ \eta_k k \in N_0$$

Provide value first few members of the series δ_k :

$$\delta_2 = \frac{4\beta + \alpha^2\pi}{(2-\alpha)^2},\quad \delta_3 = \frac{8\beta - 4\alpha\beta + \alpha^3\pi + 2\alpha^2\pi}{(2-\alpha)^3}$$

$$\delta_4 = \frac{16\beta - 16\alpha\beta + \alpha^4\pi + 2\alpha^2\pi}{(2-\alpha)^4},\ \delta_5 = \frac{32\beta - 48\alpha\beta - 8\alpha^3\pi + 32\alpha^2\pi + 2\alpha^4\pi + \alpha^5\pi}{(2-\alpha)^5}$$

By induction we prove the relation

$$\begin{aligned} \delta_k - \xi_1 &= \tfrac{\alpha^k}{(2-\alpha)^k(1-\alpha)}\gamma(\pi) \\ \pi - \xi_1 - \eta_k &= \tfrac{\alpha^{k+1}}{(2-\alpha)^k(1-\alpha)}\gamma(\pi), \quad k \in N \end{aligned} \tag{3.3}$$

On the basis of (3.3) we can write

$$[\xi_1, \pi] = \bigcup_{k=1}^{\infty}[\delta_k, \delta_{k-1}],\ [\gamma(\pi), \pi - \xi_1] = \bigcup_{k=1}^{\infty}[\eta_{k-1}, \eta_k] \tag{3.4}$$

From (3.2) follows

$$q(t_1) = (2-\alpha)\widetilde{q}\,(\theta_1) = (2-\alpha)\widetilde{q}\,(\pi + \beta - (2-\alpha)t_1),\ t_1 \in [\delta_1, \delta_0] \tag{3.4'}$$

Thus, the segment $[\delta_1, \pi]$= $[\delta_1, \delta_0]$ potential q is uniquely determined by the transitional function $\tilde{q}$ on the segment $[\gamma(\pi), \eta_0]$. For $t_1 \in [\delta_2, \delta_1)$ we have $\theta_1 \in (\eta_0, \eta_1]$ and the equality

$$(t_1) = (2-\alpha)\widetilde{q}\,(\theta_1) + \frac{2-\alpha}{\alpha}q(t_1^*),\ t_1^* = \frac{-2\beta + (2-\alpha)t_1}{\alpha} \in [\delta_1, \delta_0] \tag{3.4"}$$

The value of q at a distance $\delta_1, \delta_0)$ known from (3.4'), and $\widetilde{q}\,(\theta_1)$ is given in advance, so by using the function q (3.4") and the well defined $[\delta_2, \delta_1)$

Completely analogous to continue the process of construction of the potential q on each interval $[\delta_{k+1}, \delta_k), k \in N$ on the basis of the known potential at a distance $[\delta_k, \delta_{K-1}), k \in N$.

This result means that the task of solving the inverse observed equation boils down to determining the transitional function operator.

Note 2: In [**11**] and [**12**] used the transition functions for solving inverse tasks in homogeneous delay that is the corrected argument $\alpha x,\ 0 < \alpha < 1$.

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