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COMMON FIXED POINT THEOREMS FOR PAIRS OF SINGLE AND MULTIVALUED D-MAPS AND TANGENTIAL MULTIVALUED MAPPINGS SATISFYING CONTRACTIVE CONDITION OF INTEGRAL INEQUALITY

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Abstract. In [2, 25] the others defined a tangential property which can be used not only for a single mapping but also for a multi-valued mappings and the concept of subcomatiblity of them. Motivated by the results in [2, 25] we prove common fixed point theorems satisfying a contractive conditions for pairs of single and multivalued used D-maps and tangential multivalued mappings of integral inequality.

1. Introduction and Preliminaries

S.Banach proved a theorem which ensures under appropriate conditions, the existence and uniqueness of fixed point, in 1922 ([3],[4]). His results is called Banach's fixed point theorem. This theorem provides a method for solving a variety of applied problems in mathematical Science and Engineering. Banach contraction principle has been extended in many different directions, see [3, 24, 26-30], etc. In 1969, the Banach's Contraction Mapping Principle extended nicely to set valued or multivalued mappings, by Nadler [18]. Afterward, the study of fixed points for multi-valued contractions using the Hausdorff metric was initiated by Markin [17]. the study of fixed points of a functions satisfying certain contractive conditions has been at the center of vigorous research activity, because it has a wide range

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Key words and phrases. Weakly compatible mappings, Property (E.A), Common property (E.A), Weak tangle point, Pair-wise tangential property, Commuting and weakly commuting maps, compatible and compatible maps of type (A), (B), (C) and (P), $\delta-$ compatible maps, subcompatible maps, D- maps, integral type, common fixed point theorems, metric space .

of application in different area such as, variational, linear inequalities, differential equations, control theory, optimization and parameterize estimation problems.

In this paper, (X, d) denotes a metric space, $CB(X)$, the class of all nonempty bounded closed subsets of X and $B(X)$, the class of all nonempty bounded subsets of X, also $\mathbb{R}_+$ denotes the set of nonnegative real numbers. Sessa [23] introduced the notion of weak commutativity which generalized the notion of commutativity. Jungck [10] gave a generalization of weak commutativity by introducing the concept of compatibility later on. In [11], the others introduced the concept of compatible maps of type (A) to generalize weakly commuting maps. Pathak and Khan [21] introduced the notion of compatible maps of type (B). to extending type (A). In [19], the concept of compatible maps of type (P) was introduced and compared with compatible and compatible maps of type (A). In 1998, Pathak, Cho, Kang and Madharia [20] defined the notion of compatible maps of type (C) as another extension of compatible maps of type (A). Jungck [9] generalized all the concepts of compatibility by giving the notion of weak compatibility (subcompatibility). In [13], the authors extended the concept of compatible maps to the setting of single and multivlued maps by giving the notion of $\delta-$compatible maps. In [12], the authors extended the definition of weak compatibility to the setting of single and multivalued maps by introducing the concept of subcompatible maps. Djoudi and khemis [5] introduced the notion of $D-$maps which is a generalization of $\delta-$compatible maps.

Let (X, d) be a metric space and let $B(X)$ be the class of all nonempty bounded subsets of X. For all A, B in $B(X)$, define

$$\delta(A, B) = sup\{d(a, b) : a \in A, b \in B\}.$$

If $A = \{a\}$, we write $\delta(A, B) = \delta(a, B)$. Also, if $B = \{b\}$, it yields that

$$\delta(A, B) = d(a, b).$$

From the definition of $\delta(A, B)$, for all A, B, C in $B(X)$ it follows that

$$\delta(A, B) = \delta(B, A) > 0,$$

$$\delta(A, B) \leqslant \delta(A, C) + \delta(C, B),$$

$$\delta(A, B) = 0 \text{ iff } A = B = \{a\}$$

DEFINITION 1.1 [2, 6] A sequence $\{A_n\}$ of nonempty subsets of X is said to be convergent to a subset A of X if: for each point $a \in A$ is the limit of a convergent sequence $\{a_n\}$, where $a_n \in A_n$ for $n \in N$.

LEMMA 1.1 [2, 6, 7] If $\{A_n\}$ and $\{B_n\}$ are sequences in $B(X)$ converging to A and B in $B(X)$, respectively, then the sequence $\{\delta(A_n, B_n)\}$ converges to $\delta(A, B)$.

LEMMA 1.2 [2, 7] Let $\{A_n\}$ be a sequence in $B(X)$ and y be a point in X such that $\delta(A_n, y) \to 0$, Then the sequence $\{A_n\}$ converges to the set $\{y\}$ in $B(X)$.

DEFINITION 1.2 [2, 20] The self-maps f and g of a metric space X are said to be weakly commuting if

$$d(fgx, gfx) \leqslant d(gx, fx) \text{ for all } x \in X.$$

DEFINITION 1.3 [2, 10] The self-maps f and g of a metric space X are said to be:
(1) compatible if

$$\lim_{n\to\infty} d\ (fgx_n, gfx_n) = 0,$$

(2) compatible of type (A) if

$$\lim_{n\to\infty} d(fgx_n, g^2x_n) = 0 \ and \ \lim_{n\to\infty} d(gfx_n, f^2x_n) = 0,$$

(3) compatible of type (B) if

$$\lim_{n\to\infty} d(fgx_n, g^2x_n) \leqslant \frac{1}{2}[\lim_{n\to\infty} d(fgx_n, ft) + \lim_{n\to\infty} d(ft, f^2x_n)],$$

$$\lim_{n\to\infty} d(gfx_n, f^2x_n) \leqslant \frac{1}{2}[\lim_{n\to\infty} d(gfx_n, gt) + \lim_{n\to\infty} d(gt, g^2x_n)],$$

(4) compatible of type (C) if

$$\lim_{n\to\infty} d(fgx_n, g^2x_n) \leqslant \tfrac{1}{3}[\lim_{n\to\infty} d(fgx_n, ft) + \lim_{n\to\infty} d(ft, f^2x_n) + \lim_{n\to\infty} d(ft, g^2x_n)],$$

$$\lim_{n\to\infty} d(gfx_n, f^2x_n) \leqslant \tfrac{1}{3}[\lim_{n\to\infty} d(gfx_n, gt) + \lim_{n\to\infty} d(gt, g^2x_n) + \lim_{n\to\infty} d(gt, f^2x_n)],$$

(5) compatible of type(P) if

$$\lim_{n\to\infty} d(f^2x_n, g^2x_n) = 0,$$

whenever $\{x_n\}$ is a sequence in X such that

$$\lim_{n\to\infty} fx_n = \lim_{n\to\infty} gx_n = t$$

for some $t \in X$.

DEFINITION 1.4 [2, 16] The self-maps f and g of a metric space X are called weakly compatible if $fx = gx$, $x \in X$ implies $fgx = gfx$.

DEFINITION 1.5 [2, 17] The maps $f : X \to X$ and $F : X \to B(X)$ are $\delta-$compatible if

$$\lim_{n\to\infty} \delta(Ffx_n, fFx_n) = 0,$$

whenever $\{x_n\}$ is a sequence in X such that

$$Ffx_n \in B(X), fx_n \to t,$$

and $Fx_n \to \{t\}$ for some $t \in X$.

DEFINITION 1.6 [2, 18] the Maps $f : X \to X$ and $F : X \to B(X)$ are subcompatible if they commute at coincidence points;
i.e., for each point $u \in X$ such that

$$Fu = \{fu\}, \text{ we have } Ffu = fFu.$$

DEFINITION 1.7 [2, 5] The maps $f : X \to X$ and $F : X \to B(X)$ are said to be D-maps iff there exists a sequence $\{x_n\}$ in X such that for some $t \in X$

$$\lim_{n\to\infty} fx_n = t \text{ and } \lim_{n\to\infty} Fx_n = \{t\}.$$

DEFINITION 1.8 [25] Let $f : X \to X$ and $T : X \to CB(X)$ be a single and multivalued mapping respectively:
1. A point $x \in X$ is called a fixed point of f and T iff $fx = x$ and $x \in Tx$, the set of all fixed points of f and T is called $F(f)$ and $F(T)$ respectively.
2. A point $x \in X$ is said to be a coincidence point of f and T iff $fx \in Tx$, the set of all coincidence points of f and T is denoted by $C(f, T)$.
3. A point $x \in X$ is called a common fixed point of f and T iff $x = fx \in Tx$, the set of all common fixed points of f and T is denoted by $F(f, T)$.

DEFINITION 1.9 [25, 12] The mappings $f : X \to X$ and $A : X \to CB(X)$ are said to be weakly compatible if

$$fAx = Afx \text{ for all } x \in C(f, A).$$

Definition 1.10 [25, 10] Let $f : X \to X$ and g :$X \to X$ .The pair (f, g) satisfies property (E.A) if there exist the sequence $\{x_n\}$ in X such that

$$\lim_{n\to\infty} fx_n = \lim_{n\to\infty} g\ x_n = z \tag{1.1}$$

for some $z \in X$.

DEFINITION 1.11 [25, 16] Let $f, g, A, B : X \to X$. The pair (f, g) and (A, B) satisfy a common property (E.A) if there exist sequences $\{x_n\}$ and $\{y_n\}$ in X such that

$$\lim_{n\to\infty} fx_n = \lim_{n\to\infty} gx_n = \lim_{n\to\infty} Ay_n = \lim_{n\to\infty} By_n = z \in X. \tag{1.2}$$

REMARK 1.1 [25] If $A = f$, $B = g$ and $\{x_n\}$=$\{y_n\}$ in (2), then we get the definition of property (E.A).

DEFINITION 1.12 [25, 22] Let $f, g : X \to X$. A point $z \in X$ is said to be a weak tangent point to (f, g) if there exists sequences $\{x_n\}$ and $\{y_n\}$ in X such that

$$\lim_{n\to\infty} fx_n = \lim_{n\to\infty} gy_n = z \in X. \tag{1.3}$$

REMARK 1.2 [25] If $\{x_n\} = \{y_n\}$ in (3), we get the definition of property (E.A).

DEFINITION 1.13 [25, 22] Let $f, g, A, B : X \to X$. The pair (f, g) is called tangential with respect to the pair (A, B) if there exists sequences$\{x_n\}$ and $\{y_n\}$ in X such that

$$\lim_{n\to\infty} fx_n = \lim_{n\to\infty} gy_n = \lim_{n\to\infty} Ax_n = \lim_{n\to\infty} By_n = z \ \in X. \tag{1.4}$$

DEFINITION 1.14 [25] Let $f, g : X \to X$, and $A, B : X \to CB(X)$.

The pair (f, g) is called tangential with respect to the pair (A, B) if there exists two sequences$\{x_n\}$ and $\{y_n\}$ in X such that

$$\lim_{n\to\infty} fx_n = \lim_{n\to\infty} gy_n = z, \tag{1.5}$$

for some $z \in X$, then

$$z \in \lim_{n\to\infty} Ax_n = \lim_{n\to\infty} By_n \in CB(X) \tag{1.6}$$

EXAMPLE 1.1 [25] Let $(\mathbb{R}_+, d)$ be a metric space with usual metric d,
$f, g : \mathbb{R}_+ \to \mathbb{R}_+$ and $A, B : \mathbb{R}_+ \to CB(\mathbb{R}_+)$ mappings defined by
$fx = x + 1$, $gx = x + 2$, $Ax=[\frac{x^2}{2}, \frac{x^2}{2} + 1]$, and $Bx = [x^2 + 1, x^2 + 2]$
for all $x \in \mathbb{R}_+$.
Since there exists two sequences $x_n=2+\frac{1}{n}$ and $y_n=1+\frac{1}{n}$ such that
$\lim_{n\to\infty} fx_n = \lim_{n\to\infty} gy_n = 3$ and $3 \in [2, 3] = \lim_{n\to\infty} Ax_n = \lim_{n\to\infty} By_n$.
Thus the pair (f, g) is tangential with respect to the pair (A, B).

DEFINITION 1.15 [25] Let $f : X \to X$. and $A : X \to CB(X)$. The mapping f is called tangential with respect to the pair A if there exists two sequences $\{x_n\}$ and $\{y_n\}$ in X such that

$$\lim_{n\to\infty} fx_n = \lim_{n\to\infty} fy_n = z \tag{1.7}$$

for some $z \in X$, then

$$z \in \lim_{n\to\infty} Ax_n = \lim_{n\to\infty} Ay_n \in CB(X). \tag{1.8}$$

EXAMPLE 1.2 [25] Let $(\mathbb{R}_+, d)$ be a metric space with usual metric d,
$f : \mathbb{R}_+ \to \mathbb{R}_+$ and $A : \mathbb{R}_+ \to CB(\mathbb{R}_+)$ mappings defined by

$$fx = x + 1 \text{ and } Ax = [x^2 + 1, x^2 + 2].$$

Since there exists two sequences $x_n=1+\frac{1}{n}$ and $y_n=1-\frac{1}{n}$ such that

$$\lim_{n\to\infty} fx_n = \lim_{n\to\infty} fy_n = 2 \text{ and } 2 \in [2, 3] = \lim_{n\to\infty} Ax_n = \lim_{n\to\infty} Ay_n,$$

therefore the mapping f is tangential with respect to the mapping A.

DEFINITION 1.16 [8] A function $\phi : [0, \infty) \to [0, \infty)$ is said to be contractive modulus if $\phi(t) < t$ for $t > 0$.

DEFINITION 1.17 [8] A real valued function ϕ defined on $X$is said to be upper semi continuous if

$$\limsup_{n\to\infty}\phi(t_n) \leqslant \phi(t),$$

for every sequence $\{t_n\} \in X$ with $t_n \to t$ as $n \to \infty$.

2. Main Results

THEOREM 2.1 Let $S, T : X \to X$ and $P, Q : X \to CB(X)$ satisfy

$$(\int_0^{d(Sx,Qy)} \varphi(t)dt\ )^r + (\int_0^{d(Px,Ty)} \varphi(t)dt)^r$$

$$\leqslant \phi((\int_0^{\max\{d(Px,Qy),d(Px,Sx),d(Qy,Ty),\frac{d(Px,Ty)+d(Qy,Sx)}{2}\}} \varphi(t)dt)^r) \tag{2.1}$$

for all $x, y \in X$,where $r \geqslant 1$, $\phi : R_+ \to R_+$ is an upper semi-continuous contractive modulus and $\varphi : R_+ \to R_+$ is a Lebesgue integrable mapping which is a summable nonnegative and such that

$$\int_0^{\epsilon} \varphi(t)dt > 0, \tag{2.2}$$

for each $\epsilon > 0$. If the following conditions (a)-(d) holds:
(a) there exists a point $z \in S(X) \cap T(X)$ which is a weak tangent point to (S, T),
(b) (S, T) is tangential with respect to (P, Q),
(c) $S^2a = Sa$, $T^2b = Tb$ and $PSa = QTb$ for $a \in C(S, P)$ and $b \in C(T, Q)$,
(d) the pairs (S, P) and (T, Q) are weakly compatible.
Then S, T, P and Q have a unique common fixed point in X.

Proof. It is clearly from $z \in S(X) \cap T(X)$ that $z = Su = Tv$ for some $u, v \in X$.
Using that a point z is a weak tangent point to (S, T), there exists two sequences$\{x_n\}$ and $\{y_n\}$ in X such that

$$\lim_{n\to\infty} Sx_n = \lim_{n\to\infty} Ty_n = z. \tag{2.3}$$

Since the pair (S, T) is tangential with respect to (P, Q) and (2.3), we get

$$z \in \lim_{n\to\infty} Px_n = \lim_{n\to\infty} Qy_n \in CB(X). \tag{2.4}$$

Using the fact $z = Su = Tv$, (2.3) and (2.4), we have

$$z = Su = Tv = \lim_{n\to\infty} Sx_n = \lim_{n\to\infty} Ty_n \in \lim_{n\to\infty} Px_n = \lim_{n\to\infty} Qy_n. \tag{2.5}$$

We show that $z \in Qv$. if not, then condition (2.1) implies

$$(\int_0^{d(Sx_n,Qv)} \varphi(t)dt)^r + (\int_0^{d(Px_n,Tv)} \varphi(t)dt)^r$$

$$\leqslant \phi((\int_0^{\max\{d(Px_n,Qv),d(Px_n,Sx_n),d(Qv,Tv),\frac{d(Px_n,Tv)+d(Qv,Sx_n)}{2}\}} \varphi(t)dt)^r) \tag{2.6}$$

Taking the limit as $n \to \infty$, we have

$$(\int_0^{d(z,Qv)} \varphi(t)dt)^r \leqslant \phi((\int_0^{\max\{d(z,Qv),0,d(Qv,z),\frac{d(Qv,z)}{2}\}} \varphi(t)dt)^r) \tag{2.7}$$

$$\Longrightarrow (\int_0^{d(z,Qv)} \varphi(t)dt)^r \leqslant \phi((\int_0^{d(z,Qv)} \varphi(t)dt)^r) < (\int_0^{d(z,Qv)} \varphi(t)dt)^r \tag{2.8}$$

which is a contradiction. Therefore $z \in Qv$.
Again, we claim that $z \in Pu$. if not, then condition (2.1) implies

$$(\int_0^{d(Su,Qy_n)} \varphi(t)dt)^r + (\int_0^{d(Pu,Ty_n)} \varphi(t)dt)^r$$

$$\leqslant \phi((\int_0^{\max\{d(Pu,Qy_n),d(Pu,Su),d(Qy_n,Ty_n),\frac{d(Pu,Ty_n)+d(Qy_n,Su)}{2}\}} \varphi(t)dt)^r) \tag{2.9}$$

Letting $n \to \infty$, we get

$$(\int_0^{d(Pu,z)} \varphi(t)dt)^r \leqslant \phi((\int_0^{\max\{d(Pu,z),d(Pu,z),0,\frac{d(Pu,z)}{2}\}} \varphi(t)dt)^r) \tag{2.10}$$

$$\Longrightarrow (\int_0^{d(Pu,z)} \varphi(t)dt)^r \leqslant \phi((\int_0^{d(Pu,z)} \varphi(t)dt\ )^r) < \int_0^{d(Pu,z)} \varphi(t)dt, \tag{2.11}$$

which is a contradiction.Thus $z \in Pu$.
Now we conclude $z = Tv \in Qv$ and $z = Su \in Pu$.It follows from
$v \in C(T,Q)$, $u \in C(S,P)$ that $S^2u = Su$, $T^2v = Tv$ and $PSu = QTv$.
Hence $z = Tv = T^2v = Tz$, $z = Su = S^2u = Sz$
and $PSu = QTv \Longrightarrow Pz = Qz$.
Since the pair (T,Q) is weakly compatible, $TQv = QTv$.
Thus $z \in Qv \Longrightarrow Tz \in TQv = QTv = Qz = Pz$.

Similarly, we can prove that $Sz \in Pz$.
Consequently, $z = Sz = Tz \in Qz \in Pz$.
Therefore S, T, P and Q have a common fixed point in X .
The uniqueness of the common fixed point follows easily from conditions (2.1)
Therefore S, T, P and Q have a unique common fixed point in X .

Putting $r = 1$ in Theorem 2.1, we obtain the following Corollary:

COROLLARY 2.1 Let $S, T : X \to X$ and $P, Q : X \to CB(X)$ satisfy

$$\int_0^{d(Sx,Qy)} \varphi(t)dt + \int_0^{d(Px,Ty)} \varphi(t)dt$$

$$\leqslant \phi\Big(\int_0^{\max\{d(Px,Qy),d(Px,Sx),d(Qy,Ty),\frac{d(Px,Ty)+d(Qy,Sx)}{2}\}} \varphi(t)dt\Big), \tag{2.12}$$

for all $x, y \in X$, where $\phi : R_+ \to R_+$ is an upper semi continuous contractive modulus and $\varphi : R_+ \to R_+$ is a Lebesgue integrable mapping which is a summable nonnegative and such that

$$\int_0^{\epsilon} \varphi(t)dt > 0$$

for each $\epsilon > 0$. If the following conditions (a)-(d) holds:
(a) there exists a point $z \in S(X) \cap T(X)$ which is a weak tangent point to (S, T),
(b) (S, T) is tangential with respect to (P, Q),
(c) $S^2a = Sa$, $T^2b = Tb$ and $PSa = QTb$ for $a \in C(S, P)$ and $b \in C(T, Q)$,
(d) the pairs (S, P) and (T, Q) are weakly compatible.
Then S, T, P and$Q$ have a unique common fixed point in X.

If $\varphi(t) = 1$ in Corollary 2.1, we get the following Corollary

Corollary 2.2 Let $S, T : X \to X$ and $P, Q : X \to CB(X)$ satisfy

$$d(Sx, Qy) + d(Px, Ty)$$

$$\leqslant \phi(\max\{d(Px, Qy), d(Px, Sx), d(Qy, Ty), \frac{d(Px,Ty)+d(Qy,Sx)}{2}, \tag{2.13}$$

for all $x, y \in X$ where $\phi : R_+ \to R_+$ is an upper semi continuous-contractive modulus If the following conditions (a)-(d) holds:
(a) there exists a point $z \in S(X) \cap T(X)$ which is a weak tangent point to (S, T),
(b) (S, T) is tangential with respect to (P, Q),
(c) $S^2a = Sa$, $T^2b = Tb$ and $PSa = QTb$ for $a \in C(S, P)$ and $b \in C(T, Q)$,
(d) the pairs (S, P) and (T, Q) are weakly compatible.
Then $S, T, P,$ and$Q$ have a unique common fixed point in X.

If $\varphi(t) = 1$, $S = T$ and $P = Q$ in Corollary 2.1, we have the following Corollary:

Corollary 2.3 Let $S : X \to X$ and $P : X \to CB(X)$ satisfy

$$d(Sx, Py) + d(Px, Sy)$$

$$\leqslant \phi(\max\{d(Px, Py), d(Px, Sx), d(Py, Sy), \frac{d(Px,Sy)+d(Py,Sx)}{2}\}), \tag{2.14}$$

for all $x, y \in X$ where $\phi : R_+ \to R_+$ is an upper semi continuous contractive modulus If the following conditions (a)-(d) holds:
(a) there exists a sequence $\{x_n\}$ in X such that $\lim_{n\to\infty} Sx_n \in X$,
(b) S is tangential with respect to P,
(c) $S^2a = Sa$ for $a \in C(S, P)$,
(d) the pairs (S, P) is weakly compatible.
Then S and P have a unique common fixed point in X.

Now, we can rewrite the contractive condition of the Theorem 2.1 in the sense of $D-$maps to obtain the following Theorem:

THEOREM 2.2 Let S, T be self-maps of a metric space (X, d) and let P, Q be maps from X into $B(X)$ satisfying the following conditions:

(1) S and T are surjective,
(2) $(\int_0^{d(Sx,Qy)} \varphi(t)dt)^r + (\int_0^{d(Px,Ty)} \varphi(t)dt\ )^r$

$$\leqslant \phi((\int_0^{\max\{d(Px,Qy),d(Px,Sx),d(Qy,Ty),\frac{d(Px,Ty)+d(Qy,Sx)}{2}\}} \varphi(t)dt)^r), \tag{2.15}$$

for all $x, y \in X$, where $r \geqslant 1$, $\phi : R_+ \to R_+$ is an upper semi continuous contractive modulus and $\varphi : R_+ \to R_+$ is a Lebesgue integrable mapping which is a summable nonnegative
and such that $\int_0^{\epsilon} \varphi(t)dt > 0$, for each $\epsilon > 0$.
If either
(3) S and P are subcompatible $D-$maps; T and Q are subcompatible, or
(4) T and Q are subcompatible $D-$maps; S and P are subcompatible.
Then, S, T, P and Q have a unique common fixed point $t \in X$ such that

$$Pt = Qt = \{Tt\} = \{St\} = \{t\}.$$

Proof : Suppose that S and P are $D-$maps, then, there is a sequence $\{x_n\}$ in X such that $\lim_{n\to\infty} Sx_n = t$ and $\lim_{n\to\infty} Px_n = \{t\}$ for some $t \in X$.
By condition (1), there exist points u, v in X such that $t = Su = Tv$.
First, we show that $Qv = \{Tv\} = \{t\}$. then, by (2.15) we get

$$(\int_0^{d(Sx_n,Qv)}\varphi(t)dt)^r+(\int_0^{d(Px_n,Tv)}\varphi(t)dt)^r$$

$$\leqslant\phi((\int_0^{\max\{d(Px_n,Qv),d(Px_n,Sx_n),d(Qv,Tv),\frac{d(Px_n,Tv)+d(Qv,Sx_n)}{2}\}}\varphi(t)dt)^r). \tag{2.16}$$

Taking the limit as $n\to\infty$, one obtains

$$(\int_0^{d(Tv,Qv)}\varphi(t)dt)^r\leqslant\phi((\int_0^{\max\{d(Tv,Qv),0,d(Qv,Tv),\frac{d(Qv,Tv)}{2}\}}\varphi(t)dt)^r) \tag{2.17}$$

$$\Longrightarrow(\int_0^{d(Tv,Qv)}\varphi(t)dt)^r\leqslant\phi((\int_0^{d(Tv,Qv)}\varphi(t)dt)^r)<(\int_0^{d(Tv,Qv)}\varphi(t)dt)^r, \tag{2.18}$$

a contradiction implies that $Qv=\{Tv\}=\{t\}$.
Since the pair (T,Q) is subcompatible, then $QTv=TQv$, i.e., $Qt=Tt$
We claim that $Qt=\{Tt\}=\{t\}$. if not, then by condition (2.15) we have

$$(\int_0^{d(Sx_n,Qt)}\varphi(t)dt)^r+(\int_0^{d(Px_n,Tt)}\varphi(t)dt)^r$$

$$\leqslant\phi((\int_0^{\max\{d(Px_n,Qt),d(Px_n,Sx_n),d(Qt,Tt),\frac{d(Px_n,Tt)+d(Qt,Sx_n)}{2}\}}\varphi(t)dt)^r) \tag{2.19}$$

when $n\to\infty$ we obtain,

(2.20)

$$(\int_0^{d(t,Qt)}\varphi(t)dt)^r+(\int_0^{d(t,Qt)}\varphi(t)dt\ )^r\leqslant\phi((\int_0^{\max\{d(t,Qt),0,0,\frac{d(t,Qt)+d(Qt,t)}{2}\}}\varphi(t)dt)^r)$$

$$\Longrightarrow 2(\int_0^{d(t,Qt)}\varphi(t)dt)^r\leqslant\ \phi((\int_0^{d(t,Qt)}\varphi(t)dt\ )^r)<(\int_0^{d(t,Qt)}\varphi(t)dt)^r \tag{2.21}$$

which is a contradiction. Hence,

$$Qt=\{Tt\}=\{t\}. \tag{2.22}$$

Next, we claim that $Pu=\{Su\}=\{t\}$. If not, then, by (23) we get (letting $x=u$ and $y=t$ in (23))

$$(\int_0^{d(Su,Qt)} \varphi(t)dt)^r + (\int_0^{d(Pu,Tt)} \varphi(t)dt\ )^r$$

$$\leqslant \phi((\int_0^{\max\{d(Pu,Qt),d(Pu,Su),d(Qt,Tt),\ \frac{d(Pu,Tt)+d(Qt,Su)}{2}\}} \varphi(t)dt)^r) \tag{2.23}$$

$$\Longrightarrow (\int_0^{d(Pu,t)} \varphi(t)dt)^r \leqslant \phi((\int_0^{\max\{d(Pu,t),d(Pu,t),0,\frac{d(Pu,t)}{2}\}} \varphi(t)dt)^r) \tag{2.24}$$

$$\Longrightarrow (\ int_0^{d(Pu,t)}\varphi(t)dt)^r \leqslant ((\int_0^{d(Pu,t)} \varphi(t)dt)^r) < (\int_0^{d(Pu,t)} \varphi(t)dt)^r, \tag{2.25}$$

which is a contradiction again. Thus $Pu = \{Su\} = \{t\}$.
Since the pair (P, S) is subcompatible, then $PSu = \{SPu\}$,i.e., $Pt = \{St\}$.
Suppose that $St \neq t$, then, the use of (2.15) gives (letting $x = y = t$ in (2.15))

$$(\int_0^{d(St,Qt)} \varphi(t)dt)^r + (\int_0^{d(Pt,Tt)} \varphi(t)dt)^r$$

$$\leqslant \phi((\int_0^{\max\{d(Pt,Qt),d(Pt,St),d(Qt,Tt),\frac{d(Pt,Tt)+d(Qt,St)}{2}\}} \varphi(t)dt)^r)$$

$$2(\int_0^{d(St,t)} \varphi(t)dt)^r \leqslant \phi((\int_0^{\max\{d(t,St),0,0,\frac{d(St,t)+d(t,St)}{2}\}} \varphi(t)dt)^r)$$

$$2(\int_0^{d(St,t)} \varphi(t)dt)^r \leqslant \phi(\int_0^{d(St,t)} \varphi(t)dt)^r < (\int_0^{d(St,t)} \varphi(t)dt)^r, \tag{2.26}$$

this contradiction implies that $St = t$ and hence

$$Pt = \{St\} = \{t\}. \tag{2.27}$$

From (2.22) and (2.27), we have

$$Qt = \ Pt = \{St\} = \{Tt\} = \{t\}$$

Then, S, T, P and Q have a common fixed point. The uniqueness of the common fixed point follows easily from condition (2). We get the same conclusion if we consider (4) instead of (3).

if we put $S = T$ and $r = 1$ in Theorem 2.2, we get the following Corollary:

COROLLARY 2.4 Let (X, d) be a metric space and let $S : X \to X$; $P, Q : X \to B(X)$ be maps. Suppose that

(1) S is surjective,

(2) $$\Big(\int_0^{d(Sx,Qy)} \varphi(t)dt\Big)^r + \Big(\int_0^{d(Px,Sy)} \varphi(t)dt\ \Big)^r$$

$$\leqslant \phi\Big(\Big(\int_0^{\max\{d(Px,Qy),d(Px,Sx),d(Qy,Sy),\frac{d(Px,Sy)+d(Qy,Sx)}{2}\}} \varphi(t)dt\Big)^r\Big),$$

for all $x, y \in X$, and φ, ϕ are as in Theorem 2.2 If either,
(III) S and P are subcompatible D-maps; S and Q are subcompatible, or
(IV) S and Q are subcompatible D-maps; S and P are subcompatible.
Then S, P and Q have a unique common fixed point $t \in X$ such that

$$Pt = Qt = \{St\} = \{t\}.$$

Now, we generalize Theorem 2.2 by giving the following Theorem:

THEOREM 2.3 Let S, T be self$-$maps of a metric space (X, d) and let P_n, where $n = 1, 2, 3, ...$ be maps from X into $B(X)$ satisfying the following conditions: (1) S and T are surjective,

(2) $$\Big(\int_0^{d(Sx,P_{n+1}y)} \varphi(t)dt\Big)^r + \Big(\int_0^{d(P_nx,Ty)} \varphi(t)dt\Big)^r$$

$$\leqslant \phi\Big(\Big(\int_0^{\max\{d(P_nx,P_{n+1}y),d(P_nx,Sx),d(P_{n+1}y,Ty),\frac{d(P_nx,Ty)+d(P_{n+1}y,Sx)}{2}\}} \varphi(t)dt\Big)^r\Big)$$

for all $x, y \in X$, and φ, ϕ and r are as in Theorem 2.2. If either, (3) S and P_1 are subcompatible D-maps; T and P_2 are subcompatible, or (4) T and P_2 are subcompatible D-maps; S and $P_1$are subcompatible. Then, S, T and P_n have a unique common fixed point $t \in X$ such that
$P_nt = \{Tt\} = \{St\} = \{t\}$. for $n = 1, 2, 3, ...$

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