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# Four orthogonal polynomials connected to a two parameter function of Mittag-Leffler type

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Abstract. In the present paper, first we describe the orthogonality relations between denominator polynomials of $[n-1/n]$ Pade approximants and related power series expansion of two parameter function of Mittag-Leffler type; next we compute four orthogonal polynomials which are extracted from numerator as well as denominator polynomials of both even and odd order convergents of the regular C-fraction connected to Pade approximants. The two orthogonal polynomials extracted from denominators are shown to be classical orthogonal polynomials and two orthogonal polynomials extracted from numerator are shown to be non-classical orthogonal polynomials.

## 1. Introduction

The theory of orthogonal polynomials [**5, 9**] has one of its origins in Pade approximants given by certain types of continued fractions. Hence the orthogonal polynomials have a close link with the theory of Pade approximation. The Pade approximation to a function, $f$ represented by a power series

$$f(x) = \sum_{i=0}^{\infty} c_i x^i$$

is a type of rational fraction approximation [**1, 2, 3**] in the form $P_m(x)/Q_n(x)$ satisfying

$$Q_n(x)f(x) - P_m(x) = \mathbf{O}(x^{m+n+1})$$

and it is called the $(m, n)-$ order Pade approximant to $f(x)$, denoted by $[m/n]_f(x)$. They can be arranged in a two dimensional array called Pade table. If $m = n$, the Pade approximants occupy the main diagonal of the table. The denominator as well as numerator polynomials of Pade approximation are orthogonal with respect

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to their linear moment functional $L : \mathbb{P} \longrightarrow \mathbb{R}$ from the space of all polynomials over $\mathbb{R}$ into $\mathbb{R}$ which has $n^{th}$ moment same as the coefficient of $x^n$ in a known power series called moment generating function.

According to Favard's theorem [**6, 8, 10**] the necessary and sufficient condition for orthogonality of $P_n(x)$ is to satisfy the following three term recurrence relation:

$$\begin{aligned} P_{-1}(x) &:= 0, \qquad P_0(x) := 1, \\ P_n(x) &:= (x - c_n)P_{n-1}(x) - \lambda_n P_{n-2}(x), \; n = 1, 2, 3, 4, \ldots, \end{aligned} \tag{1.1}$$

where $c_n$ 's are real and $\lambda_n$ 's are non zero numbers. The orthogonality relation [**6, 8, 10**] is given by

$$L\{P_m(x)P_n(x)\} = \begin{cases} 0, & m \neq n; \\ \lambda_1\lambda_2 \cdots \lambda_{n+1}, & m = n. \end{cases} \tag{1.2}$$

In addition, if both $P_n(x)$ and $\dfrac{P'_{n+1}(x)}{n+1}$ are orthogonal polynomials with respect to their linear moment functionals, then the pair $\left\{P_n(x), \dfrac{P'_{n+1}(x)}{n+1}\right\}$ is called classical orthogonal polynomials [**4, 6**].

Motivated strongly by the above works, in the present paper, four orthogonal polynomials are extracted from numerator as well as denominator polynomials of both even and odd order convergents of a regular C-fraction connected to Pade approximants for power series expansion of two parameter function of Mittag-Leffler type. In Section two, we compute four sequences of orthogonal polynomials. In the last Section, the two orthogonal polynomials extracted from denominators are shown to be classical orthogonal polynomials and two orthogonal polynomials extracted from numerators are shown to be non-classical orthogonal polynomials.

## 2. Computation of four orthogonal polynomials

Two parameter function of Mittag-Leffler type [**7, 11**] which is very useful for solving fractional differential equation is given by

$$E_{\alpha,\beta}(x) = \sum_{n=0}^{\infty} \frac{x^n}{\Gamma(\alpha n + \beta)}.$$

For $\alpha = \beta = 1$,

$$E_{1,1}(x) = \sum_{n=0}^{\infty} \frac{x^n}{\Gamma(n+1)} = \sum_{n=0}^{\infty} \frac{x^n}{n!} = e^x.$$

Consider,

$$\begin{aligned} 1 - \frac{x}{1} + \frac{x^2}{1 \cdot 3} - \frac{x^3}{1 \cdot 3 \cdot 5} + \cdots &= \Gamma\left(\frac{1}{2}\right) \sum_{n=0}^{\infty} \frac{\left(\frac{-x}{2}\right)^n}{\left(\frac{1}{2} + (n-1)\right) \cdots \left(\frac{1}{2} + 1\right) \frac{1}{2} \, \Gamma\left(\frac{1}{2}\right)} \\ &= \sqrt{\pi} \sum_{n=0}^{\infty} \frac{\left(\frac{-x}{2}\right)^n}{\Gamma\left(n + \frac{1}{2}\right)} \\ &= \sqrt{\pi} \, E_{1,\frac{1}{2}}\left(\frac{-x}{2}\right). \end{aligned}$$

**Confluent hypergeometric representation [2, 9]:**

$$1-\frac{x}{1}+\frac{x^2}{1\cdot 3}-\frac{x^3}{1\cdot 3\cdot 5}+\cdots = \sum_{n=0}^{\infty}\frac{1(1+1)\cdots(1+(n-1))}{\frac{1}{2}\left(\frac{1}{2}+1\right)\cdots\left(\frac{1}{2}+(n-1)\right)}\frac{\left(\frac{-x}{2}\right)^n}{n!} = {}_1F_1\left(1,\frac{1}{2};\frac{-x}{2}\right).$$

**Regular C-fraction representation for ${}_1F_1(1,\beta;z)$ [2, 9]:**

$${}_1F_1(1,\beta;z)=\frac{1}{1}\,_{-}\,\frac{\frac{1}{\beta}z}{1}\,_{+}\,\frac{\frac{1}{\beta\ (\beta+1)}z}{1}\,_{-}\,\frac{\frac{\beta}{(\beta+1)\ (\beta+2)}z}{1}\,_{+\cdots+}\,\frac{\frac{n}{(\beta+2n-2)\ (\beta+2n-1)}z}{1}\,_{-}\,\frac{\frac{\beta+(n-1)}{(\beta+2n-2)\ (\beta+2n-1)}z}{1}\,_{+\cdots}$$

For $\beta=\frac{1}{2}$, and $z=\frac{-x}{2}$, we obtain

$$F(x)={}_1F_1\left(1,\frac{1}{2};\frac{-x}{2}\right)=\frac{1}{1}\,_{+}\,\frac{x}{1}\,_{+}\,\frac{\frac{-2}{3}x}{1}\,_{+}\,\frac{\frac{1}{15}x}{1}\,_{+\cdots+}\,\frac{\frac{-2n}{(4n-3)(4n-1)}x}{1}\,_{+}\,\frac{\frac{(2n-1)}{(4n-1)(4n+1)}x}{1}\,_{+\cdots} \tag{2.1}$$

In the context of Pade table [2, 3], the continued fraction provides a staircase sequence of Pade approximants
$[0/0]_f(x)$, $[0/1]_f(x)$, $[1/1]_f(x)$, $[1/2]_f(x)$, $[2/2]_f(x)\ldots$ $[n-1/n]_f(x)$, $[n/n]_f(x),\ldots$
which are given by

$$\frac{A_1}{B_1}=\frac{1}{1}=\frac{P_0^{(0,0)}}{Q_0^{(0,0)}},\qquad \frac{A_3}{B_3}=\frac{1-\frac{2}{3}x}{1+\frac{1}{3}x}=\frac{P_1^{(1,1)}}{Q_1^{(1,1)}},\ \ldots\ ,\frac{A_{2n+1}}{B_{2n+1}}=\frac{P_n^{(n,n)}}{Q_n^{(n,n)}}$$

and

$$\frac{A_2}{B_2}=\frac{1}{1+x}=\frac{P_0^{(0,1)}}{Q_0^{(0,1)}},\qquad \frac{A_4}{B_4}=\frac{1-\frac{3}{5}x}{1+\frac{2}{5}x+\frac{1}{15}x^2}=\frac{P_1^{(1,2)}}{Q_1^{(1,2)}},\ \ldots\ ,\frac{A_{2n+2}}{B_{2n+2}}=\frac{P_n^{(n-1,n)}}{Q_n^{(n-1,n)}}.$$

**The even order convergents:**

Let us make use of definitions of even parts of continued fraction as given in [**12**]. $[n-1/n]_f(x)$ Pade approximants can be computed using the even part of continued fraction (2.1):

$$\frac{1}{1+x}\,_{+}\,\frac{\frac{2}{3}x^2}{1-\frac{3}{5}x}\,_{+\cdots+}\,\frac{\frac{(2n-3)2n}{(4n-5)(4n-3)^2(4n-1)}\ x^2}{1-\frac{3}{(4n-3)(4n+1)}x}\,_{+\cdots}.$$

The $n^{th}$ convergent is given by

$$\frac{A_{2n+2}(x)}{B_{2n+2}(x)}=\frac{\left(1-\frac{3}{(4n-3)(4n+1)}x\right)A_{2n}(x)+\frac{(2n-3)2n}{(4n-5)(4n-3)^2(4n-1)}x^2A_{2n-2}(x)}{\left(1-\frac{3}{(4n-3)(4n+1)}x\right)B_{2n}(x)+\frac{(2n-3)2n}{(4n-5)(4n-3)^2(4n-1)}x^2B_{2n-2}(x)}$$

with

$$\frac{A_2}{B_2}=\frac{1}{1+x},\qquad \frac{A_4}{B_4}=\frac{1-\frac{3}{5}x}{1+\frac{2}{5}x+\frac{1}{15}x^2},\ n=2,3,\ldots\ .$$

**The odd order convergents:**

Let us make use of definitions of odd parts of continued fraction as given in [**12**]. $[n/n]_f(x)$ Pade approximants can be computed using the odd part of continued fraction (2.1):

$$1-\frac{x}{1+\frac{1}{3}x}{}_{+\cdots+}\frac{\frac{2n(2n-1)}{(4n-3)(4n-1)^2(4n+1)}\,x^2}{1-\frac{1}{(4n-1)(4n+3)}x}{}_{-\cdots}.$$

The $n^{th}$ convergent is given by

$$\frac{A_{2n+1}(x)}{B_{2n+1}(x)}=\frac{\left(1-\frac{1}{(4n-1)(4n+3)}x\right)A_{2n-1}(x)+\frac{2n(2n-1)}{(4n-3)(4n-1)^2(4n+1)}x^2A_{2n-3}(x)}{\left(1-\frac{1}{(4n-1)(4n+3)}x\right)B_{2n-1}(x)+\frac{2n(2n-1)}{(4n-3)(4n-1)^2(4n+1)}x^2B_{2n-3}(x)}$$

with

$$\frac{A_1}{B_1}=\frac{1}{1},\qquad \frac{A_3}{B_3}=\frac{1-\frac{2}{3}x}{1+\frac{1}{3}x},\ n=2,3,\dots\ .$$

**The desired orthogonal polynomials:**

$$\begin{aligned}p_n(x)&=x^nA_{2n+2}\left(\frac{1}{x}\right),\qquad q_n(x)=x^nB_{2n}\left(\frac{1}{x}\right),\\ r_n(x)&=x^nA_{2n+1}\left(\frac{1}{x}\right),\qquad s_n(x)=x^nB_{2n+1}\left(\frac{1}{x}\right),\\ n&=0,1,2,\dots,\ \text{where } B_0\left(\frac{1}{x}\right):=1.\end{aligned}$$

**Orthogonality of $q_n(x)$ :**

Consider the series

$$F(x)=1-x+\frac{1}{1\cdot3}x^2-\frac{1}{1\cdot3\cdot5}x^3+\cdots+(-1)^n\frac{1}{1\cdot3\cdot5\cdots(2n-1)}x^n+\cdots\ .$$

The linear moment generating function with respect to $F(x)$ denoted by $L_F$ has $n^{th}$ moment,

$$L_F\{x^n\}=\frac{(-1)^n}{1\cdot3\cdot5\cdots(2n-1)}.$$

The three term recurrence relation of $q_n(x)$ is

(2.2)

$$\begin{aligned}q_{n+1}(x)&=\left(x-\frac{3}{(4n-3)(4n+1)}\right)q_n(x)+\frac{2n(2n-3)}{(4n-5)(4n-3)^2(4n-1)}q_{n-1}(x),\\ q_0(x)&=1,\qquad q_1(x)=x+1,\ n=1,2,3,\dots\ .\end{aligned}$$

As a result of applying (1.1) and (1.2), we obtain the orthogonality of $q_n(x)$ is

$$L_F\{q_m(x)q_n(x)\}=\begin{cases}0, & m\neq n;\\ \lambda_1\lambda_2\cdots\lambda_{n+1}, & m=n.\end{cases}$$

where $\lambda_1=1$ and $\lambda_k=\dfrac{-(2k-2)(2k-5)}{(4k-9)(4k-7)^2(4k-5)},\ k=2,3,\dots,n+1.$

**Orthogonality of $s_n(x)$ :**

Following the literature [**2, 3**], we obtain the series

$$F_1(x)=\frac{1-F(x)}{x}=1-\frac{1}{1\cdot3}x+\cdots+(-1)^n\frac{1}{1\cdot3\cdot5\cdots(2n+1)}x^n+\cdots\ .$$

The linear moment generating function with respect to $F_1(x)$ denoted by $L_{F_1}$ has $n^{th}$ moment

$$L_{F_1}\{x^n\} = (-1)^n \frac{1}{1 \cdot 3 \cdot 5 \cdots (2n+1)}.$$

The three term recurrence relation of $s_n(x)$ is

(2.3)

$$\begin{aligned} s_{n+1}(x) &= \left(x - \frac{1}{(4n-1)(4n+3)}\right) s_n(x) + \frac{2n(2n-1)}{(4n-3)(4n-1)^2(4n+1)} s_{n-1}(x), \\ s_0(x) &= 1, \qquad s_1(x) = x + \frac{1}{3}, \ n = 1, 2, 3, \dots \ . \end{aligned}$$

As a result of applying (1.1) and (1.2), we obtain the orthogonality of $s_n(x)$ is

$$L_{F_1}\{s_m(x)s_n(x)\} = \begin{cases} 0, & m \neq n; \\ \lambda_1\lambda_2 \cdots \lambda_{n+1}, & m = n, \end{cases}$$

where $\lambda_1 = 1$ and $\lambda_k = \dfrac{-(2k-2)(2k-3)}{(4k-7)(4k-5)^2(4k-3)}$, $k = 2, 3, \dots, n+1$.

**Orthogonality of $r_n(x)$ :**

Following the literature [**2, 3**], we obtain the series

$$\frac{1}{F(x)} = 1 + x + d_2x^2 + d_3x^3 + d_4x^4 + \cdots + d_nx^n + \cdots \ .$$

and

$$F_2(x) = \frac{\frac{1}{F(x)} - 1}{x} = 1 + d_2x + d_3x^2 + d_4x^3 + \cdots + d_{n+1}x^n + \cdots \ .$$

The linear moment generating function with respect to $F_2(x)$ denoted by $L_{F_2}$ has $n^{th}$ moment

$$L_{F_2}\{x^n\} = d_{n+1}.$$

The three term recurrence relation of $r_n(x)$ is

$$\begin{aligned} r_{n+1}(x) &= \left(x - \frac{1}{(4n-1)(4n+3)}\right) r_n(x) + \frac{2n(2n-1)}{(4n-3)(4n-1)^2(4n+1)} r_{n-1}(x), \\ r_0(x) &= 1, \qquad r_1(x) = x - \frac{2}{3}, \ n = 1, 2, 3, \dots \end{aligned}$$

As a result of applying (1.1) and (1.2), we obtain the orthogonality of $r_n(x)$ is

$$L_{F_2}\{r_m(x)r_n(x)\} = \begin{cases} 0, & m \neq n; \\ \lambda_1\lambda_2 \cdots \lambda_{n+1}, & m = n, \end{cases}$$

where $\lambda_1 = 1$ and $\lambda_k = \dfrac{-(2k-2)(2k-3)}{(4k-7)(4k-5)^2(4k-3)}$, $k = 2, 3, \dots, n+1$.

Suppose $r_n(x) = x^n + r_{n-1}x^{n-1} + \cdots + r_1x + r_0$. Since $L_{F_2}\{r_0(x)r_n(x)\} = 0$, we can compute $d_n$ using

$$d_n = -[r_{n-1}d_{n-1} + \cdots + r_1d_1 + r_0], \ d_0 = 1, \ n = 1, 2, \dots \ .$$

**Orthogonality of $p_n(x)$ :**

Following the literature [**2, 3**], we obtain the series

$$F_3(x) = \frac{3}{2}\left(\frac{\frac{1}{F(x)} - 1 - x}{x^2}\right) = 1 + \frac{3}{5}x + e_2x^2 + e_3x^3 + \cdots + e_nx^n + \cdots \ .$$

The linear moment generating function with respect to $F_3(x)$ denoted by $L_{F_3}$ has $n^{th}$ moment

$$L_{F_3}\{x^n\} = e_n.$$

The three term recurrence relation of $p_n(x)$ is

$$\begin{aligned} p_{n+1}(x) &= \left(x - \frac{3}{(4n+1)(4n+5)}\right) p_n(x) + \frac{(2n-1)(2n+2)}{(4n-1)(4n+1)^2(4n+3)} p_{n-1}(x), \\ p_0(x) &= 1, \qquad p_1(x) = x - \frac{3}{5}, \ n = 1, 2, 3, \dots . \end{aligned}$$

As a result of applying (1) and (2), we obtain the orthogonality of $p_n(x)$ is

$$L_{F_3}\{p_m(x)p_n(x)\} = \begin{cases} 0, & m \neq n; \\ \lambda_1\lambda_2\cdots\lambda_{n+1}, & m = n, \end{cases}$$

where $\lambda_1 = 1$ and $\lambda_k = \dfrac{-2k(2k-3)}{(4k-5)(4k-3)^2(4k-1)}$, $k = 2, 3, \dots, n+1$.

Suppose $p_n(x) = x^n + p_{n-1}x^{n-1} + \cdots + p_1 x + p_0$. Since $L_{F_3}\{p_0(x)p_n(x)\} = 0$, we can compute $e_n$ using

$$e_n = -[p_{n-1}e_{n-1} + \cdots + p_1 e_1 + p_0], \ e_0 = 1, \ n = 1, 2, \dots .$$

## 3. Classical orthogonal polynomials

The following theorem [**4**], gives necessary and sufficient conditions for classical orthogonality of polynomials:

THEOREM 3.1. $\left\{P_n(x), \dfrac{d}{dx}\left(\dfrac{P_{n+1}(x)}{n+1}\right)\right\}$ *is a pair of classical orthogonal polynomials if and only if*

*A.* $P_n(x)$ *form orthogonal polynomials with respect to* $L$.

*B.* $P_n(x) = \dfrac{d}{dx}\left(\dfrac{P_{n+1}(x)}{n+1}\right) - \alpha_n \dfrac{d}{dx}\left(\dfrac{P_n(x)}{n}\right) - \alpha_{n-1}\dfrac{d}{dx}\left(\dfrac{P_{n-1}(x)}{n-1}\right)$, *where* $\alpha_n$ *and* $\alpha_{n-1}$ *are non-zero numbers.*

THEOREM 3.2. *The coefficients of* $q_n(x)$ *and* $s_n(x)$ *can be explicitly computed and hence*

$$q_n(x) = x^n + \sum_{r=1}^{n} \frac{\binom{n}{r}}{(4n-3)(4n-5)\cdots(4n-2r-1)} x^{n-r}, \ n = 0, 1, 2, \dots$$

$$\text{and } s_n(x) = x^n + \sum_{r=1}^{n} \frac{\binom{n}{r}}{(4n-1)(4n-3)\cdots(4n-2r+1)} x^{n-r}, \ n = 0, 1, 2, \dots .$$

PROOF. We obtain the result by using the recurrence relation of $q_n(x)$ and $s_n(x)$ given by (2.2) and (2.3) respectively and the principle of mathematical induction on $n$. $\square$

THEOREM 3.3. $q_n(x)$ *is a classical orthogonal polynomial because*

$A_1$. $q_n(x)$ *is orthogonal polynomial with respect to* $L_F$.

$B_1$. $$q_n(x) = \frac{d}{dx}\left(\frac{q_{n+1}(x)}{n+1}\right) + \frac{4n}{(4n+1)(4n-3)}\frac{d}{dx}\left(\frac{q_n(x)}{n}\right) + \frac{4n(n-1)}{(4n-1)(4n-3)^2(4n-5)}\frac{d}{dx}\left(\frac{q_{n-1}(x)}{n-1}\right)$$

*and $s_n(x)$ is a classical orthogonal polynomial because*

$A_2$. *$s_n(x)$ is orthogonal polynomial with respect to $L_{F_1}$.*

$B_2$.
$$s_n(x) = \frac{d}{dx}\left(\frac{s_{n+1}(x)}{n+1}\right) + \frac{4n}{(4n-1)(4n+3)}\frac{d}{dx}\left(\frac{s_n(x)}{n}\right) + \frac{4n(n-1)}{(4n+1)(4n-1)^2(4n-3)}\frac{d}{dx}\left(\frac{s_{n-1}(x)}{n-1}\right).$$

Proof. The result follows from Theorem 3.1 and Theorem 3.2. □

In Section two, we have already shown that $r_n(x)$ and $p_n(x)$ are orthogonal polynomials with respect to $L_{F_2}$ and $L_{F_3}$ respectively. Hence they satisfy the condition $A$ of Theorem 3.1. But they do not satisfy the condition $B$ of Theorem 3.1.

**For example:**

$$\begin{aligned} r_4(x) \quad = \quad & \frac{d}{dx}\left(\frac{r_5(x)}{5}\right) - \frac{41}{15\cdot 19}\frac{d}{dx}\left(\frac{r_4(x)}{4}\right) - \frac{2809}{4\cdot 5\cdot 13\cdot 15\cdot 17}\frac{d}{dx}\left(\frac{r_3(x)}{3}\right) \\ & -\frac{22051}{2\cdot 11\cdot 13\cdot 15^2\cdot 17}\frac{d}{dx}\left(\frac{r_2(x)}{2}\right) - \frac{1814713}{4\cdot 5\cdot 11\cdot 13\cdot 15\cdot 19\cdot 21}\frac{d}{dx}\left(\frac{r_1(x)}{1}\right). \end{aligned}$$

$$\begin{aligned} p_4(x) \quad = \quad & \frac{d}{dx}\left(\frac{p_5(x)}{5}\right) - \frac{16}{7\cdot 21}\frac{d}{dx}\left(\frac{p_4(x)}{4}\right) - \frac{7198}{5\cdot 7\cdot 17\cdot 19\cdot 21}\frac{d}{dx}\left(\frac{p_3(x)}{3}\right) \\ & -\frac{734456}{5^2\cdot 13\cdot 17\cdot 19\cdot 21^2}\frac{d}{dx}\left(\frac{p_2(x)}{2}\right) - \frac{2183848}{5^2\cdot 11\cdot 13^2\cdot 17\cdot 21}\frac{d}{dx}\left(\frac{p_1(x)}{1}\right). \end{aligned}$$

Hence $r_n(x)$ and $p_n(x)$ are nonclassical orthogonal polynomials.

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## References

[1] Ahmad F., *The Orthogonal Polynomials and the Pade Approximation* , Appl. Math. Mech.; 19(7)(1998), 663-668.
[2] Baker Jr. G. A. , *Essentials of Pade Approximants*, Acad. Press, New York, 1975.
[3] Baker Jr. G. A. and Graves-Morris P. , *Pade Approximants*, Camb. Univ. Press, New York, 1996.
[4] Branquihinho A. , *A Note on Semi-Classical Orthogonal Polynomials*, Bull. Belg. Math. Soc. 3(1996), 01-12.
[5] Brezinski C. , *History of Continued Fractions and Pade Approximants*, Springer - Verlag, Berlin Heidelberg, 1991.
[6] Chihara T. S. , *An Introduction to Orthogonal Polynomials*, Gordon and Breach, New York, 1978.
[7] Erdelyi A., *Higher Transcendental Functions*, McGraw - Hill, New York, 1955.

[8] Gautchi W., *Orthogonal Polynomials: Computation and Approximation*, Numerical Mathematics and Scientific Computation, Oxford Univ. Press, New York, 2004.
[9] Jones W. B. and Thron W. J., *Continued Fractions: Analytic Theory of and Continuation*, Camb. Univ. Press, New York, 1984.
[10] Hahn W., *On Differential Equations for Orthogonal Polynomials*, Funk. Ekva. 21(1978), 1-9.
[11] Podlubny I., *Fractional Differential Equations*, Acad. Press, San Diego, 1998.
[12] Wall H. S., *Analytic Theory of Continued Fractions*, D. Van Nostrand Company, New York, 1948.

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