THE AVERAGE LOWER DOMINATION NUMBER OF GRAPHS

Ersin Aslan and Alpay Kirlangic

Abstract. The average lower domination number \( \gamma_{av}(G) \) is defined as

\[
\frac{1}{|V(G)|} \sum_{v \in V(G)} \gamma_{v}(G)
\]

where \( \gamma_{v}(G) \) is the minimum cardinality of a maximal dominating set that contains \( v \). In this paper, the average lower domination number of complete \( k \)-ary tree and \( B_n \) tree are calculated. Moreover we obtain the \( \gamma_{av}(G^*) \) for thorn graph \( G^* \). Finally we compute the \( \gamma_{av}(G_1 + G_2) \) of \( G_1 \) and \( G_2 \).

1. Introduction

A network is modelled with graphs in a situation which the centers are equal to the vertex of graphs and connection lines are equal to the edges of a graph. A graph \( G \) is denoted by \( G = (V(G), E(G)) \), where \( V(G) \) and \( E(G) \) are vertex and edge sets of \( G \), respectively. Let \( v \) be a vertex in \( V(G) \).

In a graph \( G = (V(G), E(G)) \), a subset \( S \subseteq V(G) \) of vertices is a dominating set if every vertex in \( V(G) - S \) is adjacent to at least one vertex of \( S \). The domination number of \( \gamma(G) \) is the minimum cardinality of a dominating set. A dominating set of cardinality \( \gamma(G) \) is called a \( \gamma(G) \)-set.

Henning [12] introduced the concept of average domination. The lower domination number, denoted by \( \gamma_{v}(G) \) is the minimum cardinality of a dominating set of \( (G) \) that contains \( v \).

The average lower domination number \( \gamma_{av}(G) \) is defined as

\[
\frac{1}{|V(G)|} \sum_{v \in V(G)} \gamma_{v}(G)
\]

where \( \gamma_{v}(G) \) is the minimum cardinality of a maximal dominating set that contains \( v \).

2010 Mathematics Subject Classification. 05C35, 05C40, 05C69, 68M10, 68R10.

Key words and phrases. Connectivity, Network Design and Communication, Vulnerability, Average Lower Domination Number.
Clearly for a vertex \( v \) in a graph \( G \), \( \gamma(G) \leq \gamma_{av}(G) \) with equality if and only if \( v \) belongs to a \( \gamma(G) \)-set. Consequently, \( \gamma_{av}(K_n) = 1 \), while for a cycle \( C_n \) on \( n \geq 3 \) vertices, \( \gamma_{av}(C_n) = \gamma(C_n) = \left\lceil \frac{3}{2} \right\rceil \).

**Proposition 1.1** ([12]). For any graph \( G \) of order \( n \) with domination number \( \gamma \), \( \gamma_{av}(G) \leq \gamma + 1 - \frac{2}{n} \), with equality if and only if \( G \) has a unique \( \gamma(G) \)-set.

**Theorem 1.1** ([12]). If \( T \) is a tree of order \( n \geq 4 \) then \( \gamma_{av}(T) \leq \frac{n}{2} \) with equality if and only if \( T \) is the corona of a tree.

In this paper, the average lower domination number of complete \( k \)-ary tree and \( B_n \) tree are calculated. Moreover we obtain the \( \gamma_{av}(G^*) \) for thorn graph \( G^* \). Finally we compute the \( \gamma_{av}(G_1 + G_2) \) of \( G_1 \) and \( G_2 \).

2. **Average Lower Domination Number Of Some Graphs**

Firstly we give the definition of a complete \( k \)-ary tree with depth \( n \). The average lower domination number of complete \( k \)-ary tree are calculated. Moreover we obtain \( \gamma_{av}(B_n) \) for binomial tree and \( \gamma_{av}(G^*) \) for thorn graph \( G^* \).

**Definition 2.1.** ([3]) A complete \( k \)-ary tree with depth \( n \) is all leaves have the same depth and all internal vertices have degree \( k \). A complete \( k \)-ary tree has \( \frac{k^{n+1} - 1}{k-1} \) vertices and \( \frac{k^{n+1} - 1}{k-1} - 1 \) edges.

**Theorem 2.1.** Let \( G \) be a complete \( k \)-ary tree with depth \( n \). Then

\[
\gamma_{av}(G) = \begin{cases} 
\gamma(G) + 1 - \frac{\gamma(G) + k}{|V(G)|}, & n \equiv 0 \pmod{3} \\
\gamma(G) + 1 - \frac{\gamma(G)}{|V(G)|}, & \text{otherwise}
\end{cases}
\]

**Proof.** If \( G \) is a \( k \)-ary tree with depth \( n \) then \( |V(G)| = \frac{k^{n+1} - 1}{k-1} \). We have two cases for \( n \) to find the average lower average number of \( G \).

**Case 1.** If \( n \equiv 1 \pmod{3} \) or \( n \equiv 2 \pmod{3} \) then \( G \) has a unique \( \gamma(G) \)-set. The minimal domination set of \( G \) contains the vertices on the levels \( (n - 1 - 3i) \) for \( 0 \leq i \leq \left\lfloor \frac{n}{3} \right\rfloor \). Let vertices set of \( G \) be \( V(G) = V(G_1) \cup V(G_2) \) where, \( V(G_1) \): The set contains the vertices on the levels \( (n - 1 - 3i) \) for \( 0 \leq i \leq \left\lfloor \frac{n}{3} \right\rfloor \), \( V(G_2) \): The set contains the vertices of \( V(G) - V(G_1) \).

\( i \) If \( v \in V(G_1) \), then \( \gamma_v(G) = \gamma(G) \) since the vertex \( v \) is in the dominating set. Since this equality is satisfied for every vertex of \( V(G_1) \) we have

\[
\Sigma_{v \in V(G_1)} \gamma_v(G) = \gamma(G) \cdot \gamma(G).
\]

\( ii \) If \( v \in v(G_2) \), then \( \gamma_v(G) = \gamma(G) + 1 \) since the vertex \( v \) is not in the dominating set. Since this equality is satisfied for every vertex in \( V(G_2) \), we have

\[
\Sigma_{v \in V(G_2)} \gamma_v(G) = (|V(G)| - \gamma(G)) \cdot (\gamma(G) + 1).
\]

Consequently,

\[
\gamma_{av}(G) = \frac{1}{|V(G)|} (\Sigma_{v \in V(G)} \gamma_v(G) = \frac{1}{|V(G)|} (\Sigma_{v \in V(G_1)} \gamma_v(G) + \Sigma_{v \in V(G_2)} \gamma_v(G))
\]

\[
= \frac{1}{|V(G)|} [(\gamma(G) \cdot \gamma(G)) + (|V(G)| - \gamma(G)) \cdot (\gamma(G) + 1)]
\]
\[ \gamma(G) + 1 - \frac{\gamma(G)}{|V(G)|}. \tag{1} \]

**Case 2.** If \( G \) is a \( k \)-ary tree with depth \( n \) and \( n \equiv 0 \pmod{3} \), then \( G \) has \( k + 1 \) domination sets which give the domination number of \( G \). The minimal domination set of \( G \) contains the vertices on the levels \((n - 1 - 3i)\) for \( 0 \leq i \leq \left\lfloor \frac{n}{3} \right\rfloor \). But in this case the vertex on the 0th level cannot be reached. Therefore the vertex on the 0th level or one of the vertices on the 1st level should be taken to the dominating set. Hence there are \( k + 1 \) dominating sets according to the choice of vertices.

\( i) \) If \( v \in \gamma(G) \)-set, then \( \gamma_v(G) = \gamma(G) \) since the vertex \( v \) is in the dominating set. We have to repeat this process for \( k + \gamma(G) \) vertices. Therefore

\[ \sum_{v \in V(G)} \gamma_v(G) = (\gamma(G) + k) \cdot \gamma(G). \]

\( ii) \) If \( v \notin \gamma(G) \)-set, then \( \gamma_v(G) = \gamma(G) \) since the vertex \( v \) is in the dominating set. We have to repeat this process for \(|V(G)| - k - \gamma(G) \) vertices. Hence,

\[ \sum_{v \in V(G)} \gamma_v(G) = (|V(G)| - (\gamma(G) + k)) \cdot (\gamma(G) + 1). \]

As a result

\[ \gamma_{av}(G) = \frac{1}{|V(G)|} \left( (\gamma(G) + k) \cdot \gamma(G) + (|V(G)| - (\gamma(G) + k)) \cdot (\gamma(G) + 1) \right) \]

\[ = \gamma(G) + 1 - \frac{\gamma(G) + k}{|V(G)|}. \tag{2} \]

By (1) and (2) the proof is completed. \qed

**Definition 2.2.** ([3]) The binomial tree of order \( n \geq 0 \) with root \( R \) is the tree \( B_n \) defined as follows.

1) If \( n = 0 \), \( B_n = B_0 = R \), i.e., the binomial tree of order zero consists of a single node \( R \).

2) If \( n > 0 \), \( B_n = R, B_0, B_1, \ldots, B_{n-1} \), i.e., the binomial tree of order \( n > 0 \) comprises the root \( R \), and \( n \) binomial subtrees, \( B_0, B_1, \ldots, B_{n-1} \).

**Theorem 2.2.** Let \( B_n \) be a binomial tree. Then \( \gamma_{av}(B_n) = 2^{n-1} \).

**Proof.** Any binomial tree \( B_n \) consists of \( 2^n \) vertices; \( 2^{n-1} \) vertices with degree 1. While the domination set is found, all of the vertices with degree 1 or the vertices adjacent to these vertices should be taken into the set. Therefore the domination number of \( B_n \) is \( \gamma(B_n) = 2^{n-1} \). Obviously the domination set satisfying the domination number can be obtained for every element of \( B_n \). Since \( \gamma_v(B_n) = 2^{n-1} \) for every element \( v \) of \( B_n \). Hence

\[ \sum_{v \in V(B_n)} \gamma_v(B_n) = 2^{n-1} \cdot 2^n. \]

From the definition of average lower domination number we have

\[ \gamma_{av}(B_n) = \frac{1}{2^n} 2^{n-1} \cdot 2^n = 2^{n-1}. \]

\qed
2.3. Let \( p_1, p_2, \ldots, p_n \) be non-negative integers and \( G \) be such a graph, \( V(G) = n \). The thorn graph of the graph, with parameters \( p_1, p_2, \ldots, p_n \), is obtained by attaching \( p_i \) new vertices of degree 1 to the vertex \( u_i \) of the graph \( G \), \( i = 1, 2, \ldots, n \). The thorn graph of the graph \( G \) will be denoted by \( G^* \) or by \( G^*(p_1, p_2, \ldots, p_n) \), if the respective parameters need to be specified.

**Theorem 2.3.** Let \( G \) be a non complete connected graph with order \( n \) and \( G^* \) be a thorn graph of \( G \) with every \( p_i = 1 \). Then
\[
\gamma_{av}(G^*) = n
\]

**Proof.** The number of vertices of \( G^* \) is \( 2n \). While the domination set is found every vertex of degree 1 or the vertex adjacent to it must be taken into the dominating set. Therefore the domination number of \( G^* \) is \( \gamma(G^*) = n \). Thus the domination set satisfying the domination number can be obtained for every element of \( G^* \). Since \( v(G^*) = n \) for every element \( v \) of \( G^* \), therefore
\[
\sum_{v \in G^*} \gamma_e(G^*) = 2n.n.
\]
From the definition of average lower domination number we have
\[
\gamma_{av}(G^*) = \frac{1}{2n} 2n.n = n.
\]

**Theorem 2.4.** Let \( G^* \) be a thorn graph of \( G \) with every \( p_i > 1 \). Then
\[
\gamma_{av}(G^*) = |V(G)| + 1 - \frac{|V(G)|}{|V(G^*)|}
\]

**Proof.** Let \( G^* \) be a thorn graph of \( G \) with every \( p_i > 1 \). Obviously \( \gamma(G^*) = |V(G)| \), hence all of the vertices of \( G \) should be taken into the dominating set. Let vertices set of \( G^* \) be \( V(G^*) = V(G_1) \cup V(G_2) \) where,\( V(G_1) \): The set contains the vertices of graph \( G \).\( V(G_2) \): The set contains the vertices of \( V(G) - V(G_1) \) Then we have
\[
\sum_{v \in V(G^*)} \gamma_e(G^*) = \sum_{v \in V(G_1)} \gamma_e(G^*) + \sum_{v \in V(G_2)} \gamma_e(G^*)
\]

(i) If \( v \in V(G_1) \), then \( \gamma_e(G^*) = |V(G)| \). We have to repeat this process for every vertices of \( V(G_1) \). Hence
\[
\sum_{v \in V(G_1)} \gamma_e(G^*) = |V(G)||V(G)|.
\]

(ii) If \( v \in V(G_2) \), then \( \gamma_e(G^*) = |V(G)| + 1 \). We have to repeat this process for every vertices of \( V(G_2) \). So,
\[
\sum_{v \in V(G_2)} \gamma_e(G^*) = (|V(G^*)| - |V(G)|)(|V(G)| + 1).
\]
From the definition of average lower domination number we have
\[
\gamma_{av}(G^*) = \frac{1}{|V(G)|}(|V(G)| - |V(G^*)| + 1) = |V(G)| + 1 - \frac{|V(G)|}{|V(G^*)|}.
\]
\[\blacksquare\]

3. Join Operation

We give some result of average lower domination number of \( G_1 + G_2 \).

**Theorem 3.1.** If \( G_1 \) and \( G_2 \) are two graphs with domination numbers different from 1, then \( \gamma_{av}(G_1 + G_2) = 2 \).

**Proof.** The domination set of \( G_1 + G_2 \) is formed by the pairs of \((x, y)\) such that \( x \) is any vertex of the graph \( G_1 \) and \( y \) is any vertex of \( G_2 \). Since a domination set can be formed by every element \( v \) in \( G_1 + G_2 \), we have \( \gamma(v(G_1 + G_2)) = 2 \). Then by the definition
\[
\gamma_{av}(G_1 + G_2) = \frac{1}{|V(G_1 + G_2)|}2|V(G_1 + G_2)| = 2.
\]
\[\blacksquare\]

**Theorem 3.2.** Let \( G_1 \) and \( G_2 \) be two graphs with orders \( m \) and \( n \), respectively, and let \( \gamma(G_1) = 1 \) or \( \gamma(G_2) = 1 \). Let \( a \) be the number of the domination sets satisfying \( \gamma(G_1) = 1 \) and \( b \) be the number of the domination sets satisfying \( \gamma(G_2) = 1 \), then
\[
\gamma_{av}(G_1 + G_2) = \begin{cases} 
2 - \frac{a}{m+n}, & \gamma(G_1) = 1 \text{ and } \gamma(G_2) \neq 1 \\
2 - \frac{b}{m+n}, & \gamma(G_1) \neq 1 \text{ and } \gamma(G_2) = 1 \\
2 - \frac{a+b}{m+n}, & \gamma(G_1) = 1 \text{ and } \gamma(G_2) = 1
\end{cases}
\]

**Proof.** The proof is done in three cases according to the domination number of the graphs \( G_1 \) and \( G_2 \).

**Case 1:** Let \( \gamma(G_1) = 1 \) and \( \gamma(G_2) \neq 1 \). In this case,

(i) If \( v \in V(G_1) \) and an element of one of the \( a \) sets satisfying \( \gamma(G_1) = 1 \) then \( \gamma_\nu(G_1 + G_2) = 1 \) and this equality is satisfied for \( a \) vertices.

(ii) If \( v \in V(G_1 + G_2) \) which doesn’t satisfy \( \gamma(G_1) = 1 \), then \( \gamma_\nu(G_1 + G_2) = 2 \), and this equality is satisfied for \( m+n-a \) vertices. Therefore,
\[
\gamma_{av}(G_1 + G_2) = \frac{1}{m+n}.(a + (m + n - a).2) = 2 - \frac{a}{m+n}. \tag{3}
\]

**Case 2:** Let \( \gamma(G_1) \neq 1 \) and \( \gamma(G_2) = 1 \).

(i) If \( v \in V(G_2) \) and an element of one of the \( b \) sets satisfying \( \gamma(G_2) = 1 \) then \( \gamma_\nu(G_1 + G_2) = 1 \) and this equality is satisfied for \( b \) vertices.

(ii) If \( v \in V(G_1 + G_2) \) which doesn’t satisfy \( \gamma(G_2) = 1 \), then \( \gamma_\nu(G_1 + G_2) = 2 \), and this equality is satisfied for \( m+n-b \) vertices. Hence we have,
\[
\gamma_{av}(G_1 + G_2) = \frac{1}{m+n}.(b + (m + n - b).2) = 2 - \frac{b}{m+n}. \tag{4}
\]
Case 3: Let $\gamma(G_1) = 1$ and $\gamma(G_2) = 1$. Then

(i) If $v \in V(G_1)$ and an element of one of the $a$ sets satisfying $\gamma(G_1) = 1$ then $\gamma_v(G_1 + G_2) = 1$ and this equality is satisfied for $a$ vertices.

(ii) If $v \in V(G_2)$ and an element of one of the $b$ sets satisfying $\gamma(G_2) = 1$ then $\gamma_v(G_1 + G_2) = 1$ and this equality is satisfied for $b$ vertices.

(iii) If $v \in V(G_1 + G_2)$ which doesn’t satisfy $\gamma(G_1) = 1$ and $\gamma(G_2) = 1$, then $\gamma_v(G_1 + G_2) = 2$, and this equality is satisfied for $m + n - a - b$ vertices. Therefore,

$$\gamma_v(G_1 + G_2) = \frac{1}{m+n}(a + b + (m + n - a - b)2) = 2 - \frac{a+b}{m+n} \tag{5}$$

By (3), (4) and (5) the proof is completed. □

References


Received by editors 25.07.2013; Revised version 11.10.2013; Available online 11.11.2013.

Turgutlu Vocational Training School, Celal Bayar University, Turgutlu-Manisa, Turkey

E-mail address: ersin.aslan@cbu.edu.tr

Department of Mathematics, Ege University, 35100 Bornova-Izmir, Turkey

E-mail address: alpay.kirlangicc@ege.edu.tr