BULLETIN OF THE INTERNATIONAL MATHEMATICAL VIRTUAL INSTITUTE
ISSN 1840-4367
Vol. 3(2013), 123-126
Available on: www.imvibl.org

Former
BULLETIN OF THE SOCIETY OF MATHEMATICIANS BANJA LUKA
ISSN 0354-5792 (o), ISSN 1986-521X (p)

A NOTE ON CORADICAL OF CONSISTENT SUBSET OF σ-REFLEXIVE SEMIGROUPS WITH APARTNESS

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Abstract. The concept of σ-reflexive semigroups is due to Chacron and Thierrin in 1972. In the present paper we discuss σ-reflexive semigroups with apartness. We prove some fundamental properties of these semigroups.

1. Introduction and Preliminaries

This investigation is in Semigroup Theory within Constructive Algebra, in sense of the book [**1**], [**3**], [**4**] and [**13**] and papers [**8**]- [**11**]. Constructive Mathematics is developed on Constructive Logic (or Intuitionists Logic ([**13**])) - logic without the Law of Excluded Middle: $P \vee \neg P$. We have to note that 'the crazy axiom' $\neg P \Longrightarrow (P \Longrightarrow Q)$ is included in the Constructive Logic. Precisely, in Constructive Logic the 'Double Negation Law' $P \Longleftrightarrow \neg\neg P$ does not hold, but the following implication $P \Longrightarrow \neg\neg P$ holds even in the Minimal Logic. In Constructive Logic 'Weak Law of Excluded Middle' $\neg P \vee \neg\neg P$ does not hold. It is interesting that in Constructive Logic the following deduction principle $A \vee B, \neg A \vdash B$ holds, but this is impossible to prove without 'the crazy axiom'. An advantage of working in this manner is that proofs and results have more interpretations. On one hand, Bishop's Constructive Mathematics is consistent with the traditional mathematics. On the other hand, the results can be interpreted recursively or intuitively. If we are working constructively, the first problem is to obtain appropriate substitutes of the classical definitions.

Throughout this paper, $S = (S, =, \neq, \cdot)$ always denotes a semigroup with apartness in the sense of the books [**7**], [**13**] and papers [**8**]- [**11**]. The apartness "$\neq$ " on S is a binary relation with the following properties: For every elements x, y and z in S, it holds:

2010 *Mathematics Subject Classification.* Primary 05F65, 03A65, Secondary 20M10, 20M99.
Key words and phrases. Constructive mathematics, semigroup with apartness, reflexive semigroup, ...

$$\neg(x \neq x),\ x \neq y \Longrightarrow y \neq x,\ x \neq y \wedge y = z \Longrightarrow x \neq z,$$
$$x \neq z \Longrightarrow (\forall t \in S)(x \neq t \vee t \neq z).$$

It takes that the semigroup operation is strongly extensional, in the following sense

$$(\forall a, b, x, y \in S)((ay \neq by \Longrightarrow a \neq b) \wedge (xa \neq xb \Longrightarrow a \neq b)).$$

Let T be a subset of S. We say that it is:
– *strongly extensional subset* (see, for example [**1**], [**3**]) iff $(\forall x, y \in S)(x \in T \Longrightarrow x \neq y \vee y \in T)$;
– *consistent subset* (see [**2**]) iff $(\forall x, y \in S)(xy \in T \Longrightarrow x \in T \wedge y \in T)$;
– *completely prime* (see [**2**]) subset of S if for any $x, y \in S$ holds

$$xy \in T \Longrightarrow (x \in T \vee y \in T);$$

– *subsemigroup* of S iff $(\forall x, y \in S)(x \in T \wedge y \in T \Longrightarrow xy \in T)$;
– *filter* of S if T is a consistent subsemigroup of S.

It is easy to show that if T is a consistent subset of semigroup S, then T^C is an ideal of S. (See, for example Remark 1.1. in paper [**11**]) Let us note that the opposite assertion "If J is an ideal of semigroup S then J^C is a consistent subset if S." is not valid in general case.

If T is a subset of S, we define *coradical* of T by

$$cr(T) = \{x \in T : (\forall n \in N)(x^n \in T)\}.$$

It is easy to show that $cr(T) \subseteq T$ and $cr(T) = T$ if T is a filter of S. For a consistent subset T we say that it is *primary* if the implication

$$x \in T \wedge y \in cr(T) \Longrightarrow xy \in T$$

holds.

For undefined notions and notations in the Semigroup theory we refer to books [**2**] and [**6**] and in the Constructive mathematics we refer to books [**1**]- [**4**] and [**7**].

A semigroup S is called *σ-reflexive* if for any subsemigroup H of S $ab \in H (a, b \in S)$ implies $ba \in H$. Clearly, a commutative semigroup is a σ-reflexive semigroup. σ-reflexive semigroup are investigated by the following authors [**5**] and [**12**], for example. σ-reflexive semigroup has some special properties. In this paper we research these semigroups in case when they supplied with the apartness relation. In this case, coradical of a consistent subset is a consistent subset again and so-called primary consistent subset of a σ-reflexive semigroup is a filter of S.

2. Main Results

The following two results on σ-reflexive semigroups are due to Chacron and Thierrin (1972):

PROPOSITION 2.1 ([**5**], Proposition 1). *Any semigroup S is σ-reflexive if ti satisfies the condition:*

$$(\forall a, b \in S)(\exists m \geqslant 1)(ab = (ba)^m).$$

PROPOSITION 2.2 ([**5**], Proposition 2). *Let a, b be any two noncommutative elements of a σ-reflexive semigroup S. Then, for some $m > 1$, $ab = (ab)^m$.*

Generally speaking the coradical of a consistent subset of a semigroup need not to be a consistent subset also. In the first theorem we prove that the coradical of a consistent subset of a σ-reflexive semigroup is also a consistent subset of that semigroup:

THEOREM 2.1. *Coradical of a consistent subset A of a σ-reflexive semigroup S is a consistent subset of S again.*

PROOF. Let a, b be arbitrary elements of semigroup S such that $ab \in cr(A)$. Then for any natural n holds $(ab)^n \in A$. For $n = 1$ we have $a \in A$ and $b \in A$. Suppose that the implication $ab \in c(A) \Longrightarrow a^n \in A \wedge b^n \in A$ is true. Since

$$(ab)^{n+2} = a(ba)^{n+1}b = a((ab)^m)^{n+1}b = a(ab)^{m(n+1)}b = a^2(ba)^{m(n+1)-1}b^2$$

and after repeating this procedure we have $a^k(ab)^s b^k = (ab)^{n+2} \in A$ for $k \geqslant 1$ and some natural s. From this, by consistency of A, follows

$$(\forall k \leqslant n+1)(a^k \in A \wedge b^k \in A).$$

So, by induction, we have $a \in cr(A)$ and $b \in cr(A)$. Therefore, the set $cr(A)$ is a consistent subset of semigroup S. □

The σ-reflexivity of S is an essential condition in theorem above. In the following example we show that.

EXAMPLE 2.1. Let us consider set $S = \{0, a, b, c, d\}$ where the multiplication is given by

$\cdot$	0	a	b	c	d
0	0	0	0	0	0
a	0	a	0	c	0
b	0	0	b	0	d
c	0	0	c	0	c
d	0	d	0	b	0

This semigroup is not σ-reflexive, as $H = \{b\}$ is a subsemigroup of S containing product dc, but not cd. Now taking $A = \{a, b, c, d\}$, we have $cr(A) = \{a, b\}$ but this is not a consistent subset of S because, for example, $dc = b \in cr(A)$ and $\neg(c \in cr(A))$ and $\neg(d \in cr(A))$.

The following theorem is the main result of this short note.

THEOREM 2.2. *Let S be a reflexive semigroup and let Q be a primary consistent subset of S. Then $cr(Q)$ is a filter of S.*

PROOF. (i) For $a \in cr(Q) = cr(cr(Q))$ and $b \in cr(Q) \subseteq Q$ we have $(\forall i)(a^i \in cr(Q))$ and thus $ab \in cr(Q)Q \subseteq Q$.

(ii) For $a \in cr(Q) = cr(cr(Q)$ and $b \in cr(Q)$ we have $(\forall k)(a^k \in cr(Q))$ and $b^2 \in Q$. Thus, we have $Q \ni a^k b^2 = a^{k-1} \cdot (a \cdot b) \cdot b = a^{k-1} \cdot (ba)^m \cdot b = a^{k-2}(ab)^{m+1}$. So, by consistency of Q, we have to have $a^{k-2} \in Q$ and $(ab)^{m+1} \in Q$. Therefore, we have $(\forall j \leqslant m+1)((ab)^j \in Q)$.

(iii) Suppose that $(\forall k \leqslant n)((ab)^k \in Q)$ is true. Then, from $(\forall k)(a^k \in cr(Q))$ and $(\forall t)(b^t \in Q)$ we have
$Q \ni a^k b^t = a^{k-1}(ab)b^{t-1} = a^{k-1}(ba)^m b^{t-1} = a^{k-2}(ab)^{m+1}b^{t-2}$
$= a^{k-2}((ba)^s)^{m+1}b^{t-2} = a^{k-2}(ba)^{s(m+1)}b^{t-2} = a^{k-3}(ab)^{s(m+1)+1}b^{t-3}$.
After enough number of repetitions we can get $s(m+1)+1 \geqslant n+1$. Thus, by consistency of Q we have $(ab)^{n+1} \in Q$.

(iv) Finally, by induction we have $ab \in cr(Q)$. Therefore, the subset $cr(Q)$ is a filter of S. □

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Received by the editors March 23, 2010; on revised form January 4, 2011 and June 1, 2013; available online July 10, 2013

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