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Even Number of Symmetric Positive Solutions for Sturm-Liouville Two-Point Boundary Value Problems on Time Scales

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Abstract. In this paper, we are concerned with the existence of at least two symmetric positive solutions for the even order boundary value problems on time scales satisfying Sturm-Liouville two-point boundary conditions by using Avery–Henderson fixed point theorem. We also establish the existence of at least $2m$ symmetric positive solutions to the boundary value problem for an arbitrary positive integer m.

1. Introduction

This paper establishes the existence of even number of symmetric positive solutions for the even order boundary value problems on a time scales,

$$(1.1)\qquad (-1)^n y^{(\Delta\nabla)^n}(t) = f(t, y(t), y^{\Delta\nabla}(t), \cdots, y^{(\Delta\nabla)^{(n-1)}}(t)),\ t \in [a, b]_{\mathbb{T}},$$

satisfying Sturm-Liouville type two-point boundary conditions,

$$(1.2)\qquad \begin{cases} \alpha y^{(\Delta\nabla)^i}(a) - \beta y^{(\Delta\nabla)^i\Delta}(a) = 0, \\ \alpha y^{(\Delta\nabla)^i}(b) + \beta y^{(\Delta\nabla)^i\Delta}(b) = 0, \end{cases}$$

for $0 \leqslant i \leqslant n-1$, where $n \geqslant 1$, $f : [a, b]_{\mathbb{T}} \times \mathbb{R}^{+^n} \to \mathbb{R}^+$ is continuous with $a \in \mathbb{T}_{k^n}$, $b \in \mathbb{T}^{k^n}$ for a time scale $\mathbb{T}$, $\alpha > 0, \beta \geqslant 0$ and $\sigma^n(a) < \rho^n(b)$.

In recent years, there is an increasing interest in obtaining positive solutions for boundary value problems on time scales. The theory of time scales was introduced by Hilger [**18**] to unify not only continuous and discrete theory, but also provide an accurate information of phenomena that manifest themselves partly in continuous

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time and partly in discrete time. The time scale calculus would allow to explore a variety of situations in economic, biological, heat transfer, stock markets and epidemic models. Recent results indicate that considerable achievement has been made in the existence of positive solutions of the boundary value problems on time scales. However they did not further provide characteristics of positive solutions such as symmetry. Symmetry has been widely used in science, engineering. The reason is that the symmetry has not only its theoretical value in studying the metric manifolds and symmetric graph and so forth, but also its practical value, for example, we can apply this characteristic to study graph structures and chemistry structures.

The primary purpose of this investigation is to study the symmetry properties of the solutions of even order boundary value problem on time scales. For recent discussions on symmetry properties of solutions of boundary value problems associated to differential equations or finite difference equations or time scales, to mention a few papers along these lines are Davis and Henderson [**10**], Avery, Davis and Henderson [**4**], Davis, Henderson and Wong [**11**], Henderson [**14**], Henderson and Thompson [**16**], Henderson and Wong [**17**], Wong [**21**], Eloe, Henderson and Sheng [**12**] and Avery and Henderson [**6, 7**]. Recently, Henderson, Murali and Prasad [**15**] studied the multiple symmetric positive solutions for two-point even order boundary value problems on time scales,

$$(-1)^n y^{(\Delta\nabla)^n}(t) = f(y(t), y^{\Delta\nabla}(t), \cdots, y^{(\Delta\nabla)^{(n-1)}}(t)),\ t \in [a, b],$$

satisfying the boundary conditions,

$$y^{(\Delta\nabla)^i}(a) = 0 = y^{(\Delta\nabla)^i}(b),\ 0 \leqslant i \leqslant n-1,$$

by using Avery generalization of the Leggett–Williams fixed point theorem.

Motivated by the papers mentioned above, in this paper, we establish the existence of even number of symmetric positive solutions for the BVP (1.1)-(1.2), by using Avery–Henderson fixed point theorem. To establish the symmetric positive solutions for the boundary value problem on time scales, we are dealing with symmetric time scales. By an interval time scale, we mean the intersection of a real interval with a given time scale, that is, $[a, b]_{\mathbb{T}} = [a, b] \cap \mathbb{T}$. An interval time scale $\mathbb{T} = [a, b]$ is said to be a symmetric time scale, if

$$t \in \mathbb{T} \Leftrightarrow b + a - t \in \mathbb{T}.$$

By a symmetric solution $y(t)$ of the BVP (1.1)-(1.2), we mean $y(t)$ is a solution of the BVP (1.1)-(1.2) and satisfies

$$y(t) = y(b + a - t),\ \text{for } t \in [a, b]_{\mathbb{T}}.$$

The rest of the paper is organized as follows. In Section 2, we briefly describe some salient features of time scales. In Section 3, we construct the Green's function for the homogeneous problem corresponding to (1.1)-(1.2) and estimate bounds for the Green's function. In Section 4, we establish criteria for the existence of at least two symmetric positive solutions of the BVP (1.1)-(1.2) by using the Avery–Henderson fixed point theorem. We also establish the existence of at least $2m$

symmetric positive solutions to the BVP (1.1)-(1.2) for an arbitrary positive integer m. Finally as an application, we give an example to demonstrate our result.

2. Preliminaries about Time Scales

A time scale $\mathbb{T}$ is a nonempty closed subset of $\mathbb{R}$. For an excellent introduction to the overall area of dynamic equations on time scales, we refer the recent text books by Bohner and Peterson [**8, 9**], from which we cull the following definitions. The functions $\sigma, \rho : \mathbb{T} \to \mathbb{T}$ are jump operators given by

$$\sigma(t) = \inf\{s \in \mathbb{T} : s > t\} \text{ and } \rho(t) = \sup\{s \in \mathbb{T} : s < t\}$$

(supplemented by $\inf \emptyset = \sup \mathbb{T}$ and $\sup \emptyset = \inf \mathbb{T}$). The point $t \in \mathbb{T}$ is left-dense, left-scattered, right-dense, right-scattered if $\rho(t) = t, \rho(t) < t, \sigma(t) = t, \sigma(t) > t$, respectively. If $\mathbb{T}$ has a right- scattered minimum m, define $\mathbb{T}_\kappa = \mathbb{T} - \{m\}$; otherwise, set $\mathbb{T}_\kappa = \mathbb{T}$. If $\mathbb{T}$ has a left-scattered maximum M, define $\mathbb{T}^\kappa = \mathbb{T} - \{M\}$; otherwise, set $\mathbb{T}^\kappa = \mathbb{T}$.

For $f : \mathbb{T} \to \mathbb{R}$ and $t \in \mathbb{T}^\kappa$, the delta derivative of f at t, denoted $f^\Delta(t)$, is the number (provided it exists) with the property that given any $\epsilon > 0$, there is a neighborhood U of t such that

$$| f(\sigma(t)) - f(s) - f^\Delta(t)[\sigma(t) - s] | \leqslant \epsilon \mid \sigma(t) - s \mid,$$

for all $s \in U$.

For $f : \mathbb{T} \to \mathbb{R}$ and $t \in \mathbb{T}_\kappa$, the nabla derivative of f at t, denoted $f^\nabla(t)$, is the number (provided it exists) with the property that given any $\epsilon > 0$, there is a neighborhood U of t such that

$$| f(\rho(t)) - f(s) - f^\nabla(t)[\rho(t) - s] | \leqslant \epsilon \mid \rho(t) - s \mid,$$

for all $s \in U$. Define $f^{\Delta\nabla}(t)$ to be the nabla derivative of $f^\Delta(t)$, i.e., $f^{\Delta\nabla}(t) = (f^\Delta(t))^\nabla$.

A function $f : \mathbb{T} \to \mathbb{R}$ is left-dense continuous or ld-continuous, provided it is continuous at left-dense points in $\mathbb{T}$ and its right-sided limits exist (finite) at right-dense points in $\mathbb{T}$. It is known that if f is ld-continuous, then there is a function $F(t)$ such that $F^\nabla(t) = f(t)$. In this case, we define

$$\int_a^b f(t)\nabla t = F(b) - F(a).$$

3. Green's Function and Bounds

In this section, we construct the Green's function for the homogeneous BVP corresponding to (1.1)-(1.2) and we estimate bounds for the Green's function. We prove certain lemmas which are needed to establish our main result.

Let $G_1(t, s)$ be the Green's function for the homogeneous BVP,

$$-y^{\Delta\nabla}(t) = 0,\ t \in [a, b]_{\mathbb{T}},$$

$$\alpha y(a) - \beta y^\Delta(a) = 0, \quad \alpha y(b) + \beta y^\Delta(b) = 0,$$

and is given by

$$G_1(t,s) = \begin{cases} \frac{1}{d}\{\alpha(t-a)+\beta\}\{\alpha(b-s)+\beta\}, & t \leqslant s, \\ \frac{1}{d}\{\alpha(s-a)+\beta\}\{\alpha(b-t)+\beta\}, & s \leqslant t, \end{cases}$$

where $d = 2\alpha\beta + \alpha^2(b-a) > 0$.

The Green's function $G_1(t,s)$ is positive and satisfies the following inequality,

$$G_1(t,s) \leqslant G_1(s,s), \text{ for all } t,s \in [a,b]_{\mathbb{T}}.$$

Let $I = \left[\frac{3a+b}{4}, \frac{a+3b}{4}\right]_{\mathbb{T}}$. Then

$$G_1(t,s) \geqslant kG_1(s,s), \text{ for all } (t,s) \in I \times [a,b]_{\mathbb{T}},$$

where

$$k = \frac{\alpha(b-a)+4\beta}{4[\alpha(b-a)+\beta]} < 1. \tag{3.1}$$

For $2 \leqslant j \leqslant n$, we can recursively define

$$G_j(t,s) = \int_a^b G_{j-1}(t,r)G_1(r,s)\nabla r, \text{ for all } t,s \in [a,b]_{\mathbb{T}}. \tag{3.2}$$

Then $G_n(t,s)$ is the Green's function for the homogeneous BVP corresponding to (1.1)-(1.2). It is clear that $G_n(t,s) \geqslant 0$, for all $t,s \in [a,b]_{\mathbb{T}}$.

Let $D = \{v | v : C[a,b]_{\mathbb{T}} \to \mathbb{R}^+\}$. For each $1 \leqslant j \leqslant n-1$, define the operator $T_j : D \to D$ by

$$T_j v(t) = \int_a^b G_j(t,s)v(s)\nabla s, \; t \in [a,b]_{\mathbb{T}},$$

and these integrals are converges. By the construction of T_j and the properties of $G_j(t,s)$, it is clear that

$$(-1)^j (T_j v)^{(\Delta\nabla)^j}(t) = v(t), \; t \in [a,b]_{\mathbb{T}},$$

$$\alpha(T_j v)^{(\Delta\nabla)^i}(a) - \beta(T_j v)^{(\Delta\nabla)^i\Delta}(a) = 0,$$

$$\alpha(T_j v)^{(\Delta\nabla)^i}(b) + \beta(T_j v)^{(\Delta\nabla)^i\Delta}(b) = 0,$$

for $0 \leqslant i \leqslant j-1$. Hence, we see that the BVP (1.1)-(1.2) has a solution if and only if the following BVP has a solution,

$$v^{\Delta\nabla}(t) + f(t, T_{n-1}v(t), T_{n-2}v(t), \cdots, T_1 v(t), v(t)) = 0, \; t \in [a,b]_{\mathbb{T}}, \tag{3.3}$$

$$\alpha v(a) - \beta v^{\Delta}(a) = 0, \quad \alpha v(b) + \beta v^{\Delta}(b) = 0. \tag{3.4}$$

Indeed, if y is a solution of the BVP (1.1)-(1.2), then $v(t) = y^{(\Delta\nabla)^{(n-1)}}(t)$ is a solution of the BVP (3.3)-(3.4). Conversely, if v is a solution of the BVP (3.3)-(3.4), then $y(t) = T_{n-1}v(t)$ is a solution of the BVP (1.1)-(1.2). In fact, we have the representation,

$$y(t) = \int_a^b G_{n-1}(t,s)v(s)\nabla s,$$

where

$$v(s) = \int_a^b G_1(s,\tau) f(\tau, T_{n-1}v(\tau), T_{n-2}v(\tau), \cdots, T_1 v(\tau), v(\tau)) \nabla \tau.$$

The following lemmas are useful for establishing the main result.
Denote

$$\phi = \int_a^b G_1(s,s) \nabla s.$$

LEMMA 3.1. *For $t, s \in [a,b]_{\mathbb{T}}$, the Green's function $G_j(t,s)$ satisfies the following inequality,*

$$G_j(t,s) \leqslant \phi^{j-1} G_1(s,s), \ \ j = 1, 2, \cdots, n. \tag{3.5}$$

PROOF. The proof is by induction. For $j = 1$, the inequality (3.5) is obvious. Next, for fixed j, assume that the inequality (3.5) holds. Then from (3.2), we have

$$\begin{aligned} G_{j+1}(t,s) &= \int_a^b G_j(t,r) G_1(r,s) \nabla r \\ &\leqslant \int_a^b \phi^{j-1} G_1(r,r) G_1(s,s) \nabla r \\ &= \phi^j G_1(s,s). \end{aligned}$$

Hence, by induction the proof is completed. □

LEMMA 3.2. *For $(t,s) \in I \times [a,b]_{\mathbb{T}}$, the Green's function $G_j(t,s)$ satisfies the following inequality,*

$$G_j(t,s) \geqslant k^j \phi^{j-1} G_1(s,s), \ \ j = 1, 2, \cdots, n. \tag{3.6}$$

PROOF. Again, the proof is by induction. First, for $j = 1$, the inequality (3.6) is obvious. Next, for fixed j, assuming that the inequality (3.6) holds. Then from (3.2), we have

$$\begin{aligned} G_{j+1}(t,s) &= \int_a^b G_j(t,r) G_1(r,s) \nabla r, \text{ for all } t, s \in [a,b]_{\mathbb{T}} \\ &\geqslant \int_a^b G_j(t,r) G_1(r,s) \nabla r, \text{ for all } (t,s) \in I \times [a,b]_{\mathbb{T}} \\ &\geqslant \int_a^b k^j \phi^{j-1} G_1(r,r) k G_1(s,s) \nabla r \\ &= k^{j+1} \phi^j G_1(s,s). \end{aligned}$$

Hence, by induction the proof is completed. □

LEMMA 3.3. *For $t, s \in [a,b]_{\mathbb{T}}$, the Green's function $G_j(t,s)$ satisfies the symmetric property,*

$$G_j(t,s) = G_j(b+a-t, \ b+a-s). \tag{3.7}$$

PROOF. By the definition of $G_j(t,s)(2 \leqslant j \leqslant n)$,

$$G_j(t,s) = \int_a^b G_{j-1}(t,r)G_1(r,s)\nabla r, \text{ for all } t,s \in [a,b]_{\mathbb{T}}.$$

Clearly, we can see that $G_1(t,s) = G_1(b+a-t, b+a-s)$. Now, the proof is by induction. Next, assume that (3.7) is true, for fixed $j \geqslant 2$. Then from (3.2) and using the transformation $r_1 = b+a-r$, we have

$$\begin{aligned}
G_{j+1}(t,s) &= \int_a^b G_j(t,r)G_1(r,s)\nabla r\\
&= \int_a^b G_j(b+a-t, b+a-r)G_1(b+a-r, b+a-s)\nabla r\\
&= \int_a^b G_j(b+a-t, r_1)G_1(r_1, b+a-s)\nabla r_1\\
&= G_{j+1}(b+a-t, b+a-s).
\end{aligned}$$

□

LEMMA 3.4. *For $t,s \in [a,b]_{\mathbb{T}}$, the operator T_j satisfies the symmetric property,*

$$T_j v(t) = T_j v(b+a-t).$$

PROOF. By the definition of T_j and using the transformation $s_1 = b+a-s$, we have

$$\begin{aligned}
T_j v(t) &= \int_a^b G_j(t,s)v(s)\nabla s\\
&= \int_a^b G_j(b+a-t, b+a-s)v(s)\nabla s\\
&= \int_a^b G_j(b+a-t, s_1)v(s_1)\nabla s_1\\
&= T_j v(b+a-t).
\end{aligned}$$

□

4. Multiple Symmetric Positive Solutions

In this section, we establish the existence of at least two symmetric positive solutions for the BVP (1.1)-(1.2), by using Avery–Henderson functional fixed point theorem [**5**]. And then, we establish the existence of at least $2m$ symmetric positive solutions for an arbitrary positive integer m.

Let B be a real Banach space. A nonempty closed convex set $P \subset B$ is called a cone, if it satisfies the following two conditions,

(i) $y \in P, \lambda \geqslant 0$ implies $\lambda y \in P$,
(ii) $y \in P$ and $-y \in P$ implies $y = 0$.

Let ψ be a nonnegative continuous functional on a cone P of the real Banach space B. Then for a positive real number c', we define the sets

$$P(\psi, c') = \{y \in P : \psi(y) < c'\}$$

and

$$P_a = \{y \in P :\| y \|< a\}.$$

In obtaining multiple symmetric positive solutions of the BVP (1.1)-(1.2), the following Avery–Henderson functional fixed point theorem will be the fundamental tool.

THEOREM 4.1. [**5**] *Let P be a cone in a real Banach space B. Suppose α and γ are increasing, nonnegative continuous functionals on P and θ is nonnegative continuous functional on P with $\theta(0) = 0$ such that, for some positive numbers c' and k, $\gamma(y) \leqslant \theta(y) \leqslant \alpha(y)$ and $\|y\| \leqslant k\gamma(y)$, for all $y \in \overline{P(\gamma, c')}$. Suppose that there exist positive numbers a' and b' with $a' < b' < c'$ such that $\theta(\lambda y) \leqslant \lambda\theta(y)$, for all $0 \leqslant \lambda \leqslant 1$ and $y \in \partial P(\theta, b')$. Further, let $T : \overline{P(\gamma, c')} \to P$ is completely continuous operator such that*
(B1) $\gamma(Ty) > c'$, for all $y \in \partial P(\gamma, c')$,
(B2) $\theta(Ty) < b'$, for all $y \in \partial P(\theta, b')$,
(B3) $P(\alpha, a') \neq \emptyset$ and $\alpha(Ty) > a'$, for all $y \in \partial P(\alpha, a')$.
Then T has at least two fixed points $y_1, y_2 \in \overline{P(\gamma, c')}$ such that $a' < \alpha(y_1)$ with $\theta(y_1) < b'$ and $b' < \theta(y_2)$ with $\gamma(y_2) < c'$.

Let $B = \{v | v \in C[a, b]_{\mathbb{T}}\}$ be the Banach space equipped with the norm

$$\|v\| = \max_{t\in[a,b]_{\mathbb{T}}} |v(t)|.$$

Define the cone $P \subset B$ by

$$P = \left\{ \begin{array}{l} v \in B : v(t) \geqslant 0,\ v(t) = v(b + a - t),\ \text{for } t \in [a, b]_{\mathbb{T}} \\ \text{and } \min_{t\in I} v(t) \geqslant k\|v\| \end{array} \right\},$$

where k is given as in (3.1). Define the nonnegative, increasing, continuous functionals $\gamma,\ \theta$ and α on the cone P by

$$\gamma(v) = \min_{t\in I} v(t),\ \theta(v) = \max_{t\in I} v(t) \text{ and } \alpha(v) = \max_{t\in[a,b]_{\mathbb{T}}} v(t).$$

We observe that for any $v \in P$,

$$\gamma(v) \leqslant \theta(v) \leqslant \alpha(v) \tag{4.1}$$

and

$$\|v\| \leqslant \frac{1}{k} \min_{t\in I} v(t) = \frac{1}{k}\gamma(v) \leqslant \frac{1}{k}\theta(v) \leqslant \frac{1}{k}\alpha(v). \tag{4.2}$$

We are now ready to present the main result of this section.

THEOREM 4.2. *Suppose there exist $0 < a' < b' < c'$ such that f satisfies the following conditions:*

(D1) $f(t, u_{n-1}, u_{n-2}, \cdots, u_1, u_0) > \frac{c'}{k\phi}$, *for all* $(t, u_{n-1}, u_{n-2}, \cdots, u_1, u_0)$ *in* $I \times \Pi_{j=n-1}^{1}[c'k^j\phi^j, \frac{c'\phi^j}{k}] \times [c', \frac{c'}{k}]$,

(D2) $f(t, u_{n-1}, u_{n-2}, \cdots, u_1, u_0) < \frac{b'}{\phi}$, *for all* $(t, u_{n-1}, u_{n-2}, \cdots, u_1, u_0)$ *in* $[a, b]_{\mathbb{T}} \times \Pi_{j=n-1}^{1}[b'k^j\phi^j, \frac{b'\phi^j}{k}] \times [b', \frac{b'}{k}]$,

(D3) $f(t, u_{n-1}, u_{n-2}, \cdots, u_1, u_0) > \frac{a'}{k\phi}$, *for all* $(t, u_{n-1}, u_{n-2}, \cdots, u_1, u_0)$ *in* $I \times \Pi_{j=n-1}^{1}[a'k^j\phi^j, \frac{a'\phi^j}{k}] \times [a', \frac{a'}{k}]$.

Then the BVP (1.1)-(1.2) *has at least two symmetric positive solutions.*

PROOF. Define the operator $T : P \to B$ by

$$Tv(t) = \int_a^b G_1(t, s)f(s, T_{n-1}v(s), T_{n-2}v(s), \cdots, T_1v(s), v(s))\nabla s. \tag{4.3}$$

It is obvious that a fixed point of T is the solution of the BVP (3.3)-(3.4). We seek two fixed points $v_1, v_2 \in P$ of T. First, we show that $T : P \to P$. Let $v \in P$. Clearly, $Tv(t) \geqslant 0$ on $[a, b]_{\mathbb{T}}$. Further, since $G_j(t, s) = G_j(b + a - t, b + a - s)$, we see that $T_jv(t) = T_jv(b+a-t)$, $1 \leqslant j \leqslant n-1$, for $t \in [a, b]_{\mathbb{T}}$. Hence, it follows that $Tv(t) = Tv(b + a - t)$, for $t \in [a, b]_{\mathbb{T}}$. Also noting that $\alpha(Tv)(a) - \beta(Tv)^{\Delta}(a) = 0 = \alpha(Tv)(b) + \beta(Tv)^{\Delta}(b)$. Then, we have

$$\begin{aligned}
\min_{t\in I} Tv(t) =& \min_{t\in I} \int_a^b G_1(t, s)f(s, T_{n-1}v(s), T_{n-2}v(s), \cdots, T_1v(s), v(s))\nabla s \\
\geqslant& k \int_a^b G_1(s, s)f(s, T_{n-1}v(s), T_{n-2}v(s), \cdots, T_1v(s), v(s))\nabla s \\
\geqslant& k \int_a^b G_1(t, s)f(s, T_{n-1}v(s), T_{n-2}v(s), \cdots, T_1v(s), v(s))\nabla s \\
\geqslant& k\|Tv\|.
\end{aligned}$$

Hence $Tv \in P$ and so $T : P \to P$. Moreover, T is completely continuous. From (4.1) and (4.2), for each $v \in P$, we have $\gamma(v) \leqslant \theta(v) \leqslant \alpha(v)$ and $\|v\| \leqslant \frac{1}{k}\gamma(v)$. Also, for any $0 \leqslant \lambda \leqslant 1$ and $v \in P$, we have $\theta(\lambda v) = \max_{t\in I}(\lambda v)(t) = \lambda \max_{t\in I} v(t) = \lambda\theta(v)$. It is clear that $\theta(0) = 0$. We now show that the remaining conditions of Theorem 4.1 are satisfied.

Firstly, we shall verify the condition $(B1)$ of Theorem 4.1 is satisfied. Since $v \in \partial P(\gamma, c')$, from (4.2), we have that $c' = \min_{t\in I} v(t) \leqslant \|v\| \leqslant \frac{c'}{k}$. Using Lemma 3.1, for $1 \leqslant j \leqslant n - 1$ and $t \in [a, b]_{\mathbb{T}}$, we have

$$\begin{aligned}
T_jv(t) =& \int_a^b G_j(t, s)v(s)\nabla s \\
\leqslant& \frac{c'}{k} \int_a^b G_j(t, s)\nabla s \\
\leqslant& \frac{c'}{k}\phi^{j-1} \int_a^b G_1(s, s)\nabla s = \frac{c'\phi^j}{k}.
\end{aligned}$$

Using Lemma 3.2, for $1 \leqslant j \leqslant n-1$ and $t \in I$, we have

$$\begin{aligned}T_j v(t) =& \int_a^b G_j(t,s)v(s)\nabla s \\ \geqslant& c' \int_a^b G_j(t,s)\nabla s \\ \geqslant& c' k^j \phi^{j-1} \int_a^b G_1(s,s)\nabla s = c' k^j \phi^j.\end{aligned}$$

We may now use condition $(D1)$ to obtain,

$$\begin{aligned}\gamma(Tv) =& \min_{t\in I} \int_a^b G_1(t,s) f(s, T_{n-1}v(s), T_{n-2}v(s), \cdots, T_1 v(s), v(s))\nabla s \\ >& \frac{c'}{\phi} \int_a^b G_1(s,s)\nabla s = c'.\end{aligned}$$

Therefore, we have shown that $\gamma(Tv) > c'$, for all $v \in \partial P(\gamma, c')$.

Now, we shall show that condition $(B2)$ of Theorem 4.1 is satisfied. Since $v \in \partial P(\theta, b')$, from (4.2) that $b' = \max_{t\in I} v(t) \leqslant \|v\| \leqslant \frac{b'}{k}$, for $t \in [a,b]_{\mathbb{T}}$. Using Lemma 3.1, for $1 \leqslant j \leqslant n-1$ and $t \in [a,b]_{\mathbb{T}}$, we have

$$\begin{aligned}T_j v(t) =& \int_a^b G_j(t,s)v(s)\nabla s \\ \leqslant& \frac{b'}{k} \int_a^b G_j(t,s)\nabla s \\ \leqslant& \frac{b'}{k} \phi^{j-1} \int_a^b G_1(s,s)\nabla s = \frac{b'\phi^j}{k}.\end{aligned}$$

Using Lemma 3.2, for $1 \leqslant j \leqslant n-1$ and $t \in I$, we have

$$\begin{aligned}T_j v(t) =& \int_a^b G_j(t,s)v(s)\nabla s \\ \geqslant& b' \int_a^b G_j(t,s)\nabla s \\ \geqslant& b' k^j \phi^{j-1} \int_a^b G_1(s,s)\nabla s = b' k^j \phi^j.\end{aligned}$$

We may now use condition $(D2)$ to obtain,

$$\begin{aligned}\theta(Tv) =& \max_{t\in I} \int_a^b G_1(t,s) f(s, T_{n-1}v(s), T_{n-2}v(s), \cdots, T_1 v(s), v(s))\nabla s \\ <& \frac{b'}{\phi} \int_a^b G_1(s,s)\nabla s = b'.\end{aligned}$$

Therefore, we have shown that $\theta(Tv) < b'$, for all $v \in \partial P(\theta, b')$.

Finally, we shall show that $(B3)$ of Theorem 4.1 is satisfied. Since $a' > 0$ and $\frac{a'}{2} \in P(\alpha, a')$, $P(\alpha, a') \neq \emptyset$. Since $v \in \partial P(\alpha, a')$, from (4.2) that $a' =$

$\max_{t\in[a,b]_{\mathbb{T}}} v(t) \leqslant \|v\| \leqslant \frac{a'}{k}$, for $t \in I$. Using Lemma 3.1, for $1 \leqslant j \leqslant n-1$ and $t \in [a,b]_{\mathbb{T}}$, we have

$$\begin{aligned}T_j v(t) =& \int_a^b G_j(t,s)v(s)\nabla s\\ \leqslant& \frac{a'}{k}\int_a^b G_j(t,s)\nabla s\\ \leqslant& \frac{a'}{k}\phi^{j-1}\int_a^b G_1(s,s)\nabla s = \frac{a'\phi^j}{k}.\end{aligned}$$

Using Lemma 3.2, for $1 \leqslant j \leqslant n-1$ and $t \in I$, we have

$$\begin{aligned}T_j v(t) =& \int_a^b G_j(t,s)v(s)\nabla s\\ \geqslant& a'\int_a^b G_j(t,s)\nabla s\\ \geqslant& a'k^j\phi^{j-1}\int_a^b G_1(s,s)\nabla s = a'k^j\phi^j.\end{aligned}$$

We may now use condition (D3) to obtain,

$$\begin{aligned}\alpha(Tv) =& \max_{t\in[a,b]_{\mathbb{T}}}\int_a^b G_1(t,s)f(s,T_{n-1}v(s),T_{n-2}v(s),\cdots,T_1v(s),v(s))\nabla s\\ >& \frac{a'}{\phi}\int_a^b G_1(s,s)\nabla s = a'.\end{aligned}$$

Therefore, we have shown that $\alpha(Tv) > a'$, for all $v \in \partial P(\alpha, a')$. We have proved that all the conditions of Theorem 4.1 are satisfied and so there exist at least two symmetric positive solutions $v_1, v_2 \in \overline{P(\gamma, c')}$ for the BVP (3.3)-(3.4). Therefore, the BVP (1.1)-(1.2) has at least two symmetric positive solutions y_1, y_2 of the form,

$$y_i(t) = T_{n-1}v_i(t) = \int_a^b G_{n-1}(t,s)v_i(s)\nabla s,\ i = 1,2.$$

This completes the proof of the theorem. □

THEOREM 4.3. *Let m be an arbitrary positive integer. Assume that there exist numbers $a_r(r = 1,2,\cdots,m+1)$ and $b_s(s = 1,2,\cdots,m)$ with $0 < a_1 < b_1 < a_2 < b_2 < \cdots < a_m < b_m < a_{m+1}$ such that*

$$(4.4)\qquad \begin{cases} f(t,u_{n-1},u_{n-2},\cdots,u_1,u_0) > \dfrac{a_r}{k\phi},\ \textit{for all}\ (t,u_{n-1},u_{n-2},\cdots,u_1,u_0)\\ \qquad \textit{in}\ I \times \Pi_{j=n-1}^{1}[a_rk^j\phi^j, \dfrac{a_r\phi^j}{k}] \times [a_r, \dfrac{a_r}{k}],\ r = 1,2,\cdots,m+1,\end{cases}$$

$$(4.5)\quad \begin{cases} f(t,u_{n-1},u_{n-2},\cdots,u_1,u_0) < \dfrac{b_s}{\phi},\ \textit{for all}\ (t,u_{n-1},u_{n-2},\cdots,u_1,u_0) \\ \qquad \textit{in}\ [a,b]_{\mathbb{T}} \times \Pi_{j=n-1}^{1}[b_s k^j \phi^j, \dfrac{b_s\phi^j}{k}] \times [b_s, \dfrac{b_s}{k}],\ s=1,2,\cdots,m. \end{cases}$$

Then the BVP (1.1)-(1.2) *has at least* $2m$ *symmetric positive solutions in* $\overline{P}_{a_{m+1}}$.

PROOF. We use induction on m. For $m = 1$, we know that from (4.4) and (4.5) that $T : \overline{P}_{a_2} \to P_{a_2}$, then, it follows from Avery–Henderson fixed point theorem that the BVP (1.1)-(1.2) has at least two symmetric positive solutions in $\overline{P}_{a_2}$. Next, we assume that this conclusion holds for $m = l$. In order to prove this conclusion holds for $m = l + 1$. We suppose that there exist numbers $a_r(r = 1, 2, \cdots, l + 2)$ and $b_s(s = 1, 2, \cdots, l + 1)$ with $0 < a_1 < b_1 < a_2 < b_2 < \cdots < a_{l+1} < b_{l+1} < a_{l+2}$ such that

$$(4.6)\quad \begin{cases} f(t,u_{n-1},u_{n-2},\cdots,u_1,u_0) > \dfrac{a_r}{k\phi},\ \text{for all}\ (t,u_{n-1},u_{n-2},\cdots,u_1,u_0) \\ \qquad \text{in}\ I \times \Pi_{j=n-1}^{1}[a_r k^j \phi^j, \dfrac{a_r\phi^j}{k}] \times [a_r, \dfrac{a_r}{k}],\ r=1,2,\cdots,l+2, \end{cases}$$

$$(4.7)\quad \begin{cases} f(t,u_{n-1},u_{n-2},\cdots,u_1,u_0) < \dfrac{b_s}{\phi},\ \text{for all}\ (t,u_{n-1},u_{n-2},\cdots,u_1,u_0) \\ \qquad \text{in}\ [a,b]_{\mathbb{T}} \times \Pi_{j=n-1}^{1}[b_s k^j \phi^j, \dfrac{b_s\phi^j}{k}] \times [b_s, \dfrac{b_s}{k}],\ s=1,2,\cdots,l+1. \end{cases}$$

By assumption, the BVP (1.1)-(1.2) has at least $2l$ symmetric positive solutions $u_i(i = 1, 2, \cdots, 2l)$ in $\overline{P}_{a_{l+1}}$. At the same time, it follows from Theorem 4.2, (4.6) and (4.7) that the BVP (1.1)-(1.2) has at least two symmetric positive solutions u, v in $\overline{P}_{a_{l+2}}$ such that $a_{l+1} < \alpha(u)$ with $\theta(u) < b_{l+1}$ and $b_{l+1} < \theta(v)$ with $\gamma(v) < a_{l+2}$. Obviously u and v are different from $u_i(i = 1, 2, \cdots, 2l)$. Therefore, the BVP (1.1)-(1.2) has at least $2l + 2$ symmetric positive solutions in $\overline{P}_{a_{l+2}}$, which shows that this conclusion holds for $m = l + 1$. □

5. Example

Let us introduce an example to illustrate the usage of Theorem 4.2. Let $\mathbb{T} = [0, \frac{3}{2}] \cup [2, 3]$, $n = 2$, $a = 0$, $b = 3$, $\alpha = \frac{4}{5}$, $\beta = \frac{3}{2}$. Now, consider the BVP,

$$(5.1)\quad y^{(\Delta\nabla)^2}(t) = f(t, y(t), y^{\Delta\nabla}(t)),\ t \in [0, 3]_{\mathbb{T}},$$

satisfying the boundary conditions,

$$(5.2)\quad \begin{cases} \dfrac{4}{5}y(0) - \dfrac{3}{2}y^{\Delta}(0) = 0,\ \dfrac{4}{5}y(3) + \dfrac{3}{2}y^{\Delta}(3) = 0, \\ \dfrac{4}{5}y^{\Delta\nabla}(0) - \dfrac{3}{2}y^{(\Delta\nabla)\Delta}(0) = 0,\ \dfrac{4}{5}y^{\Delta\nabla}(3) + \dfrac{3}{2}y^{(\Delta\nabla)\Delta}(3) = 0, \end{cases}$$

and

$$f(t, y, y^{\Delta\nabla}) = \frac{250(y + 1)^4}{73(y^2 + 999)}.$$

Then the Green's function $G_1(t,s)$ is

$$G_1t,s) = \begin{cases} \frac{(8t+15)(39-8s)}{432}, & t \leqslant s, \\ \frac{(8s+15)(39-8t)}{432}, & s \leqslant t. \end{cases}$$

By direct calculations, we have $k = 0.5384615385$ and $\phi = 3.891589506$. Clearly, f is continuous and increasing on $[0,\infty)$. If we choose $a' = 0.001$, $b' = 0.04$ and $c' = 20$ then $0 < a' < b' < c'$ and f satisfies

(i) $f(t,y,y^{\Delta\nabla}) > 9.544392358 = \frac{c'}{k\phi}$, for $(t,y,y^{\Delta\nabla}) \in [\frac{3}{4},\frac{9}{4}]_{\mathbb{T}} \times [41.90942545, 144.5447531] \times [20, 37.14285714]$,

(ii) $f(t,y,y^{\Delta\nabla}) < 0.01027857639 = \frac{b'}{\phi}$, for $(t,y,y^{\Delta\nabla}) \in [0,3]_{\mathbb{T}} \times [0.0838188509, 0.2890895061] \times [0.04, 0.07428571428]$,

(iii) $f(t,y,y^{\Delta\nabla}) > 0.0004772196179 = \frac{a'}{k\phi}$, for $(t,y,y^{\Delta\nabla}) \in [\frac{3}{4},\frac{9}{4}]_{\mathbb{T}} \times [0.002095471273, 0.007227237653] \times [0.001, 0.001857142857]$.

Then all the conditions of Theorem 4.2 are satisfied. Thus, by Theorem 4.2, the BVP (5.1)-(5.2) has at least two symmetric positive solutions.

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