BULLETIN OF INTERNATIONAL MATHEMATICAL VIRTUAL INSTITUTE
ISSN 1840-4367
Vol. 3(2013), 7-13

Former
BULLETIN OF SOCIETY OF MATHEMATICIANS BANJA LUKA
ISSN 0354-5792 (o), ISSN 1986-521X (p)

EQUITABLE EDGE DOMINATION IN GRAPHS

Anwar Alwardi and N. D. Soner

Abstract. A subset D of $V(G)$ is called an equitable dominating set of a graph G if for every $v \in (V - D)$, there exists a vertex $u \in D$ such that $uv \in E(G)$ and $|deg(u) - deg(v)| \leqslant 1$. The minimum cardinality of such a dominating set is denoted by $\gamma_e(G)$ and is called equitable domination number of G. In this paper we introduce the equitable edge domination and equitable edge domatic number in a graph, exact value for the some standard graphs bounds and some interesting results are obtained.

1. Introduction

All the graphs considered here are finite and undirected with no loops and multiple edges. As usual $p = |V|$ and $q = |E|$ denote the number of vertices and edges of a graph G, respectively. In general, we use $\langle X \rangle$ to denote the subgraph induced by the set of vertices X. $N(v)$ and $N[v]$ denote the open and closed neighbourhood of a vertex v, respectively. A set D of vertices in a graph G is a dominating set if every vertex in $V - D$ is adjacent to some vertex in D. The domination number $\gamma(G)$ is the minimum cardinality of a dominating set of G. A set $S \subseteq V$ is a neighbourhood set of G, if $G = \cup_{v \in S} \langle N[v] \rangle$, where $\langle N[v] \rangle$ is the subgraph of G induced by v and all vertices adjacent to v. The neighbourhood number $\eta(G)$ of G is the minimum cardinality of a neighbourhood set of a graph G. A neighbourhood set $S \subseteq V$ is a minimal neighbourhood set, if $S - v$ for all $v \in S$, is not a neighbourhood set of G. A graph $G = (V, E)$ is called bi regular graph with regularity l, m if the degree of any vertex in V is either l or m.

For terminology and notations not specifically defined here we refer reader to [**4**]. For more details about domination number and its related parameters, we refer to [**5**], [**10**], and [**11**].

2010 *Mathematics Subject Classification.* 05C69.

Key words and phrases. Equitable edge dominating set, Equitable edge domatic number, Equitable edge domination number.

Partially supported by UGC-SAP, DRS-1, NO: F.510/2/DRS/2011(SAP-I). .

A subset D of $V(G)$ is called an equitable dominating set of a graph G if for every $v \in (V-D)$, there exists a vertex $u \in D$ such that $uv \in E(G)$ and $|deg(u)-deg(v)| \leqslant 1$. The minimum cardinality of such a dominating set is denoted by $\gamma_e(G)$ and is called an equitable domination number of G. D is minimal if for any vertex $u \in D$, $D-\{u\}$ is not an equitable dominating set of G. If a vertex $u \in V$ be such that $|d(u)-d(v)| \geqslant 2$ for all $v \in N(u)$ then u is in equitable dominating set. Such points are called equitable isolates. The equitable neighbourhood of u denoted by $N_e(u)$ is defined as $N_e(u) = \{v \in N(u), |deg(u)-deg(v)| \leqslant 1\}$. The cardinality of $N_e(u)$ is denoted by $deg_e(u)$. The maximum and minimum equitable degree of a vertex in G are denoted respectively by $\Delta_e(G)$ and $\delta_e(G)$. That is $\Delta_e(G) = max_{u\in V(G)}|N_e(u)|$, $\delta_e(G) = min_{u\in V(G)}|N_e(u)|$.
A subset S of V is called an equitable independent set, if for any $u \in S$, $v \notin N_e(u)$, for all $v \in S-\{u\}$.

In this paper we introduce the equitable edge domination in a graph, exact value for the some standard graphs and bounds for equitable edge domination number are obtained.

2. Equitable Edge Domination Number

Let $G = (V, E)$ be a graph. for any edge $f \in E$ The degree of $f = uv$ in G is defined by $deg(f) = deg(u) + deg(v) - 2$. A set $S \subseteq E$ of edges is equitable edge dominating set of G if every edge f not in S is adjacent to at least one edge $f^{'} \in S$ such that $|deg(f)-deg(f^{'})| \leqslant 1$. The minimum cardinality of such equitable edge dominating set is denoted by $\gamma_e^{'}(G)$ and is called equitable edge domination number of G. S is minimal if for any edge $f \in S$, $S-\{f\}$ is not an equitable edge dominating set of G. A subset S of E is called an equitable edge independent set, if for any $f \in S$, $f \notin N_e(g)$, for all $g \in S-\{f\}$. If an edge $f \in E$ be such that $|deg(f)-deg(g)| \geqslant 2$ for all $g \in N(f)$ then f is in any equitable dominating set. Such edges are called equitable isolates. The equitable neighbourhood of f denoted by $N_e(f)$ is defined as $N_e(f) = \{g \in N(f), |deg(f)-deg(g)| \leqslant 1\}$. The cardinality of $N_e(f)$ is called the equitable degree of f and denoted by $deg_e(f)$. The maximum and minimum equitable degree of edge in G are denoted respectively by $\Delta_e^{'}(G)$ and $\delta_e^{'}(G)$. That is $\Delta_e^{'}(G) = max_{f\in E(G)}|N_e(f)|$, $\delta_e^{'}(G) = min_{f\in E(G)}|N_e(f)|$. The equitable degree of an edge f in a graph G denoted by $deg_e(f)$ is equal to the number of edges which is equitable adjacent with f. the minimum equitable edge dominating set is denoted by $\gamma_e^{'}$-set. In this paper if f and g any two edges in $E(G)$ we say that f and g are equitable adjacent if f and g are adjacent and $|deg(f)-deg(g)| \leqslant 1$ where $deg(f), deg(g)$ is the degree of the edges f and g respectively. The degree of the edge $f = uv$, $deg(f) = deg(v) + deg(u) - 2$.
An edge dominating set X is called an equitable independent edge dominating set if no two edges in X are equitable adjacent. The equitable independent edge domination number $\gamma_{ei}^{'}(G)$ is the minimum cardinality taken over all equitable independent edge dominating sets of G. The edge independence number $\beta_e^{'}(G)$ is defined to be the number of edges in a maximum equitable independent set of edges of G.

For a real number x; $\lfloor x \rfloor$ denotes the greatest integer less than or equal to x and $\lceil x \rceil$ denotes the smallest integer greater than or equal to x.

THEOREM 2.1. *An equitable edge dominating set F is minimal if and only if for each edge $f \in F$ one of the following conditions holds*
(i) $N_e(f) \cap F = \phi$
(ii) there exist an edge $g \in E - F$ such that $N_e(g) \cap F = \{f\}$.

PROOF. Suppose that F is a minimal equitable edge dominating set. Assume that (i) and (ii) do not hold. Then for some $f \in F$ there exist an edge $g \in N_e(f) \cap F$ and for every edge $h \in E - F$, $N_e(h) \cap F \neq \{f\}$. Therefore $F - \{f\}$ is an equitable edge dominating set contradiction to the minimality of F. Therefor (i) or (ii) holds. Conversely, suppose for every $f \in F$ one of the conditions holds. Suppose F is not minimal. Then there exist $f \in F$ such that $F - \{f\}$ is an equitable edge dominating set. Therefore there exist an edge $g \in F - \{f\}$ such that $g \in N_e(f)$. Hence f does not satisfy (i). Then f must satisfy (ii). Then there exist an edge $g \in E - F$ such that $N_e(g) \cap F = \{f\}$. Since $F - \{f\}$ is an equitable edge dominating set there exist an edge $f^{'} \in F - \{f\}$ such that $f^{'}$ is equitable adjacent to g. Therefore $f^{'} \in N_e(g) \cap F$ and $f^{'} \neq f$, a contradiction to $N_e(g) \cap F = \{f\}$. Hence F is minimal equitable edge dominating set. □

In the following proposition we determine the equitable edge domination number for some standard graphs

PROPOSITION 2.1. *(i) For the path P_p on p vertices*

$$\gamma_e^{'}(P_p) = \begin{cases} \frac{p}{3}, & \text{if } p \equiv 0(mod3); \\ \frac{p-1}{3}, & \text{if } p \equiv 1(mod3); \\ \frac{p+1}{3}, & \text{if } p \equiv 2(mod3). \end{cases}$$

(ii) For the cycle C_p on $p \geqslant 4$ vertices, $\gamma_e^{'}(C_p) = \lceil \frac{p}{3} \rceil$.
(iii)$\gamma_e^{'}(K_{m,n}) = min\{m, n\}$.
(iv)$\gamma_e^{'}(G \cup H) = \gamma_e^{'}(G) + \gamma_e^{'}(H)$.

It is clear for any graph G any equitable edge dominating set is edge dominating set then the following proposition follows

PROPOSITION 2.2. *For any graph G, $\gamma_e^{'}(G) \geqslant \gamma^{'}(G)$.*

PROPOSITION 2.3. *For any graph G without any equitable isolated edges, if F is minimal equitable edge dominating set then $E - F$ is equitable edge dominating set.*

PROOF. Let F be minimal equitable edge dominating set of G. Suppose $E - F$ is not an equitable edge dominating set. Then there exist an edge f such that $f \in F$ is not equitable adjacent to any edge in $E - F$. Since G has no equitable isolated edges then f is equitable dominated by at least one edge in $F - \{f\}$. Thus $F - \{f\}$ is equitable edge dominating set a contradiction to the minimality of F. Therefore $E - F$ is is equitable edge dominating set. □

PROPOSITION 2.4. *An equitable independent set F is maximal equitable independent set if and only if it is equitable edge independent and equitable edge dominating set of G.*

PROOF. Suppose an equitable independent set F is maximal. Then for every edge $f \in E-F$, the set $F \cup \{f\}$ is not equitable independent, that is for every edge $f \in E-F$, there is an edge $g \in F$ such that f is equitable adjacent to g. Thus F is equitable edge dominating set. Hence F is both equitable edge independent and equitable edge dominating set of G.
Conversely, suppose F is both equitable edge independent and equitable edge dominating set of G. Suppose F is not maximal equitable edge independent set. Then there exist an edge $f \in E-F$ such that $F \cup \{f\}$ is equitable independent, then there is no edge in F equitable adjacent to f. Hence F is not equitable edge dominating set which is a contradiction. Hence F is maximal equitable independent set. □

THEOREM 2.2. *For any graph G, $\gamma_e^{'}(G) \leqslant q - \Delta_e^{'}(G)$.*

PROOF. Let f be an edge in G of equitable degree $\Delta_e^{'}(G)$. Then clearly $E(G) - N_e(f)$ is an equitable edge dominating set. Hence $\gamma_e^{'}(G) \leqslant q - \Delta_e^{'}(G)$. □

THEOREM 2.3. *For any $\gamma_e^{'}$-set F of a graph $G=(V,E)$, $|E-F| \leqslant \sum_{f\in F} deg_e(f)$ and the equality holds if and only if*
(i) F is equitable independent
(ii) for every edge $f \in E-F$, there exists only one edge $g \in F$ such that $N_e(f) \cap F = \{g\}$.

PROOF. Since each edge in $E-F$ is equitable adjacent to at least one edge of F. Therefore each edge in $E-F$ contributes at least one to the sum of the equitable degrees of the edges of F. Hence

$$|E-F| \leqslant \sum_{f\in F} deg_e(f).$$

Let $|E-F| = \sum_{f\in F} d_e(f)$ and suppose that F is not equitable independent. Clearly each edge in $E-F$ is counted in the sum $\sum_{f\in F} deg_e(f)$. Hence if f_1 and f_2 are equitable adjacent, then f_1 is counted in $deg_e(f_1)$ and vice versa. Then the sum exceeds $|E-F|$ by at least two, contrary to the hypothesis. Hence F must be equitable independent.
Now suppose (ii) is not true. Then $N_e(f) \cap F \geqslant 2$ for some edge $f \in E-F$. Let f_1 and f_2 belong to $N_e(f) \cap F$, hence $\sum_{f\in F} deg_e(f)$ exceeds $E-F$ by at least one since f_1 counted twice once in $deg_e(f_1)$ and the once in $deg_e(f_2)$. Hence if the equality holds then the condition (i) and (ii) must be true. The converse is obvious. □

THEOREM 2.4. *If $G=(V,E)$ is with out equitable isolated edges, then for every minimal equitable edge dominating set F, $E-F$ is also equitable edge dominating set of G.*

Proof. Let F be minimal equitable edge dominating set of G. Suppose that $E - F$ is not equitable edge dominating set of G. Then there exists an edge h such that h is not equitable adjacent by any edge in $E - F$. We have G has no equitable isolated edges, then h is equitable adjacent by at least one edge in $F - \{h\}$. Therefore $F - \{h\}$ is equitable edge dominating set which contradicts the minimality of F. Hence $E - F$ is equitable edge dominating set of G. □

Theorem 2.5. *For any (p,q) graph G,* $\lceil \frac{q}{\Delta_e'(G)+1} \rceil \leqslant \gamma_e^{'}(G) \leqslant q - \beta_e^{'} + q_0$, *where* q_0 *is the number of equitable isolated edges.*

Proof. From the previous proposition $|E - F| \leqslant \sum_{f\in F} deg_e(f) \leqslant \gamma_e^{'}(G)\Delta_e^{'}$. Hence

$$q - \gamma_e^{'}(G) \leqslant \gamma_e^{'}(G)\Delta_e^{'}$$

Therefore

$$\lceil \frac{q}{\Delta_e'(G)+1} \rceil \leqslant \gamma_e^{'}(G). \tag{2.1}$$

Let $G^{'} = \langle E(G) - I_e(G) \rangle$ where $I_e(G)$ is the set of equitable isolated edges of G. Let S be the maximal equitable independent set of edges of $G^{'}$. Hence S is also equitable edge dominating set of $G^{'}$. Since $G^{'}$ does not have equitable isolated edges $E(G^{'}) - S$ is also equitable edge dominating set of $G^{'}$. Therefore $\gamma_e^{'}(G^{'}) \leqslant |E(G^{'}) - S| = q(G^{'}) - \beta_e^{'}(G^{'})$. But $\gamma_e^{'}(G^{'}) = \gamma_e^{'}(G) - q_0$ and $q(G^{'}) = q(G) - q_0$ and $\beta_e^{'}(G^{'}) = \beta_e^{'}(G) - q_0$. Hence

$$\gamma_e^{'}(G) - q_0 \leqslant q(G) - q_0 - (\beta_e^{'}(G) - q_0).$$

Therefore

$$\gamma_e^{'}(G) \leqslant q - \beta_e^{'} + q_0 \tag{2.2}$$

From 2.1 and 2.2 we have

$$\lceil \frac{q}{\Delta_e'(G)+1} \rceil \leqslant \gamma_e^{'}(G) \leqslant q - \beta_e^{'} + q_0.$$

□

3. Equitable Edge Domatic Number

Definition 3.1. The maximal order of partition of the edges $E(G)$ into equitable edge dominating sets is called equitable edge domatic number of G and denoted by $d_e^{'}(G)$.

Proposition 3.1. *For any graph G,* $d_e^{'}(G) \leqslant d^{'}(G)$.

Proof. Let $G = (V, E)$ be a graph. It is clear that any partition of $E(G)$ into equitable edge domination set is also partition of $E(G)$ into edge dominating set. Hence $d_e^{'}(G) \leqslant d^{'}(G)$ □

Proposition 3.2. *(i) For any cycle C_p with p vertices*

$$d_e^{'}(C_p) = \begin{cases} 3, & \text{if } p \equiv 0(mod3); \\ 2, & \text{otherwise.} \end{cases}$$

(ii) For any path P_p with $p \geqslant 3$,

$$d_e^{'}(P_p) = 2.$$

(iii) For any complete bipartite graph $K_{m,n}$,

$$d_e^{'}(K_{m,n}) = max\{m, n\}.$$

(iv) For any regular graph or bi regular with bi regularity $(k, k+1)$, $d_e^{'}(G) = d(G)$.
(v) $d_e^{'}(G) = 1$ if and only if G has at least one equitable isolated edge.

Proposition 3.3. *Let G be a complete graph K_p with $p \geqslant 2$ vertices. Then*

$$d_e^{'}(G) = \begin{cases} p-1, & \text{if } n \text{ is even;} \\ p, & \text{if } n \text{ is odd.} \end{cases}$$

Proof. If p is even, then K_p can be decomposed into $p-1$ pairwise edge-disjoint linear factors. The edge set of each these factors is an edge dominating set in K_p. Since all the edge in the complete graph K_p has the same degree $2(p-2)$, then any edge dominating set is also equitable edge dominating set in K_p. Hence $d_e^{'}(K_p) \geqslant p-1$. Now if we consider ζ is an equitable edge domatic partition of K_p with p classes the mean value of the orders of these classes has at most $\frac{p-1}{2}$. This implies that at least one of the classes has at most $[\frac{p-1}{2}] = \frac{p}{2} - 1$ edges. Since the classes, say C, covers at most $p-2$ vertices, there are such two vertices u, w satisfying that $uw \notin C$ and there is no edge in C equitable adjacent to uw. Hence there is no edge in ζ equitable adjacent to the edge joining these vertices. Therefore $d_e^{'}(K_p) = p-1$ if p is even.
Now let p is odd. By labeling the vertices of K_p by $x_1, x_2, ..., x_p$. We can make the following domatic partition of the vertices $E_1, ...E_p$, where $E_i = x_{i+j}x_{i-j+1}$, where $j = 1, ..., \frac{p-1}{2}$ and the scripts will be taken modulo p. Hence $d_e^{'}(K_p) \geqslant p$. If we suppose that $d_e^{'}(K_p) \geqslant p+1$, then in the same way of the first case we can prove that there exist an equitable edge domatic partition one of whose classes has at most $\frac{p-3}{2}$ edges this set cover at most $p-3$ vertices and it is not an equitable edge dominating set a contradiction. Hence $d_e^{'}(K_p) = p$ if p is odd. □

Theorem 3.1. *For any graph G with p vertices, $d_e^{'}(G) \geqslant \frac{q}{q-\delta_e^{'}(G)}$.*

Proof. Assume that $d_e^{'}(G) = d$ and $\{D_1, D_2, ..., D_d\}$ is a partition of $E(G)$ into d equitable edge dominating sets, clearly $|D_i| \geqslant \gamma_e^{'}(G)$ for $i = 1, 2, ..., d$ and we have $q = \sum_{i=1}^{d} |D_i| \geqslant d\delta_e^{'}(G)$.
Hence $d_e^{'}(G) \geqslant \frac{q}{q-\delta_e^{'}(G)}$. □

References

[1] J. Bondy and U. Murthy, *Graph Theory with applications*, North Holland, New York, (1976).

[2] Venkatasubramanian Swaminathan and K. M. Dharmalingam, *Degree Equitable Domination on Graphs*, Kragujevac Journal of Mathematics, V. 35 No 1 (2011), 191-197.

[3] C. Godsil and G. Royle, *Algebraic graph theory*, vol. 207 of Graduate Texts in Mathematics, Springer-Verlag, New York, 2001.

[4] F. Harary, *Graph theory*, Addison-Wesley, Reading Mass (1969).

[5] T. W. Haynes, S. T. Hedetniemi and P. J. Slater, *Fundamentals of domination in graphs*, Marcel Dekker, New York (1998).

[6] S. M. Hedetneimi, S. T. Hedetneimi, R. C. Laskar, L. Markus and P. J. Slater. *Disjoint dominating sets in graphs*. Proc. Int. Conf. on Disc.Math., IMI-IISc, Bangalore (2006) 8 - 101.

[7] S. R. Jayaram, *Line domination in graphs*, Graphs Combin. **3** (1987), 357-363.

[8] V. R. Kulli and S. C. Sigarkanti, *Further results on the neighborhood number of a graph.* Indian J. Pure and Appl. Math.**23 (8)** (1992) 575 -577.

[9] S. Mitchell and S. T. Hedetniemi, *Edge domination in trees*, Congr. Numer.**19** (1977), 489-509.

[10] E. Sampathkumar and P. S. Neeralagi, *The neighborhood number of a graph*, Indian J. Pure and Appl. Math.**16 (2)** (1985) 126 - 132.

[11] H. B. Walikar, B. D. Acharya and E. Sampathkumar, *Recent developmentsin the theory of domination in graphs*, Mehta Research instutute, Alahabad, MRI Lecture Notes in Math. **1** (1979).

Received 08.04.2012; available online 10.12.2012

Department of Studies in Mathematics, University of Mysore, Mysore, India
E-mail address: a_wardi@hotmail.com, ndsoner@yahoo.co.in