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ON THE ENERGY OF DIGRAPHS

Shariefuddin Pirzada, Mushtaq A. Bhat, Ivan Gutman, and Juan Rada

Abstract. The energy of a digraph D with eigenvalues $z_1, z_2, \ldots, z_n$ is defined as $E(D) = \sum_{j=1}^{n} |\Re z_j|$, where $\Re z_j$ is the real part of the complex number z_j. In this paper, we characterize some positive reals that cannot be the energy of a digraph. We also obtain a sharp lower bound for the energy of strongly connected digraphs.

1. Introduction

A digraph D consists of a non-empty finite set V of elements called vertices and a finite set A of ordered pairs of distinct vertices called arcs [**15**]. Throughout this paper we assume that D has no loops and no multiple arcs. Two vertices are called adjacent if they are connected by an arc. An arc from a vertex u to a vertex v is written as uv; u is the initial and v the terminal vertex of this arc. A path of length $n-1$, $(n \geqslant 2)$, denoted by P_n, is a digraph with n vertices $v_1, v_2, \ldots, v_n$ and with $n-1$ arcs $v_i v_{i+1}$, $i = 1, 2, \ldots, n$. A cycle of length n, denoted by C_n, is the digraph with vertex set $\{v_1, v_2, \ldots, v_n\}$ having arcs $v_i v_{i+1}$, $i = 1, 2, \ldots, n-1$, and $v_n v_1$. A digraph D is said to be strongly connected if for every pair u and v of vertices, there is a path both from u to v and from v to u. The strong components of a digraph are the maximally strongly connected subdigraphs. The outdegree (respectively, indegree) of a vertex is the number of arcs of which it is the initial (respectively, terminal) vertex.

A complex number z is said to be an algebraic number (respectively, an algebraic integer) if it is a zero of some monic polynomial with rational (respectively, integral) coefficients. A complex number is said to be transcendental if it is not an algebraic number (see [**16**]).

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Assume that the vertices of D are $v_1, v_2, \ldots, v_n$. The adjacency matrix of D is the $(0,1)$-matrix $\mathbf{A} = \|a_{ij}\|$ of order n, such that

$$a_{ij} = \begin{cases} 1 & \text{if there is an arc from } v_i \text{ to } v_j \\ \\ 0 & \text{otherwise.} \end{cases}$$

The characteristic polynomial of $\mathbf{A}$ is the characteristic polynomial of D and the eigenvalues of $\mathbf{A}$ are the eigenvalues of D. In general, the adjacency matrix $\mathbf{A}$ of D is not symmetric and therefore the eigenvalues $z_1, z_2, \ldots, z_n$ can be complex numbers. These eigenvalues form the spectrum of D, denoted as $Spec(D) = \{z_1, z_2, \ldots, z_n\}$ or shorter, $SpecD = \{z_i\}$.

We usually assume that $|z_1| \geqslant |z_2| \geqslant \cdots \geqslant |z_n|$. The spectral radius of D is denoted by $\rho = \rho(\mathbf{A})$ and is defined to be the largest absolute value of the eigenvalues of $\mathbf{A}$ i.e., $\rho(\mathbf{A}) = |z_1|$. Since $\mathbf{A}$ is a non-negative matrix, by the Perron–Frobenius theorem, $\rho(\mathbf{A})$ is a non-negative real number i.e., $\rho = |z_1| = z_1$. If D is strongly connected, then ρ is positive [**3, 10**].

The energy of a graph G was defined by one of the present authors [**7**] as

$$E(D) = \sum_{j=1}^{n} |\lambda_j|$$

where n is the number of vertices of G and $\lambda_1, \lambda_2, \ldots, \lambda_n$ are the eigenvalues of G which are real numbers. The energy of a graph has been extensively studied in the literature [**7, 8, 9**]. Recently, this concept was extended to digraphs [**11, 13**] as

$$E(D) = \sum_{j=1}^{n} |\Re z_j|$$

where $z_1, z_2, \ldots, z_n$ are the eigenvalues of D.

If D is a digraph with adjacency matrix $\mathbf{A}$ and $z_1, z_2, \ldots, z_n$ are its eigenvalues, then

$$Trace\mathbf{A} = \sum_{j=1}^{n} z_j = 0$$

i.e.,

$$\sum_{j=1}^{n} \Re z_j + i \sum_{j=1}^{n} \Im z_j = 0$$

i.e.,

$$\sum_{j=1}^{n} \Re z_j = \sum_{j=1}^{n} \Im z_j = 0$$

which implies

$$\sum_{+} \Re z_j = -\sum_{-} \Re z_j\,.$$

Now,

$$E(D) = \sum_{j=1}^{n} |z_j| = \sum_{+} \Re z_j - \sum_{-} \Re z_j$$

where $\sum\limits_{+}$ and $\sum\limits_{-}$ indicate, respectively, summation over all eigenvalues with positive and negative real parts. This finally yields

$$E(D) = 2 \sum_{+} \Re z_j \,. \tag{1.1}$$

Note that Brualdi [**3**] calls $E(D)$ the *low energy* of D.

Using the terminology and notations from [**6**], we define two operations on digraphs. Let E and F be two digraphs with vertex sets $U = \{u_i\}$ and $V = \{v_i\}$. The following two operations define digraphs having $U \times V$ as their vertex sets:

Conjunction $D = E \wedge F$. Here $((u_1, v_1), (u_2, v_2))$ is an arc of D whenever (u_1, u_2) and (v_1, v_2) are arcs in E and F.

Cartesian product $D = E \times F$. Here $((u_1, v_1), (u_2, v_2))$ is an arc of D whenever $u_1 = u_2$ and (v_1, v_2) is an arc of F and $v_1 = v_2$ and (u_1, u_2) is an arc of E.

Lemma 1.1. [**6**] *Let D_1 and D_2 be two digraphs of order n_1 and n_2 with spectra $Spec(D_1) = \{\lambda_i\}$ and $Spec(D_2) = \{\mu_j\}$. Then the spectrum of the conjunction and the Cartesian product of digraphs is a*

$$Spec(D_1 \wedge D_2) = \{\lambda_i \times \mu_j\} \qquad \text{and} \qquad Spec(D_1 \times D_2) = \{\lambda_i + \mu_j\}$$

for $i = 1, 2, \ldots, n_1$ and $j = 1, 2, \ldots, n_2$.

Any rational root of a monic polynomial having integral coefficients is an integer [**1**]. Since the characteristic polynomial of a digraph is a monic polynomial with integral coefficients, we have:

Lemma 1.2. *If the eigenvalue of a digraph is rational, then it is an integer.*

The next observation shows that the energy of a digraph is contributed only by its strong components.

Lemma 1.3. [**11**] *Let D be a digraph and $S_1, S_2, \ldots, S_k$ be its strong components. Then $E(D) = \sum\limits_{j=1}^{k} E(S_j)$.*

2. Real numbers that are not digraph energies

We now show that the energy of a digraph, if it is a rational number, must be an even positive integer (for a proof in the undirected case, using the concept of additive compounds, see [**2**]. Our proof applies well to the undirected case and is a graph theoretical one.)

Theorem 2.1. *If the energy of a digraph is a rational number, then it is an even integer.*

PROOF. Let D be a digraph with n vertices and let $z_1, z_2, \ldots, z_m$ be the eigenvalues of D with positive real parts. Then by Eq. (1.1),

$$E(D) = 2\sum_{i=1}^{m} \Re z_i\,. \tag{2.1}$$

If $z = \sum_{i=1}^{m} z_i$, then $z = \sum_{i=1}^{m} \Re z_i$, as the imaginary roots occur in conjugate pairs. By the second part of Lemma 1.1, z is an eigenvalue of some digraph H isomorphic to the the digraph obtained by repeated application of Cartesian product on m disjoint copies of D. By Lemma 1.2, if z is rational, then z is necessarily an integer and therefore by (2.1), whenever $E(D)$ is rational, $E(D)$ is necessarily an even integer. □

It has been shown by two of the present authors [**14**] that the energy of a graph is never the square root of an odd integer. The same argument holds true in the case of digraphs, as can be seen in the following theorem.

THEOREM 2.2. *The energy of a digraph cannot be the square root of an odd integer.*

PROOF. Let D be a digraph with n vertices and let $z_1, z_2, \ldots, z_m$ be the eigenvalues of D with positive real parts. Then Eq. (2.1) holds.

Same as in the previous proof, if $z = \sum_{i=1}^{m} z_i$, then $z = \sum_{i=1}^{m} \Re z_i$, and z is an eigenvalue of some digraph H obtained by repeated applications of the Cartesian product on m disjoint copies of D. By the first part of Lemma 1.1, z^2 is an eigenvalue of the digraph F isomorphic to the conjunction of two disjoint copies of H. Now,

$$z^2 = \left(\sum_{j=1}^{m} z_j\right)^2 = \left(\sum_{j=1}^{m} \Re z_j\right)^2 = r\,, \quad \text{(say)}$$

where $r > 0$.

If α is an odd integer and if $E(D) = \sqrt{\alpha}$, then

$$2\sum_{i=1}^{m} \Re z_i = 2\sum_{i=1}^{m} z_i = \sqrt{\alpha}\,.$$

That is,

$$\sum_{i=1}^{m} z_i = \frac{\sqrt{\alpha}}{2} \qquad \text{i.e.,} \qquad \left(\sum_{i=1}^{m} z_i\right)^2 = \frac{\alpha}{4}$$

implying

$$z^2 = \frac{\alpha}{4} \qquad \text{i.e.,} \qquad r = \frac{\alpha}{4}\,.$$

If α is odd, then $\frac{\alpha}{4}$ is a non-integral rational number. But $z^2 = r$ is an eigenvalue of the digraph F. Therefore r is a non-integral rational eigenvalue of digraph F, a contradiction to Lemma 1.2. Consequently, the digraph energy cannot be square root of an odd integer. □

Let H be the same digraph as in Theorem 2.2. Let H^* be the digraph isomorphic to the digraph obtained by repeated application of conjunction of r disjoint copies of H. By the first part of Lemma 1.1, z^r is an eigenvalue of H^*, where z is an eigenvalue of the digraph H as shown in Theorem 2.2.

Suppose that $E(D) = q^{1/r}$, where q is some non-negative integer. Then

$$2\sum_{i=1}^{m} \Re z_i = 2\sum_{i=1}^{n} z_i = 2z = q^{1/r}$$

i.e.,

$$z^r = \frac{q}{2^r}\,.$$

If q would not be divisible by 2^r, then z^r would be a non-integral rational number, that is, z^r would be a non-integral rational eigenvalue of the digraph H^*, a contradiction to Lemma 1.2. Therefore, we have proved the following generalization of Theorem 2.2.

Theorem 2.3. *Let r and s be integers such that $r \geqslant 1$, $0 \leqslant s \leqslant r-1$ and let q be an odd integer. Then $E(G)$ cannot be of the form $(2^s\,q)^{1/r}$.*

For $s = 0, r = 2$ we get Theorem 2.2 and for $s = 0, r = 1$ we get Theorem 2.1

In Theorem 2.1, we proved that the energy of a digraph cannot be a non-integral rational. We now prove that it cannot be the square root of a non-integral rational. The same proof applies in the undirected case.

Theorem 2.4. *The energy of a digraph cannot be the square root of a non-integral rational number.*

Proof. Let D be a digraph with n vertices and let $z_1, z_2, \ldots, z_m$ be the eigenvalues of D with positive real parts. Then Eq. (2.1) is applicable.

Same as in the above proofs, if $z = \sum\limits_{i=1}^{m} z_i$, then $z = \sum\limits_{i=1}^{m} \Re z_i$, and z is an eigenvalue of some digraph H obtained by repeated application of Cartesian product on m disjoint copies of D.

Assume that $E(D) = \sqrt{p/q}$, with p/q being a non-integral positive rational number, $(p, q) = 1$, where for positive integers a and b the symbol (a, b) denotes their gcd.

Then $2z = \sqrt{p/q}$, that is, $z^2 = p/(2q)$.

Since $(p, q) = 1$, so $(p, 2q) = 1$, that is, $z^2 = p/(2q)$ is a non-integral rational number. But z^2 is an eigenvalue of the digraph F isomorphic to conjunction of two disjoint copies of H, and hence z^2 is a non-integral rational eigenvalue of the digraph F, a contradiction to Lemma 1.2. This contradiction shows that the energy of a digraph cannot be the square root of a non-integral positive rational number. □

The following theorem is the generalization of Theorem 2.4.

Theorem 2.5. *The energy of a digraph cannot be the r-th root of a non-integral positive rational, that is*

$$E(D) \neq \left(\frac{p}{q}\right)^{1/r}$$

where r is a positive integer.

PROOF. Let D be a digraph with n vertices and let $z_1, z_2, \ldots, z_m$ be the eigenvalues of D with positive real parts. Then Eq. (2.1) is applicable.

Same as in the earlier proofs, if $z = \sum\limits_{i=1}^{m} z_i$, then $z = \sum\limits_{i=1}^{m} \Re z_i$, and z is an eigenvalue of some digraph H obtained by repeated application of Cartesian product on m disjoint copies of D.

Assume that $E(D) = (p/q)^{1/r}$ where $r \geqslant 1$. Then $2z = (p/q)^{1/r}$, implying $z^r = p/(2^r\, q)$.

As $(p,q) = 1$ then $(p, 2^r\, q) = 1$. Therefore $p/(2^r\, q)$ is a non-integral positive rational. Now z^r is the eigenvalue of the digraph H^* defined in Theorem 2.3. From above we see that z^r is non-integral rational eigenvalue of the digraph H^*, a contradiction to Lemma 1.2. This contradiction shows that the energy of a digraph cannot be the r-th root of a non-integral rational number. $\square$

Next we show that the energy of a digraph can never be a transcendental number.

THEOREM 2.6. *The energy of a digraph is an algebraic number.*

PROOF. Let $z_1, z_2, \ldots, z_m$ be the eigenvalues of D with positive real parts. Then Eq. (2.1) is applicable.

We know that $z = \sum\limits_{i=1}^{m} z_i$ is an algebraic number, since it is the eigenvalue of a digraph. Hence $E(D) = 2\sum\limits_{i=1}^{m} \Re z_i = 2\sum\limits_{i=1}^{m} z_i$ is an algebraic number.

$\square$

COROLLARY 2.1. *The energies of all digraphs with $n \geqslant 1$ vertices, form a countable set.*

PROOF. In Theorem 2.6, we have proved that the energy of a digraph is an algebraic number. The set $\mathcal{A}$ of all algebraic numbers is countable. The set of energies of all digraphs with $n \geqslant 1$ vertices is a subset of $\mathcal{A}$. Therefore, it is also countable. $\square$

REMARK 2.1. In Theorem 2.6, we proved that a transcendental number cannot be the energy of a digraph. Besides, by our above results we have shown that not all algebraic numbers are digraph energies. As the set of all transcendental numbers is an uncountable set, it follows that there is an uncountable set of positive real numbers that are not digraph energies.

Here we note that the technique of the proof of Theorem 2.6 and Corollary 2.1 in the undirected case is the same.

3. A lower bound for the energy of digraphs

In the following result, we obtain a sharp lower bound for the energy of a strongly connected digraph.

THEOREM 3.1. *If D is a strongly connected digraph, then $E(D) \geqslant 2$ with equality if and only if $D = C_r$, $r = 2, 3, 4$.*

PROOF. Since D is a strongly connected digraph then $d_i^+ \geqslant 1$ for all $i = 1, 2, \ldots, n$. Now it is well known [**3**] that $\rho \geqslant \min\{d_1^+, d_2^+, \ldots, d_n^+\}$ which implies $\rho \geqslant 1$. Hence

$$E(D) = 2\sum_{i=1}^{m} \Re z_i \geqslant 2\Re z_1 = 2\rho \geqslant 2\,.$$

If $E(D) = 2$ then $\rho = 1$. Since D is a strongly connected digraph then from part (c) of Theorem (2.1) in [**3**], D is a cycle, say $D = C_r$. It was shown in Theorem (3.1) of [**11**] that $E(C_k) > 2$ for all $k \geqslant 5$. Consequently $r = 2, 3$ or 4.

□

COROLLARY 3.1. *No positive real number less than two can be the energy of a digraph.*

PROOF. In [**12**] it was demonstrated that a digraph is acyclic if and only if its energy is zero. By Lemma 1.3, the energy of a digraph is the sum of energies of its strong components [**11**]. By Theorem 3.1, the energy of a strong component is always greater than or equal to two. □

EXAMPLE 3.1. The golden ratio $\tau = \dfrac{1+\sqrt{5}}{2}$, cannot be the energy of a digraph.

PROOF. $\tau \approx 1.618 < 2\,.$ □

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Department of Mathematics, University of Kashmir, Srinagar, India
E-mail address: `sdpirzada@yahoo.co.in`

Department of Mathematics, University of Kashmir, Srinagar, India
E-mail address: `mushtaqab1125@gmail.com`

Faculty of Science, University of Kragujevac, Kragujevac, Serbia
E-mail address: `gutman@kg.ac.rs`

Instituto de Matemáticas, Universidad de Antioquia, Medellín, Colombia
E-mail address: `rada.juanpa@gmail.com`