BULLETIN OF INTERNATIONAL MATHEMATICAL VIRTUAL INSTITUTE
ISSN 1840-4367
Vol. 3(2013), 49-57
Available on: www.imvibl.org

Former
BULLETIN OF SOCIETY OF MATHEMATICIANS BANJA LUKA
ISSN 0354-5792 (o), ISSN 1986-521X (p)

INEQUALITIES INVOLVING CERTAIN BIVARIATE MEANS

EDWARD NEUMAN

Abstract. Inequalities involving two Seiffert means, logarithmic mean, and the Neuman-Sándor mean are established. Those results are utilized to obtain four inequalities which have structure of the Wilker and Huygens inequalities for the trigonometric and hyperbolic functions.

1. INTRODUCTION

In recent years a significant progress has been made in theory of means with a special emphasis on the inequalities satisfied by means under discussion. Comparison results were always in the center of attention of many researchers. Inequalities obeyed by particular means were often used to obtain inequalities satisfied either by elementary or higher transcendental functions. A list of published research in this area is too long to be cited here.

This paper deals with inequalities satisfied by certain bivariate means. Those included here are two Seiffert means, logarithmic mean, and a mean which recently has been called by several researches the Neuman-Sándor mean. Their definitions are recalled in Section 2. Therein we also provide definitions of other bivariate means used in this paper. Two families of one-parameter means are introduced in Section 3. Some elementary properties of those means are also included in this section. Four inequalities involving means, mentioned in Sections 2 and 3, are

2000 *Mathematics Subject Classification.* 26E60, 26D07, 26D05.

Key words and phrases. Logarithmic mean, Seiffert means, Neuman-Sándor mean, Wilker and Huygens type inequalities.

established in Section 4. These results are further utilized in Section 5 to obtain Wilker and Huygens type inequalities for trigonometric and hyperbolic functions.

2. DEFINITIONS AND PRELIMINARIES

In this section we provide definitions of several bivariate means used in the subsequent sections of this paper.

Let $a, b > 0$. In order to avoid trivialities we will always assume that $a \neq b$. The unweighted arithmetic mean A of a and b is defined as

$$A = \frac{a+b}{2}.$$

The bivariate means discussed in this paper include the first and the second Seiffert means, denoted by P and T, respectively, the Neuman-Sándor mean M, and the logarithmic mean L. Recall that

$$\begin{aligned} P &= A\frac{v}{\sin^{-1} v}, \quad T = A\frac{v}{\tan^{-1} v}, \\ M &= A\frac{v}{\sinh^{-1} v}, \quad L = A\frac{v}{\tanh^{-1} v}, \end{aligned} \tag{2.1}$$

where

$$v = \frac{a-b}{a+b}. \tag{2.2}$$

(see [16], [17], [11]). Clearly $0 < |v| < 1$.

Other unweighted bivariate means used in this paper are the harmonic mean H, geometric mean G, root-square mean Q and the contra-harmonic mean C which are defined in usual way

$$H = \frac{2ab}{a+b}, \quad G = \sqrt{ab}, \quad Q = \sqrt{\frac{a^2+b^2}{2}}, \quad C = \frac{a^2+b^2}{a+b}. \tag{2.3}$$

One can easily verify that the means defined in (2.3) all can be expressed in terms of A and v. We have

$$\begin{aligned} H &= A(1-v^2), \quad G = A\sqrt{1-v^2}, \\ Q &= A\sqrt{1+v^2}, \quad C = A(1+v^2). \end{aligned} \tag{2.4}$$

All the means mentioned above are comparable. It is known that

$$H < G < L < P < A < M < T < Q < C \tag{2.5}$$

(see, e.g., [11]).

The four means listed in (2.1) are special cases of the Schwab-Borchardt mean SB which is defined as follows

$$SB(a,b) \equiv SB = \begin{cases} \dfrac{\sqrt{b^2 - a^2}}{\cos^{-1}(a/b)} & \text{if } a < b, \\ \dfrac{\sqrt{a^2 - b^2}}{\cosh^{-1}(a/b)} & \text{if } b < a. \end{cases}$$

(see, e.g., [2], [3]). This mean has been studied extensively in [11], [12], and in [7]. It is well known that the mean SB is strict, nonsymmetric and homogeneous of degree one in its variables.

It has been pointed out in [11] that

$$\begin{aligned} P &= SB(G,A), \quad T = SB(A,Q), \\ M &= SB(Q,A), \quad L = SB(A,G). \end{aligned} \tag{2.6}$$

3. MEANS Λ_k AND Ω_k

In this section we introduce two one-parameter families of bivariate means which involve, as particular cases, the harmonic mean H, contraharmonic mean C, and the centroidal mean D.

In what follows, let $k \geq 1$. We define

$$\Lambda_k \equiv \Lambda_k(a,b) = A(1 - \frac{1}{k}v^2) \tag{3.1}$$

and

$$\Omega_k \equiv \Omega_k(a,b) = A(1 + \frac{1}{k}v^2). \tag{3.2}$$

It follows from (3.1) that the function $k \to \Lambda_k$ is strictly increasing. Similarly, (3.2) implies that the function $k \to \Omega_k$ is strictly decreasing. Means Λ_k and Ω_k are convex combinations of H and C. Indeed substituting (2.2) into (3.1) and (3.2) we obtain

$$\Lambda_k = \frac{k+1}{2k}H + \frac{k-1}{2k}C \tag{3.3}$$

and

$$\Omega_k = \frac{k-1}{2k}H + \frac{k+1}{2k}C. \tag{3.4}$$

Some classical bivariate means belong to one of the family of means defined in this section. For instance, $H = \Lambda_1$ and $C = \Omega_1$. One can easily verify that the cendroidal mean

$$D = \frac{2(a^2 + b^2 + ab)}{3(a+b)}$$

satisfies $D = \Omega_3$.

It is an elementary task to show that

$$H \le \Lambda_k < A < \Omega_k \le C \tag{3.5}$$

for all $k \ge 1$.

Refinements of these inequalities are obtained in the following

Proposition 3.1. *Let*

$$\gamma = 1 + \sqrt{1 - v^2} \ \textit{and} \ \ \delta = 1 + \sqrt{1 + v^2}, \tag{3.6}$$

where v is defined in (2.2). Then

$$H < \Lambda_k < G < \Lambda_l < A < \Omega_n < Q < \Omega_m < C, \tag{3.7}$$

provided $1 < k < \gamma$, $l > \gamma$, $1 < m < \delta$, *and* $n > \delta$.

Proof. In the chain of inequalities (3.7) only the second, third, sixth, and the seventh inequalities must be established. The remaining ones follow from inequalities (3.5). For the proof of the second inequality in (3.7) we utilize a second formula in (2.4) and (3.1) to obtain

$$1 - \frac{1}{k}v^2 < \sqrt{1 - v^2}.$$

Hence, after a little algebra

$$k^2 - 2k + v^2 < 0. \tag{3.8}$$

Since the only positive root of the quadratic polynomial $k^2 - 2k + v^2$ is equal to γ, the domain of validity of the second inequality in (3.7) follow. The domain of validity of the third inequality in (3.7) consists of all positive numbers for which the direction of the sign of inequality (3.8) is reversed. This yields the asserted result. Inequalities sixth and seventh in (3.7) can be established in a similar fashion. Making use of the third formula in (2.4) and (3.2), with k replaced by n, yields

$$1 + \frac{1}{n}v^2 < \sqrt{1 + v^2}.$$

Hence

$$n^2 - 2n - v^2 < 0. \tag{3.9}$$

It is easy to see that the inequality (3.9) is satisfied for positive values of n such $n > \delta$. To complete the proof it suffices to solve the inequality $m^2 - 2m - v^2 > 0$ for all values of m that are greater than 1. We omit further details. $\square$

4. INEQUALITIES INVOLVING MEANS AND THEIR RECIPROCALS

The goal is to establish four inequalities involving means defined in (2.1). Before we will state and prove the first result of this section let us recall two inequalities established in [12]

$$LT < A^2 \text{ and } PM < A^2.$$

Inequalities obtained in the following theorem bear resemblance of the last two ones.

Theorem 4.1. *The following inequalities*

$$L\Omega_3 < A^2, \tag{4.1}$$

$$T\Lambda_3 < A^2, \tag{4.2}$$

$$P\Omega_6 < A^2, \tag{4.3}$$

$$M\Lambda_6 < A^2 \tag{4.4}$$

are valid.

Proof. It follows from the last equation of (2.1) and from [1, 4.6.33] that

$$\frac{A}{L} = \frac{\tanh^{-1} v}{v} = 1 + \frac{1}{3}v^2 + \frac{1}{5}v^4 + \ldots,$$

where the nonzero terms of this series are all positive. This implies that

$$\frac{A}{L} > 1 + \frac{1}{3}v^2 = \Omega_3.$$

The proof of (4.1) is complete. The last inequality also appears in [14]. See also [15]. For the proof of (4.2) we appeal to (2.1) again and use [1, 4.4.62] to obtain

$$\frac{A}{T} = \frac{\tan^{-1} v}{v} = 1 - \frac{1}{3}v^2 + \frac{1}{5}v^4 - \ldots,$$

where the nonzero members of this series alternate in sign. This in turn yields

$$\frac{A}{T} > 1 - \frac{1}{3}v^2 = \Lambda_3$$

which gives the asserted inequality (4.2). The remaining two inequalities (4.3) and (4.4) can be established in a similar manner using two formulas of (2.1) together with the series expansion for the function $\sin^{-1}(v)/v$ (see [1, 4.4.40]) and the power series expansion for the function $\sinh^{-1}(v)/v$ (see [1, 4.6.31]). We omit further details. □

A quadruple of inequalities, involving reciprocals of means which appear in the last theorem, are contained in the following

Corollary 4.2. *We have*

$$\frac{1}{A} < \frac{1}{2}\Big(\frac{1}{L} + \frac{1}{\Omega_3}\Big), \quad \frac{1}{A} < \frac{1}{2}\Big(\frac{1}{T} + \frac{1}{\Lambda_3}\Big),$$
$$\frac{1}{A} < \frac{1}{2}\Big(\frac{1}{P} + \frac{1}{\Omega_6}\Big), \quad \frac{1}{A} < \frac{1}{2}\Big(\frac{1}{M} + \frac{1}{\Lambda_6}\Big). \tag{4.5}$$

Proof. First inequality in (4.5) follows from (4.1). To this aim we raise both sides of the latter to the power of $-1/2$ and next apply inequality of the arithmetic and geometric means to obtain

$$\frac{1}{A} < \Big(\frac{1}{L} \cdot \frac{1}{\Omega_3}\Big)^{1/2} < \frac{1}{2}\Big(\frac{1}{L} + \frac{1}{\Omega_3}\Big).$$

The remaining three inequalities in (4.5) can be derived in the same way using inequalities (4.2)-(4.4). We omit further details. $\square$

5. APPLICATIONS TO WILKER AND HUYGENS TYPE INEQUALITIES

In this section we will utilize results of previous section to establish Wilker and Huygens type inequalities. Those results are derived with the aid of four inequalities obtained in the previous section.

We begin giving a short overview of the Wilker and Huygens inequalities. The following result

$$\left(\frac{\sin t}{t}\right)^2 + \frac{\tan t}{t} > 2 \tag{5.1}$$

($0 < |t| < \frac{\pi}{2}$) is due to Wilker [18]. Several Wilker type inequalities appear in mathematical literature. For more details see [5,6,8,9,13,19, 20] and the references therein. A hyperbolic counterpart of Wilker's inequality

$$\left(\frac{\sinh\, t}{t}\right)^2 + \frac{\tanh\, t}{t} > 2 \tag{5.2}$$

($t \neq 0$) has been established by L. Zhu [21]. See also [22] and [13].

Another inequality which recently has been studied extensively is due to Huygens [4]

$$2\frac{\sin t}{t} + \frac{\tan t}{t} > 3 \tag{5.3}$$

($0 < |t| < \frac{\pi}{2}$). Huygens inequality for the hyperbolic functions

$$2\frac{\sinh\, t}{t} + \frac{\tanh\, t}{t} > 3 \tag{5.4}$$

($t \neq 0$) was established by Neuman and Sándor in [13].

For generalizations and refinements of inequalities (5.1) - (5.4) the interested reader is referred to [19], [8], [10], [13] and the references therein.

In the proofs of the inequalities in this section we will utilize a result obtained in [10]. In order to present this result let us introduce more notation.

Throughout the sequel the letters r and s will stand for two positive numbers which satisfy the following conditions

$$r < 1 < s \tag{5.5}$$

and

$$1 < r^{\alpha} s^{\beta}, \tag{5.6}$$

where the last inequality must be satisfied for some positive numbers α and β.

In the proofs of the main results of this section we will utilize the following version of Theorem 3.1 in [10]:

Theorem A. Let $\lambda > 0$ and let $\mu > 0$. Then

$$1 < \frac{\lambda}{\lambda+\mu} r^p + \frac{\mu}{\lambda+\mu} s^q \tag{5.7}$$

if

$$q > 0 \qquad \text{and} \qquad p \le q\frac{\alpha\mu}{\beta\lambda}. \tag{5.8}$$

We are in a position to establish the following

Theorem 5.1. *Let λ and μ be positive numbers and let the numbers p and q satisfy the following conditions*

$$q > 0 \ \textit{and} \ \ p \le q\frac{\mu}{\lambda}. \tag{5.9}$$

Then the following inequalities

$$1 < \frac{\lambda}{\lambda+\mu}\Big(\frac{1}{1+\frac{1}{3}\tanh^2 t}\Big)^p + \frac{\mu}{\lambda+\mu}\Big(\frac{t}{\tanh t}\Big)^q \tag{5.10}$$

$(t \neq 0)$,

$$1 < \frac{\lambda}{\lambda+\mu}\Big(\frac{t}{\tan t}\Big)^p + \frac{\mu}{\lambda+\mu}\Big(\frac{1}{1-\frac{1}{3}\tan^2 t}\Big)^q \tag{5.11}$$

$(0 < |t| < \pi/4)$,

$$1 < \frac{\lambda}{\lambda+\mu}\Big(\frac{1}{1+\frac{1}{6}\sin^2 t}\Big)^p + \frac{\mu}{\lambda+\mu}\Big(\frac{t}{\sin t}\Big)^q \tag{5.12}$$

$(0 < |t| < \pi/2)$, *and*

$$1 < \frac{\lambda}{\lambda+\mu}\Big(\frac{t}{\sinh t}\Big)^p + \frac{\mu}{\lambda+\mu}\Big(\frac{1}{1-\frac{1}{6}\sinh^2 t}\Big)^q \tag{5.13}$$

$(0 < |t| < \sinh^{-1}(1))$ *are valid.*

Proof. We shall establish inequality (5.13) only. The remaining ones (5.10)-(5.12) can be established by the same method as the one employed below. Let

$$r = \frac{A}{M} \text{ and } s = \frac{A}{\Lambda_6}.$$

It follows from (4.4) that r and s satisfy conditions (5.5) and (5.6), the latter with $\alpha = \beta$. Making use of the third formula in (2.1) and (3.1) with $k = 6$ we obtain

$$r = \frac{\sinh^{-1} v}{v} \text{ and } s = \frac{1}{1-\frac{1}{6}v^2},$$

where $0 < |v| < 1$. With the substitution $v = \sinh t$ $(0 < |t| < \sinh^{-1}(1))$ formulas for r and s become

$$r = \frac{t}{\sinh t} \text{ and } s = \frac{1}{1-\frac{1}{6}\sinh^2 t}$$

To obtain the desired result it suffices to apply Theorem A. □

Acknowledgments

The author is indebted to an anonymous referee for calling his attention to references [14] and [15].

References

[1] M. Abramowitz, I.A. Stegun, *Handbook of Mathematical Functions*, Dover Publ., New York, 1970.

[2] J.M. Borwein, P.B. Borwein, *Pi and AGM: A Study in Analytic Number Theory and Computational Complexity*, John Wiley and Sons, New York, 1987.

[3] B.C. Carlson, *Algorithms involving arithmetic and geometric means*, Amer. Math. Monthly **78** (1971), 496–505.

[4] C. Huygens, *Oeuvres Completes 1888–1940*, Sociéte Hollondaise des Science, Haga.

[5] E. Neuman, *One- and two-sided inequalities for Jacobian elliptic functions and related results*, Integral Transform. Spec. Funct., **21** (2010), No. 6, 399–407.

[6] E. Neuman, *Inequalities involving inverse circular and inverse hyperbolic functions II*, J. Math. Inequal., **4** (2010), 11–14.

[7] E. Neuman, *Inequalities for the Schwab - Borchardt mean and their applications*, J. Math. Inequal. **5** (2011), 601–609.

[8] E. Neuman, *On Wilker and Huygens type inequalities*, Math. Inequal. Appl., **15** (2012), No. 2, 271–279.

[9] E. Neuman, *Refinements and generalizations of certain inequalities involving trigonometric and hyperbolic functions*, Adv. Inequal. Appl. **1** (2012), 1–11.

[10] E. Neuman, *Inequalities for weighted sums of powers and their applications*, Math. Inequal. Appl. **15** (2012), No. 4, 995–1005.

[11] E. Neuman, J.Sándor, *On the Schwab - Borchardt mean*, Math. Pannon. **14** (2003), 253–266.

[12] E. Neuman, J.Sándor, *On the Schwab - Borchardt mean II*, Math. Pannon. **17** (2006), 49–59.

[13] E. Neuman, J. Sándor *On some inequalities involving trigonometric and hyperbolic functions with emphasis on the Cusa-Huygens, Wilker and Huygens inequalities*, Math. Inequal. Appl. **13** (2010), No. 4, 715–723.

[14] J. Sándor, *On certain identities for means*, Studia Univ. Babes-Bolyai, Math. **38** (1993), No.4, 7–14.

[15] J. Sándor, T. Trif, *Some new inequalities for means of two arguments*, Int. J. Math. Math. Sci. **25** (2001), 525–532.

[16] H.-J. Seiffert, *Problem 887*, Nieuw. Arch. Wisk. **11** (1993), 176.

[17] H.-J. Seiffert, *Aufgabe 16*, Würzel **29** (1995), 87.

[18] J. B. Wilker, Problem E 3306, *Amer. Math. Monthly* **96** (1989), 55.

[19] S. -H. Wu, H. M. Srivastava, A weighted and exponential generalization of Wilker's inequality and its applications, *Integral Transform. Spec. Funct.* **18** (2007), No. 8, 525–535.

[20] L. Zhu, A new simple proof of Wilker's inequality, *Math. Inequal. Appl.*, **8** (2005), No. 4, 749–750.

[21] L. Zhu, On Wilker-type inequalities, *Math. Inequal. Appl.* **10** (2007), No. 4, 727–731.

[22] L. Zhu, Some new Wilker type inequalities for circular and hyperbolic functions, *Abstract Appl. Analysis*, Vol. 2009, Article ID 485842, 9 pages.

Received by editors January 09, 2013; in revised form March 10, 2013;
available online March 18, 2013

Department of Mathematics, Mailcode 4408, Southern Illinois University, 1245 Lincoln Drive, Carbondale, IL 62901, USA

E-mail address: `edneuman@siu.edu`