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FINITELY QUASI-CONJUGATIVE RELATIONS

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Abstract. In a former article, the first author introduces and analyzes quasi-conjugative relations. In this paper, following Jiang Guanghao and Xu Luoshan's concepts of finitely conjugative and finitely dual normal relations on sets, the concept of finitely quasi-conjugative relations is introduced. A characterization of finitely quasi-conjugative relations is obtained. Particulary we show when the anti-order relation $\nleqslant$ is finitely quasi-conjugative.

1. Introduction

The concept of conjugative relations was introduced by Guanghao Jiang and Luoshan Xu ([**1**]), and the concept of dually normal relations was introduced and analyzed by Jiang Guanghao and Xu Luoshan in [**2**]. In former article [**4**], the first author introduced and analyzed a new class of relations in set - class of *quasi-conjugative relations* on sets. In this article, as a continuation of article [**4**], following Jiang Guanghao and Xu Luoshan's concepts of finitely conjugative and finitely dual normal relations, introduced in articles [**1**] and [**2**], we introduce and analyze the notion of finitely quasi-conjugative relations.

For a set X, we call ρ a relation on X, if $\rho \subseteq X \times X$. Let $\mathcal{B}(X)$ be denote the set of all binary relations on X. For $\alpha, \beta \in \mathcal{B}(X)$, define

$$\beta \circ \alpha = \{(x,z) \in X \times X : (\exists y \in X)((x,y) \in \alpha \wedge (y,z) \in \beta)\}.$$

The relation $\beta \circ \alpha$ is called the composition of α and β. It is well known that $(\mathcal{B}(X), \circ)$ is a semigroup. The latter family, with the composition, is not only a semigroup, but also a monoid. Namely, $Id_X = \{(x,x) : x \in X\}$ is its identity element. For a relation α on a set X, define $\alpha^{-1} = \{(x,y) \in X \times X : (y,x) \in \alpha\}$ and $\alpha^C = (X \times X) \setminus \alpha$.

Let A and B be subsets of X. For $\alpha \in \mathcal{B}(X)$, set

$$A\alpha = \{y \in X : (\exists a \in A)((a,y) \in \alpha)\}, \ \ \alpha B = \{x \in X : (\exists b \in B)((x,b) \in \alpha)\}.$$

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It is easy to see that $A\alpha = \alpha^{-1}A$ holds. Specially, we put $a\alpha$ if $\{a\}\alpha$ and αb if $\alpha\{b\}$.

2. Quasi-conjugative relations

The following classes of elements in the semigroup $\mathcal{B}(X)$ have been investigated.

Definition 2.1. For a relation $\alpha \in \mathcal{B}(X)$ we say that it is:
– *regular* if there exists a relation $\beta \in \mathcal{B}(X)$ such that
$$\alpha = \alpha \circ \beta \circ \alpha.$$
– *normal* ([**3**]) if there exists a relation $\beta \in \mathcal{B}(X)$ such that
$$\alpha = \alpha \circ \beta \circ (\alpha^C)^{-1}.$$
– *dually normal* ([**2**]) if there exists a relation $\beta \in \mathcal{B}(X)$ such that
$$\alpha = (\alpha^C)^{-1} \circ \beta \circ \alpha.$$
– *conjugative* ([**1**]) if there exists a relation $\beta \in \mathcal{B}(X)$ such that
$$\alpha = \alpha^{-1} \circ \beta \circ \alpha.$$
– *dually conjugative* ([**1**]) if there exists a relation $\beta \in \mathcal{B}(X)$ such that
$$\alpha = \alpha \circ \beta \circ \alpha^{-1}.$$

The notion of *quasi-conjugative relation* is introduced in paper [**4**] by the following definition.

Definition 2.2. For a relation $\alpha \in \mathcal{B}(X)$ we say that it is a *quasi-conjugative* relation on X if there exists a relation $\beta \in \mathcal{B}(X)$ such that
$$\alpha = \alpha^{-1} \circ \beta \circ \alpha^C.$$

The latter term is not convenient since thus a conjugative relation need not be quasi-conjugative. The same remark can be applied to the terms normal and regular too.

In the following propositions we give characterization of quasi-conjugative relations. Here, for a relation α, we use the notation α^* to sign the maximal elements of all family of relation β such that $\alpha^{-1} \circ \beta \circ \alpha^C \subseteq \alpha$.

Proposition 2.1 ([**4**], Lemma 2.1). *For a binary relation $\alpha \in \mathcal{B}(X)$, relation*
$$\alpha^* = ((\alpha \circ \alpha^C \circ (\alpha^C)^{-1})^C$$
is the maximal element in family of all relation $\beta \in \mathcal{B}(X)$ such that
$$\alpha^{-1} \circ \beta \circ \alpha^C \subseteq \alpha.$$

Proposition 2.2 ([**4**], Theorem 2.1). *For a binary relation α on a set X, the following conditions are equivalent:*
(1) *α is a quasi-conjugative relation.*
(2) *For all $x, z \in X$, if $(x, y) \in \alpha$, there exists $u, v \in X$ such that:*
(a) $(x, u) \in \alpha^C \wedge (y, v) \in \alpha$,
(b) $(\forall s, t \in X)((s, u) \in \alpha^C \wedge (t, v) \in \alpha \Longrightarrow (s, t) \in \alpha)$.
(3) $\alpha \subseteq \alpha^{-1} \circ \alpha^* \circ \alpha^C$.

Proposition 2.3 ([**4**], Corollary 2.1). *Let* $(X, \leqslant)$ *be a poset. Then* $\nleqslant$ *is a quasi-conjugative relation on* X *if and only if for all* $x, y \in X$ *such that* $x \nleqslant y$ *there exist elements* $u, v \in X$ *such that :*
(a') $x \leqslant u \wedge y \nleqslant v$, *and*
(b') $(\forall z \in X)(z \nleqslant u \vee z \leqslant v)$.

3. Finitely quasi-conjugative relations

In this section we introduce the concept of finitely quasi-conjugative relations and give a characterization of this relations. For that we need the concept of *finite extension* of a relation. That notion and belonging notation we borrow from articles [**1**] and [**2**]. For any set X, let $X^{(<\omega)} = \{F \subseteq X : F \text{ is finite and nonempty } \}$.

Definition 3.1. ([**1**], Definition 3.3; [**2**], Definition 3.4) Let α be a binary relation on a set X. Define a binary relation $\alpha^{(<\omega)}$ on $X^{(<\omega)}$, called the *finite extension* of α, such that

$$(\forall F, G \in X^{(<\omega)})((F, G) \in \alpha^{(<\omega)} \Longleftrightarrow G \subseteq F\alpha).$$

From Definition 3.1, we immediately obtain that

$$(\forall F, G \in X^{(<\omega)})((F, G) \in (\alpha^C)^{(<\omega)} \Longleftrightarrow G \subseteq F\alpha^C).$$

and

$$(\forall F, G \in X^{(<\omega)})((F, G) \in (\alpha^{-1})^{(<\omega)} \Longleftrightarrow G \subseteq F\alpha^{-1} = \alpha F).$$

Now, we can introduce concept of *finitely quasi-conjugative relation.*

Definition 3.2. A relation α on a set X is called *finitely quasi-conjugative* if there exists a relation β on X such that

$$\alpha^{(<\omega)} = (\alpha^{-1})^{(<\omega)} \circ \beta^{(<\omega)} \circ (\alpha^C)^{(<\omega)}.$$

Although it seems, in accordance with Definition 2.2, it would be better to coll a relation α on X to be finitely quasi-conjugative if its finite extension to $X^{(<\omega)}$ is a quasi-conjugative relation, we will not use that option. That concept is different from our concept given by Definition 3.2.

Now we give an essential characterization of finitely quasi-conjugative relations.

Theorem 3.1. *A relation* α *on a set* X *if a finitely quasi-conjugative relation if and only if for all* $F, G \in X^{(<\omega)}$, *if* $G \subseteq F\alpha$, *then there are* $U, V \in X^{(<\omega)}$, *such that*
(i) $U \subseteq F\alpha^C$, $G \subseteq \alpha V$, *and*
(ii) *for all* $S, T \in X^{(<\omega)}$, *if* $U \subseteq S\alpha^C$ *and* $T \subseteq \alpha V$ *then* $T \subseteq S\alpha$.

Proof. ($\Longrightarrow$) Let α be a finitely quasi-conjugative relation on set X. Then there is a relation $\beta^{(<\omega)} \subseteq X^{(<\omega)} \times X^{(<\omega)}$ such that $(\alpha^{-1})^{(<\omega)} \circ \beta^{(<\omega)} \circ (\alpha^C)^{(<\omega)} = \alpha^{(<\omega)}$. For all $(F, G) \in (X^{(<\omega)})^2$, if $G \subseteq F\alpha$, i.e., $(F, G) \in \alpha^{(<\omega)}$, thus $(F, G) \in (\alpha^{-1})^{(<\omega)} \circ \beta^{(<\omega)} \circ (\alpha^C)^{(<\omega)}$. Whence there is $(U, V) \in (X^{(<\omega)})^2$ such that $(F, U) \in (\alpha^C)^{(<\omega)}$, $(U, V) \in \beta^{(<\omega)}$ and $(V, G) \in (\alpha^{-1})^{(<\omega)}$, i.e., $U \subseteq F\alpha^C$, $G \subseteq V\alpha^{-1} = \alpha V$. Hence we get the condition (i).

Now we check the condition (ii). For all $(S,T) \in (X^{(<\omega)})^2$, if $U \subseteq S\alpha^C$ and $T \subseteq \alpha V$, i.e., $(S,U) \in (\alpha^C)^{(<\omega)}$ and $(V,T) \in (\alpha^{-1})^{(<\omega)}$, then by $(U,V) \in \beta^{(<\omega)}$, we have $(S,T) \in (\alpha^{-1})^{(<\omega)} \circ \beta^{(<\omega)} \circ (\alpha^C)^{(<\omega)}$, i.e., $(S,T) \in \alpha^{(<\omega)}$. Hence $T \subseteq S\alpha$.

($\Longleftarrow$) Let α be a relationon a set X such that for $F, G \in X^{(<\omega)}$ with $G \subseteq F\alpha$ there are $U, V \in X^{(<\omega)}$ such that conditions (i) and (ii) hold. Define a binary relation $\beta^{(<\omega)} \subseteq X^{(<\omega)} \times X^{(<\omega)}$ by

$$(F,G) \in \beta \Longleftrightarrow (\forall S,T \in X^{(<\omega)})((F \subseteq S\alpha^C \wedge T \cap \alpha G \neq \emptyset) \Longrightarrow T \cap S\alpha \neq \emptyset).$$

First, check that (a) $(\alpha^{-1})^{(<\omega)} \circ \beta^{(<\omega)} \circ (\alpha^C)^{(<\omega)} \subseteq \alpha^{(<\omega)}$ holds. For all $H, W \in X^{(<\omega)}$, if $(H,W) \in (\alpha^{-1})^{(<\omega)} \circ \beta^{(<\omega)} \circ (\alpha^C)^{(<\omega)}$, then there are $F, G \in X^{(<\omega)}$ with $(H,F) \in (\alpha^C)^{(<\omega)}$, $(F,G) \in \beta^{(<\omega)}$ and $(G,W) \in (\alpha^{-1})^{(<\omega)}$. Then $F \subseteq H\alpha^C$ and $W \subseteq G\alpha^{-1} = \alpha G$. For all $w \in W$, let $S = H$, $T = \{w\}$. Then $F \subseteq S\alpha^C$ and $\alpha G \cap T \neq \emptyset$ because $w \in T$ and $w \in \alpha G$. Since $(F,G) \in \beta^{(<\omega)}$, we have that $F \subseteq S\alpha^C \wedge \alpha G \cap T \neq \emptyset$ implies $T \cap S\alpha \neq \emptyset$. Hence, $w \in S\alpha$, i.e. $W \subseteq S\alpha$. So, we have $(H,W) = (S,W) \in \alpha^{(<\omega)}$. Therefore, we have $(\alpha^{-1})^{(<\omega)} \circ \beta^{(<\omega)} \circ (\alpha^C)^{(<\omega)} \subseteq \alpha^{(<\omega)}$.

The second, check that (b) $\alpha^{(<\omega)} \subseteq (\alpha^{-1})^{(<\omega)} \circ \beta^{(<\omega)} \circ (\alpha^C)^{(<\omega)}$ holds. For all $H, W \in X^{(<\omega)}$, if $(H,W) \in \alpha^{(<\omega)}$ (i.e., $W \subseteq H\alpha$), there are $A, B \in X^{(<\omega)}$ such that:
(i') $A \subseteq H\alpha^C$, $W \subseteq \alpha B$, and
(ii') for all $S, T \in X^{(<\omega)}$, if $A \subseteq S\alpha^C$ and $T \subseteq \alpha B$, then $T \subseteq S\alpha$.

Now, we have to show that $(A,B) \in \beta^{(<\omega)}$. Let be for all $(C,D) \in (X^{(<\omega)})^2$ the following $A \subseteq D\alpha^C$ and $D \cap \alpha B \neq \emptyset$ hold. From $D \cap \alpha B \neq \emptyset$ follows that htere exists an element $d \in D \cap \alpha B (\neq \emptyset)$. So, $d \in D$ and $d \in \alpha B$. Put $S = C$ and $T = \{d\}$. Then, by (ii'), we have

$$(A \subseteq S\alpha^C \wedge T = \{d\} \subseteq \alpha B) \Longrightarrow \{d\} = T \subseteq S\alpha,$$

i.e. $\emptyset \neq D \cap S\alpha = T \cap S\alpha$. Therefore, $(A,B) \in \beta^{(<\omega)}$ by definition of $\beta^{(<\omega)}$. Finally, for $(H,A) \in (\alpha)^{(<\omega)}$, $(A,B) \in \beta^{(<\omega)}$ and $(B,W) \in (\alpha^{-1})^{(<\omega)}$ follows that $(H,W) \in (\alpha^{-1})^{(<\omega)} \circ \beta^{(<\omega)} \circ (\alpha^C)^{(<\omega)}$.

By assertion (a) and (b), finally we have $\alpha^{(<\omega)} = (\alpha^{-1})^{(<\omega)} \circ \beta^{(<\omega)} \circ (\alpha^C)^{(<\omega)}$ □

Corollary 3.1. *Let α be a relation on a set X. Then α is a finitely quasi-conjugative on X if and only if for all elements $x, y \in X$ such that $(x,y) \in \alpha$ there are finite subsets $U, V \in X^{(<\omega)}$ such that*
(1^0) $(\forall u \in U)((x,u) \in \alpha^C) \wedge (\exists v \in V)((y,v) \in \alpha)$, *and*
(2^0) *for all $S \in X^{(<\omega)}$ and $t \in X$ holds*

$$(U \subseteq S\alpha^C \wedge (\exists v \in V)((y,v) \in \alpha)) \Longrightarrow (\exists s \in S)((s,t) \in \alpha)\ .$$

Proof. Let α be a finitely quasi-conjugative relation on X and let x, y be elements of X such that $(x,y) \in \alpha$. If we put $F = \{x\}$ and $G = \{y\}$ in Theorem

3.1 then there exist finite U and V of $X^{(<\omega)}$ such that conditions (1^0) and (2^0) hold.

Opposite, let for all elements $x, y \in X$ such that $(x, y) \in \alpha$ be there are U and V of $X^{(<\omega)}$ such that conditions (1^0) and (2^0) hold. Define binary relation $\beta^{<\omega} \subseteq X^{<\omega} \times X^{<\omega}$ by

$$(A, B) \in \beta^{(<\omega)} \iff (\forall S \in X^{<\omega})(\forall t \in X)((A \subseteq S\alpha^C \wedge t \in \alpha B) \Longrightarrow t \in S\alpha).$$

The proof that the equality $(\alpha^{-1})^{(<\omega)} \circ \beta^{(<\omega)} \circ (\alpha^C)^{(<\omega)} = \alpha^{(<\omega)}$ holds is some as in the Theorem 3.1. So, the relation α is a finitely quasi-conjugative. □

THEOREM 3.2. *Let $(L, \leqslant)$ be a poset. Then the relation $\nleqslant$ on L is a finitely quasi-conjugative relation if and only if for all $x, y \in L$ such that $x \nleqslant y$, there exist finite subsets U and V of L such that*
(a) $(\forall u \in U)(x \leqslant u)$ *and* $(\exists v \in V)(y \nleqslant v)$ *and*
(b) $(\forall z \in L)((\exists u \in U)(z \nleqslant u) \vee (\forall v \in V)(z \leqslant v))$.

PROOF. Let $x, y \in L$ such that $x \nleqslant y$. Then by the finitely quasi-conjugativity of $\nleqslant$, there exist finite subsets U and V of L such that
(1) $(\forall u \in U)(x \leqslant u) \wedge (\exists v \in V)(v \nleqslant y)$, and
(2) for all $S \in L^{(<\omega)}$ and $t \in L$ the following holds

$$((\forall u \in U)(\exists s \in S)(s \leqslant u) \wedge (\exists v \in V)(t \nleqslant v)) \Longrightarrow (\exists s' \in S)(s' \nleqslant t).$$

For $z \in L$, let $S = \{z\} = \{t\}$. Then by (2), from

$$(\forall u \in U)(z \leqslant u) \wedge (\exists v \in V)(z \nleqslant v)$$

implies $z \nleqslant z$. It is a contradiction. Hence, we have

$$\neg((\forall u \in U)(z \leqslant u) \wedge (\exists v \in V)(z \nleqslant v)).$$

So, finally, we have

$$(\exists u \in U)(z \nleqslant u) \vee (\forall v \in V)(z \leqslant v).$$

Let for $(x, y) \in L^2$ be $x \nleqslant y$ holds and let there exist finite subsets U and V of L satisfying conditions (a) and (b). So, the condition (a) is the condition (1^0) in Corollary 3.1.

Let $S \in L^{(<\omega)}$ and $t \in L$ with $(\forall u \in U)(\exists s \in S)(s \leqslant u)$ and $(\exists v \in V)(t \nleqslant v)$ holds. Suppose that $(\forall s \in S)(s \leqslant t)$ holds. Then, by (b), for $S = \{s\}$ and $z = s$, we have

$$(\exists u \in U)(s \nleqslant u) \vee (\forall v \in V)(s \leqslant v).$$

The first option is impossible because $s \leqslant u$. Then from $(\exists v \in V)(t \nleqslant v)$ and $s \leqslant t$ follows $(\exists v \in V)(s \nleqslant v)$. It is in contradiction with $(\forall v \in V)(s \leqslant v)$. So, must to be $\neg(\forall s \in S)(s \leqslant t)$. Thus $(\exists s \in S)(s \nleqslant t)$. Hence $\nleqslant$ satisfies also condition (2^0) in Corollary 3.1. Finally, the relation $\nleqslant$ is a finitely quasi-conjugative relation on L. □

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