BULLETIN OF THE INTERNATIONAL MATHEMATICAL VIRTUAL INSTITUTE
ISSN 1840-4367
Vol. 3(2013), 103-111
Available on www.imvibl.org

Former
BULLETIN OF THE SOCIETY OF MATHEMATICIANS BANJA LUKA
ISSN 0354-5792 (o), ISSN 1986-521X (p)

APPLICABILITY OF LAPLACE DECOMPOSITION METHOD FOR SOLVING CERTAIN DIFFERENTIAL - DIFFERENCE EQUATION OF ORDER $(1, 2)$

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Abstract. Laplace decomposition method is based on Laplace transform method and Adomian decomposition method. In this paper we show that the method is applicable to certain successive interval valued linear as well as nonlinear differential-difference equations of order $(1, 2)$, that means the differential is of order one and the difference is of order two. It is also shown that the method gives exact solution for linear problems and suitable approximate solution for nonlinear problems. Three problems are selected to illustrate the applicability of the method.

1. Introduction

In the recent times, many researchers [6, 8] are interested in solving a singularly perturbed second order integro-differential-difference equation with one interval condition involving left extreme point of the boundary and another boundary condition at the right extreme of the boundary. The singular perturbation parameter and the delay parameter are selected as small as possible. Such problems play an important role in variety of physical problems such as microscale heat transfer, diffusion in polymers, control of chaotic systems and so on (relevant references quoted in [8]). In the present paper, we formulate a different problem, namely, an integro-differential-difference equation with differential order one and difference of order two with only interval condition. This can be done by considering the

1991 *Mathematics Subject Classification.* Primary NECESSARY!!!; Secondary OPTIONAL.
Key words and phrases. Differential-difference equation, Laplace decomposition method, Laplace transform method, Successive interval valued problem.

following integro-differential-difference equation of order $(2, 2)$:

$$\begin{aligned}\epsilon\, u''(t) &= u'(t) - [f(t) + F_1(u(t-\omega), u'(t-\omega)) + F_2(u(t-2\omega), u'(t-2\omega)) \\ &\quad + \int_0^t G(u(t_1 - 2\omega))dt_1].\end{aligned}$$

We allow some times $G \equiv 0$. Then it will be a differential-difference equation of order $(2, 2)$. Further, if we set $\epsilon = 0$, then it becomes differential-difference equation of order $(1, 2)$. Since it is a first order differential-difference equation, we avoid the boundary condition at the right extreme and work with only one interval condition:

$$u(t) = k, \qquad t \in (0, 2\omega).$$

Again only interval condition is used for integro-differential-difference equation of order $(1, 2)$.

In the present paper, we apply Laplace decomposition method for such problems. The method is motivated by Adomian decomposition method for solving differential equations [1,2,5,7,10], Laplace transform method for solving differential-difference equations [4] and Laplace decomposition method as well as Laplace decomposition with Pade approximation for solving integro-differential equations [3]. It is shown in [9] that the method gives exact solution for linear problems and suitable approximate solution for nonlinear problems related to integro-differential-difference equation with both differential and difference of order one as well as one interval condition.

The plan of the paper is as follows. In Section 2, we formulate the interval valued problem (2.1) and describe the Laplace decomposition method. In Section 3, we illustrate the method with three distinct problems. In the final Section, we give concluding remarks about the suitability of the method for both linear and nonlinear problems.

2. Formulation of the interval valued problem and description of the Laplace decomposition method

Let us consider the following integro-differential-difference equation with differential order one and difference of order two:

$$\begin{aligned}u'(t) = f(t) &+ F_1(u(t-\omega), u'(t-\omega)) + F_2(u(t-2\omega), u'(t-2\omega)) \\ &+ \int_0^t G(u(t_1 - 2\omega))dt_1 \qquad t > 2\omega,\end{aligned} \tag{2.1}$$

and the following interval condition:

$$u(t) = k, \qquad 0 \leqslant t \leqslant 2\omega. \tag{2.2}$$

In the above equations (2.1) and (2.2), $\omega > 0$ and k are known constants, the functions f, F_1, F_2 and G are either linear or nonlinear functions depending upon the particular problem discussed. In order to apply Laplace decomposition method, further they are selected in such a way that, they can be approximated by Adomian polynomial suitable for the iterative computation of Laplace transform as well as inverse Laplace transform for $u(t)$.

First we note that,

$\int_0^{2\omega} u'(t)\ e^{-st}\ dt = 0$ as a result we have $\int_{2\omega}^{\infty} u'(t)\ e^{-st}\ dt = L\{u'(t)\}$.

Hence multiply both sides of (2.1) by e^{-st} and integrate between 2ω and ∞ to get

$$\begin{aligned}\int_{2\omega}^{\infty} u'(t)\ e^{-st}\ dt &= \int_{2\omega}^{\infty} f(t)\ e^{-st}\ dt + \int_{2\omega}^{\infty} F_1(u(t-\omega), u'(t-\omega))\ e^{-st}\ dt \\ &+ \int_{2\omega}^{\infty} F_2(u(t-2\omega), u'(t-2\omega))\ e^{-st}\ dt \\ &+ \int_{2\omega}^{\infty} e^{-st} \int_0^t G(u(t_1-2\omega))dt_1\ dt.\end{aligned}$$

Let us apply suitable shifting of variables to obtain

$$\begin{aligned}L\{u'(t)\} &= e^{-2\omega s} L\{f(t+2\omega)\} + e^{-\omega s} L\{F_1(u(t), u'(t))\} - \frac{\lambda\ e^{-\omega s}}{s}(1-e^{-\omega s}) \\ &+ e^{-2\omega s} L\{F_2(u(t), u'(t))\} +\ e^{-2\omega s} L\left\{\int_0^t G(u(t_1))dt_1\right\},\end{aligned}$$

where $\lambda = F_1(k, 0)$ and note that, $\int_{2\omega}^{t+2\omega} G(u(t_1-2\omega))dt_1 = \int_0^t G(u(t_1))dt_1$.
Finally, we arrive at

$$\begin{aligned}L\{u(t)\} &= \frac{k}{s} - \frac{\lambda\ e^{-\omega s}}{s^2} + \frac{\lambda\ e^{-2\omega s}}{s^2} + \frac{e^{-2\omega s}}{s} L\{f(t+2\omega)\} \\ &+ \frac{e^{-\omega s}}{s} L\{F_1(u(t), u'(t))\} + \frac{e^{-2\omega s}}{s} L\{F_2(u(t), u'(t))\} \\ &+ \frac{e^{-2\omega s}}{s^2} L\{G(u(t))\}. \qquad (2.3)\end{aligned}$$

In this paper, we seek the following type of decomposition for the series solution $u(t)$:

$$u(t) = \sum_{n=0}^{\infty} u_n(t-n\omega)e(t-n\omega), \qquad (2.4)$$

where $e(t)$ is a unit step function, given by

$$\begin{aligned}e(t-c) &= 0, \qquad t < c, \\ e(t-c) &= 1, \qquad t > c.\end{aligned}$$

By using (2.4), $u(t)$ takes the following form in each of the following intervals:

$$u(t) = \sum_{n=0}^{N} u_n(t-n\omega),\ N\omega \leqslant t \leqslant (N+1)\omega \qquad (2.5)$$

$$N = 0, 1, 2, \ldots\ .$$

On applying Laplace transformation, we obtain the following Laplace decompositions:

$$L\{u(t)\} = \sum_{n=0}^{\infty} e^{-n\omega s} L\{u_n(t)\}, \tag{2.6}$$

$$L\{F_1(u(t), u'(t))\} = \sum_{n=0}^{\infty} e^{-n\omega s} L\{A_n(t)\}, \tag{2.7}$$

$$L\{F_2(u(t), u'(t))\} = \sum_{n=0}^{\infty} e^{-n\omega s} L\{B_n(t)\} \tag{2.8}$$

$$\text{and } L\{G(u(t))\} = \sum_{n=0}^{\infty} e^{-n\omega s} L\{C_n(t)\}. \tag{2.9}$$

In (2.7), A_n $'s$ are the n^{th} Adomian Polynomials [3] of $F_1(u(t), u'(t))$ as given below:

$$\begin{aligned} A_0(t) &= F_1(x,y)\,|_{(u_0(t),u_0'(t))}, \\ A_1(t) &= \left.\frac{\partial F_1}{\partial x}\right|_{(u_0(t),u_0'(t))} u_1(t) + \left.\frac{\partial F_1}{\partial y}\right|_{(u_0(t),u_0'(t))} u_1'(t), \\ A_2(t) &= \left.\frac{\partial F_1}{\partial x}\right|_{(u_0(t),u_0'(t))} u_2(t) + \left.\frac{\partial F_1}{\partial y}\right|_{(u_0(t),u_0'(t))} u_2'(t) \\ &\quad + \frac{1}{2!}\left[\left.\frac{\partial^2 F_1}{\partial x^2}\right|_{(u_0(t),u_0'(t))} u_1^2(t) + 2\left.\frac{\partial^2 F_1}{\partial x \partial y}\right|_{(u_0(t),u_0'(t))} u_1(t)\, u_1'(t) \right. \\ &\quad \left. + \left.\frac{\partial^2 F_1}{\partial y^2}\right|_{(u_0(t),u_0'(t))} (u_1'(t))^2 \right] \end{aligned}$$

and so on. In (2.8), B_n $'s$ are the n^{th} Adomian Polynomials [3] of $F_2(u(t), u'(t))$. Let us note that $B_0, B_1, B_2, \ldots$ are same as $A_0, A_1, A_2, \ldots$ except for the fact that F_1 should be replaced by F_2 throughout. In (2.9), C_n $'s$ are the n^{th} Adomian Polynomials [3] of $G(u(t))$ as given below:

$$\begin{aligned} C_0(t) &= G(u_0(t)), \\ C_1(t) &= G'((u_0(t)))u_1(t), \\ C_2(t) &= G'((u_0(t)))u_2(t) + \frac{1}{2!}G''((u_0(t)))u_1^2(t) \end{aligned}$$

and so on. Applying the Laplace decompositions (2.6) – (2.9) in (2.3), we obtain

$$\begin{aligned} \sum_{n=0}^{\infty} e^{-n\omega s} L\{u_n(t)\} &= \frac{k}{s} - \frac{\lambda\, e^{-\omega s}}{s^2} + \frac{\lambda\, e^{-2\omega s}}{s^2} + \frac{e^{-2\omega s}}{s} L\{f(t+2\omega)\} \\ &\quad + \frac{e^{-\omega s}}{s}\sum_{n=0}^{\infty} e^{-n\omega s} L\{A_n(t)\} + \frac{e^{-2\omega s}}{s}\sum_{n=0}^{\infty} e^{-n\omega s} L\{B_n(t)\} \\ &\quad + \frac{e^{-2\omega s}}{s^2}\sum_{n=0}^{\infty} e^{-n\omega s} L\{C_n(t)\}. \end{aligned} \tag{2.10}$$

One may compute $L\{u_n(t)\}$ iteratively as follows :

$$L\{u_0(t)\} = \frac{k}{s}\ ;\ L\{u_1(t)\} = -\frac{k}{s^2} + \frac{1}{s}L\{A_0(t)\}$$

$$L\{u_2(t)\} = \frac{\lambda}{s^2} + \frac{1}{s}L\{f(t+2\omega)\} + \frac{1}{s}L\{A_1(t)\} + \frac{1}{s}L\{B_0(t)\} + \frac{1}{s^2}L\{C_0(t)\}$$

$$L\{u_{n+1}(t)\} = \frac{1}{s}L\{A_n(t)\} + \frac{1}{s}L\{B_{n-1}(t)\} + \frac{1}{s^2}L\{C_{n-1}(t)\}, \quad n = 2, 3, 4, \dots .$$

The exact or approximate solution is obtained by using inverse laplace transform.

3. Illustrative examples

In this section, three test problems are worked out to illustrate the applicability of the method.

EXAMPLE 3.1. Consider the following linear differential-difference equation with differential order one and difference of order two:

$$2u'(t) - u(t-\omega) = u(t-2\omega), \qquad t > 2\omega, \tag{3.1}$$

with the interval condition

$$u(t) = 1, \qquad 0 \leqslant t \leqslant 2\omega. \tag{3.2}$$

Following the method for (3.1) – (3.2) , we directly arrive at

$$L\{u(t)\} = \frac{1}{s} - \frac{e^{-\omega s}}{2s^2} + \frac{e^{-2\omega s}}{2s^2} + \frac{e^{-\omega s}}{2s}L\{u(t)\} + \frac{e^{-2\omega s}}{2s}L\{u(t)\}. \tag{3.3}$$

Now applying the Laplace decomposition (2.6) in (3.3), we obtain

$$\sum_{n=0}^{\infty} e^{-n\omega s}L\{u_n(t)\} = \frac{1}{s} - \frac{e^{-\omega s}}{2s^2} + \frac{e^{-2\omega s}}{2s^2} + \frac{e^{-\omega s}}{2s}\sum_{n=0}^{\infty} e^{-n\omega s}L\{u_n(t)\} + \frac{e^{-2\omega s}}{2s}\sum_{n=0}^{\infty} e^{-n\omega s}L\{u_n(t)\}. \tag{3.4}$$

Equating the terms with co-efficient of $e^{-n\omega s}$ on both sides of (3.4) we get $L\{u_n(t)\}$. An application of inverse Laplace transform will yield $u_n(t)$:

$$u_0(t) = 1\ ;\ u_1(t) = 0 \text{ and for } n \geqslant 2 \text{ we have,}$$

$$u_n(t) = \sum_{r=1}^{\lfloor \frac{n}{2} \rfloor} \binom{n-r-1}{r-1} \frac{t^{n-r}}{2^{n-r-1}\cdot(n-r)!}, \qquad n \geqslant 2. \tag{3.5}$$

As the next step of the method, using (3.5), we have

$$u(t) = 1 + \sum_{n=2}^{\infty}\sum_{r=1}^{\lfloor \frac{n}{2} \rfloor} \binom{n-r-1}{r-1} \frac{(t-n\omega)^{n-r} e(t-n\omega)}{2^{n-r-1}\cdot(n-r)!},\ t > 0. \tag{3.6}$$

Further, using (3.6) we can find the exact solution of (3.1) in the interval wise:

$$u(t) = 1+\sum_{n=2}^{N}\sum_{r=1}^{\lfloor \frac{n}{2}\rfloor}\binom{n-r-1}{r-1}\frac{(t-n\omega)^{n-r}}{2^{n-r-1}\cdot(n-r)!},\ N\omega \leqslant t \leqslant (N+1)\omega$$
$$N=2,3,4,\ldots\ .$$

When $\omega \to 0$, equation (3.1) becomes first order differential equation and when we use (3.6), the solution becomes

$$\begin{aligned} u(t) &= 1+\sum_{n=1}^{\infty}\left[\binom{n-1}{0}+\binom{n-1}{0}+\cdots+\binom{n-1}{n-1}\right]\frac{t^n}{2^{n-1}\cdot n!} \\ &= 1+\sum_{n=1}^{\infty}\frac{t^n}{n!} \ = \ e^t. \end{aligned}$$

Example 3.2. Consider the following nonlinear differential-difference equation with differential order one and difference of order two.

$$u'(t) = 2-u(t-\omega)+au^3(t-2\omega), \qquad t>2\omega, \tag{3.7}$$

with the interval condition:

$$u(t) = 1, \qquad 0 \leqslant t \leqslant 2\omega. \tag{3.8}$$

Following the method for (3.7) − (3.8) , we directly arrive at

$$L\{u(t)\} = \frac{1}{s}+\frac{e^{-\omega s}}{s^2}+\frac{e^{-2\omega s}}{s^2}-\frac{e^{-\omega s}}{s}L\{u(t)\}-a\frac{e^{-2\omega s}}{s}L\left\{u^3(t)\right\}. \tag{3.9}$$

Let us compute Laplace decomposition series

$$L\left\{u^3(t)\right\} = \sum_{n=0}^{\infty}e^{-n\omega s}L\{A_n(t)\}, \tag{3.10}$$

where $A_i\ 's$ are Adomian Polynomials,

$$\begin{aligned} A_0(t) &= u_0^3(t), \\ A_1(t) &= 3u_0^2(t)u_1(t), \\ A_2(t) &= 3u_0^2(t)u_2(t)+3u_0(t)u_1^2(t), \\ A_3(t) &= 3u_0^2(t)u_3(t)+3u_0(t)u_1(t)u_2(t)+u_1^3(t) \text{ and so on.} \end{aligned}$$

Now by using (2.6) and (3.10) in (3.9), we get,

$$\begin{aligned} \sum_{n=0}^{\infty}e^{-n\omega s}L\{u_n(t)\} &= \frac{1}{s}+\frac{e^{-\omega s}}{s^2}+\frac{e^{-2\omega s}}{s^2}-\frac{e^{-\omega s}}{s}\sum_{n=0}^{\infty}e^{-n\omega s}L\{u_n(t)\} \\ &\quad -a\frac{e^{-2\omega s}}{s}\sum_{n=0}^{\infty}e^{-n\omega s}L\{A_n(t)\}. \end{aligned} \tag{3.11}$$

Equating the terms with co-efficient of $e^{-n\omega s}$ on both sides of (3.11) we get

$L\{u_n(t)\}$. An application of inverse Laplace transform will yield $u_n(t)$.
For $4\omega \leqslant t \leqslant 5\omega$, the approximate solution is

$$\begin{aligned} u(t) &= \sum_{n=0}^{4} u_n(t-n\omega) \\ &= 1+(1-a)(t-2\omega)-(1-a)\frac{(t-3\omega)^2}{2!} \\ &\quad +(1-a)\frac{(t-4\omega)^3}{3!}+3a(1-a)\frac{(t-4\omega)^2}{2!}. \end{aligned} \tag{3.12}$$

When $\omega \to 0$ and $a = 0$, equation (3.7) becomes linear first order differential equation,

$$u'(t) = 2 - u(t) \tag{3.13}$$

and the exact solution is given by $2 - e^{-t}$.

In this case applying Laplace decomposition method to (3.13), we get

$$u_n(t) = (-1)^n \frac{t^{n-1}}{(n-1)!},\ n \geqslant 2$$

and hence

$$\begin{aligned} u(t) &= 1+\sum_{n=2}^{\infty} u_n(t) \\ &= 1+t-\frac{t^2}{2!}+\frac{t^3}{3!}-\frac{t^4}{4!}+\cdots+(-1)^n\frac{t^{n-1}}{(n-1)!}+\cdots \\ &= 2-e^{-t}. \end{aligned}$$

Example 3.3. Consider the following integro-differential-difference equation with differential order one and difference of order two.

$$u'(t) = u(t-\omega)u'(t-\omega) + \int_0^t \sin(u(t_1-2\omega))\, dt_1, \qquad t > 2\omega, \tag{3.14}$$

with the interval condition:

$$u(t) = 1, \qquad 0 \leqslant t \leqslant 2\omega. \tag{3.15}$$

Following the method for (3.14) – (3.15) , we directly arrive at

$$L\{u(t)\} = \frac{1}{s}+\frac{e^{-\omega s}}{s}L\{u(t)u'(t)\}+\frac{e^{-2\omega s}}{s^2}L\{\sin(u(t))\}. \tag{3.16}$$

The next step is to compute the following Laplace decomposition series for $L\{u(t)u'(t)\}$ and $L\{\sin(u(t))\}$:

$$L\{u(t)u'(t)\} = \sum_{n=0}^{\infty} e^{-n\omega s} L\{B_n(t)\}, \tag{3.17}$$

where $B_i\ 's$ are Adomian Polynomials,

$$B_n(t) = u_0(t)u'_n(t) + u_1(t)u'_{n-1}(t) + \cdots + u_n(t)u'_0(t), \text{ for } n \geqslant 0$$

$$\text{and} \qquad L\{\sin(u(t))\} = \sum_{n=0}^{\infty} e^{-n\omega s} L\{C_n(t)\}, \tag{3.18}$$

where $C_i\ 's$ are Adomian Polynomials given below,

$$\begin{aligned} C_0(t) &= \sin(u_0(t)). \\ C_1(t) &= u_1(t)\cos(u_0(t)). \\ C_2(t) &= u_2(t)\cos(u_0(t)) - \frac{1}{2}u_1^2(t)\sin(u_0(t)). \\ C_3(t) &= u_3(t)\cos(u_0(t)) - u_1(t)u_2(t)\sin(u_0(t)) - \frac{1}{6}u_1^3(t)\cos(u_0(t)) \end{aligned}$$

and so on.

Now by using (2.6), (3.17) and(3.18) in (3.16), we get,

$$\begin{aligned} \sum_{n=0}^{\infty} e^{-n\omega s} L\{u_n(t)\} &= \frac{1}{s} + \frac{e^{-\omega s}}{s} \sum_{n=0}^{\infty} e^{-n\omega s} L\{B_n(t)\} \\ &+ \frac{e^{-2\omega s}}{s^2} \sum_{n=0}^{\infty} e^{-n\omega s} L\{C_n(t)\}. \end{aligned} \tag{3.19}$$

Equating the terms with co-efficient of $e^{-n\omega s}$ on both sides of (3.19) we get $L\{u_n(t)\}$. An application of inverse Laplace transform will yield $u_n(t)$.
For $4\omega \leqslant t \leqslant 5\omega$, the approximate solution is

$$\begin{aligned} u(t) &\approx \sum_{n=0}^{4} u_n(t - n\omega) \\ &= 1 + \sin(1)\frac{(t-2\omega)^2}{2!} + \sin(1)\frac{(t-3\omega)^2}{2!} \\ &+ \sin(1)\frac{(t-4\omega)^2}{2!} + \sin(1)\cos(1)\frac{(t-4\omega)^4}{4!}. \end{aligned} \tag{3.20}$$

In this way, we can continue and workout higher level approximate solutions.

4. Conclusion

The above three illustrative examples clearly demonstrate the fact that Laplace decomposition method transforms a differential-difference equation or a integro-differential-difference equation with differential order one and difference order two with a given interval condition into an algebraic equation suitable for applying inverse Laplace transformation technique. Finally this results into a series expression involving unit step functions which represents the solution. It is interesting to note that one can get exact solution in the case of a linear problem. However in the case of a nonlinear problem, one can compute iteratively approximate solutions without any hassles. If the nonlinear problem has a closed form solution, then after certain stage, every iteration leads to the same exact solution. Hence this method is suitable for both linear and nonlinear problems.

Acknowledgements The authors thank SERC division, Government of India, Ministry of Science and Technology, Department of Science and Technology for encouraging this research work under the project No. SR/S2/HEP-006/2009.

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Received by editors 18.04.2013. Available online 17.06.2013

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