BULLETIN OF INTERNATIONAL MATHEMATICAL VIRTUAL INSTITUTE
ISSN 1840-4367
Vol. 2(2012), 219-221

Former
BULLETIN OF SOCIETY OF MATHEMATICIANS BANJA LUKA
ISSN 0354-5792 (o), ISSN 1986-521X (p)

ON A LOGARITHMIC INEQUALITY

József Sándor

Abstract. We offer a new proof of a logarithmic inequality used in the theory of quasiconformal mappings and norm inequalities for vector functions [1].

1. Introduction

In the recent paper [1], the following logarithmic inequality has been proved (see Lemma 2.7 of [1]):

Theorem 1.1. *For any $k \geqslant 1$ and $t \in [t_0, 1)$, where $t_0 = \dfrac{e-1}{e+1}$ one has:*

$$\log\left(\frac{1+t^{1/k}}{1-t^{1/k}}\right) \leqslant k \log\left(\frac{1+t}{1-t}\right). \tag{1}$$

The proof of (1) given in [1] is very complicated, based on more subsequent Lemmas on various hyperbolic functions. We note that (1) has important applications in the study of quasiconformal mappings and related vector function inequalities [1].

The aim of this note is to offer a very simple proof of (1), and in fact to obtain a more general result.

2. The proof

Our method will be based on the study of monotonicity of a certain function, combined with a well-known result related to the logarithmic mean

$$L = L(x,y) = \frac{x-y}{\log x - \log y}\ (x \neq y),\ L(x,x) = x.$$

The following result is well-known (see e.g. [2]):

2010 *Mathematics Subject Classification.* Primary 11A25.
Key words and phrases. Logarithmic function, logarithmic mean.

Lemma 2.1. *One has $L > G$ for any $x, y > 0$, $x \neq y$, where*

$$G = G(x, y) = \sqrt{xy}$$

denotes the geometric mean of x and y.

Put now $t = \dfrac{1}{p}$, where $1 < p \leqslant \dfrac{e+1}{e-1}$ and $\dfrac{1}{k} = x$ in (1). Then the inequality becomes

$$f(x) = x \log\left(\frac{p^x + 1}{p^x - 1}\right) \leqslant f(1), \text{ where } 0 < x \leqslant 1,$$

and $f(1) = \log\left(\dfrac{p+1}{p-1}\right) \geqslant 1$.

Now the following result will be proved:

Theorem 2.1. *Assuming the above conditions, the function $f(x)$ is strictly increasing on $(0, 1]$.*

Particularly, one has $f(x) \leqslant f(1)$ for $0 < x \leqslant 1$.

Proof. An easy computation gives

$$f'(x) = \log\left(\frac{p^x + 1}{p^x - 1}\right) - \frac{2xp^x \log p}{p^{2x} - 1} = \log\left(\frac{a+1}{a-1}\right) - \frac{2a \log a}{a^2 - 1} = g(a),$$

where $a = p^x$. Since $0 < x \leqslant 1$, $a \leqslant p$ and as $p \leqslant \dfrac{e+1}{e-1}$, one has $a \leqslant \dfrac{e+1}{e-1}$, i.e. $\log\left(\dfrac{a+1}{a-1}\right) \geqslant 1$. This implies

$$g(a) \geqslant 1 - \frac{2a \log a}{a^2 - 1} = 1 - \frac{a \log a^2}{a^2 - 1} > 0,$$

as this is equivalent with $L(a^2, 1) < G(a^2, 1)$ of the Lemma.

Since $f'(x) > 0$, the function f is strictly increasing, and the proof of Theorem 2 is finished.

Remark 2.1. (1) Particularly, by letting $p_0 = \dfrac{e+1}{e-1}$ we get $f(1) = 1$, and the inequality

$$\log\left(\frac{p_0^x + 1}{p_0^x - 1}\right) \leqslant \frac{1}{x} \tag{2}$$

follows. For $x = \dfrac{1}{k}$ and $p_0 = \dfrac{1}{t_0}$, with the use of (2) an easier proof of Lemma 2.9 of [1] can be deduced.

(2) Let $0 < x \leqslant y \leqslant 1$. Then

$$x \log\left(\frac{p^x + 1}{p^x - 1}\right) \leqslant y \log\left(\frac{p^y + 1}{p^y - 1}\right) \leqslant \log\left(\frac{p+1}{p-1}\right). \tag{3}$$

This offers an extension of inequality (1) for $x = \dfrac{1}{k}$ and $p = \dfrac{1}{t}$.

References

[1] B.A. Bhayo, V. Božin, D. Kalaj, M. Vuorinen, *Norm inequalities for vector functions*, J. Math. Anal. Appl., **380**(2011), 768-781.

[2] J. Sándor, *On the identric and logarithmic means*, Aequationes Math., **40**(1990), 261-270.

Received 04.04.2012; available on internet 12.10.2012

Babeş-Bolyai University, Department of Mathematics, Str. Kogălniceanu nr. 1, 400084 Cluj-Napoca, Romania

E-mail address: jsandor@math.ubbcluj.ro