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WEAK SEMI COMPATIBILITY AND FIXED POINT THEOREMS

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Abstract. The aim of the present paper is to introduce the notion of weak semi compatibility and obtain fixed point theorems by employing the new notion. The new notion is the proper generalization of semi compatibility and is applicable to compatible and commuting type mappings. Our result generalizes several fixed point theorems.

1. Introduction and Preliminaries

In 1995, authors [**5**] introduced the concept of semi compatibility and obtained the first result that established a situation in which a collection of mappings has a fixed point. They defined a pair of self maps (S,T) to be semi compatible if
(a) $Sx = Tx \Rightarrow STx = TSx$ and
(b) $\lim_{n\to\infty} Sx_n = \lim_{n\to\infty} Tx_n = x$, for some $x \in X$, implies $\lim_{n\to\infty} STx_n = Tx$ holds.
B.Sing and S.Jain [**16**] observe that (b)implies(a). Hence they defined the semi compatibility by condition (b) only.

Let (X,d) be a metric space and let f and g be two maps from (X,d) into itself. f and g are commuting maps if $fgx = gfx$ for all x in X.
To generalize the notion of commuting maps, Sessa[**15**] introduced the concept of weakly commuting maps. He defines f and g to be weakly commuting if $d(fgx, gfx) \leqslant d(fx, gx)$ for all $x \in X$.
Obviously, commuting maps are weakly commuting but the converse is not true.
In 1986, Jungck[**10**] gave more generalized commuting and weakly commuting maps called compatible maps. f and g are called compatible if

$$\lim_{n\to\infty} d(fgx_n, gfx_n) = 0 \tag{1.1}$$

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Whenever $\{x_n\}$ is a sequence in Xsuch that $\lim_{n\to\infty} fx_n = \lim_{n\to\infty} gx_n = t$, for some $t \in X$. Clearly, weakly commuting maps are compatible, But the implication is not reversible (see [**10**]).
Afterward, the same author with Murthy and Cho [**8**] made another generalization of weakly commuting maps by introducing the concept of compatible maps of type (A). Previous f and g are said to be compatible of type (A) if in place of (1.1) we have two following conditions:
$\lim_{n\to\infty} d(fgx_n, ggx_n) = 0$ and $\lim_{n\to\infty} d(gfx_n, ffx_n) = 0$
It is clear to see that weakly commuting maps are compatible type (A), from [**8**] it follows that the implication is not reversible.

Two self mappings f and g of a metric space (X, d) are called R- weakly commuting [**13**] at a point x in X if $d(fgx, gfx) \leqslant Rd(fx, gx)$, for some$R > 0$.
The two self maps f and $g$of a metric space (X, d) are called $R-$weakly commuting of type (A_g) [**14**] if there exist some positive real number R such that $d(ffx, gfx) \leqslant Rd(fx, gx)$ for all x in X.
The two self maps f and g of a metric space (X, d) are called $R-$ weakly commuting of type (A_f) [**14**] if there exist some positive real number R such that $d(fgx, ggx) \leqslant Rd(fx, gx)$ for all x in X.It may be noted that compatible mappings f and g can be $R-$weakly commuting of type (A_f) and (A_g).
Pant, Bisht and Arora [**12**] introduced a notion of weak reciprocal continuity as follows,
Two self mappings f and g of metric space (X, d) will be called weakly reciprocally continuous if $\lim fgx_n = ft$or $\lim gfx_n = gt$, whenever $\{x_n\}$ is a sequence in X such that $\lim fx_n = \lim gx_n = t$ for some $t$in X.
We now generalize the semi compatibility by introducing the notion of weak semi compatibility as follows:

Definition 1.1. Two self mappings f and g of a metric space (X, d) will be called weakly semi compatible mappings if $\lim_{n\to\infty} fgx_n = gt$ or $\lim_{n\to\infty} gfx_n = ft$, whenever $\{x_n\}$ is a sequence in X such that $\lim_{n\to\infty} fx_n = \lim_{n\to\infty} gx_n = t$ for some $t$in X.

Example 1.1. Let $X = [0, 1]$ and d be the usual metric on$X$. Define $f, g : X \to X$ by

$$fx = x \quad \text{for all } x \in [0, 1]$$

$$gx = \begin{cases} 0 & ,x\in(0,1] \\ 1 & ,x=0 \end{cases}$$

Taking $\{x_n\} = 1/n$, since $\lim fx_n = \lim gx_n = 0$
also

$$\lim fgx_n = \lim f(0) = 0 \neq g(0) \tag{1.2}$$

$$\lim gfx_n = \lim g(x_n) = \lim g\left(1/n\right) = 0 = f(0) \tag{1.3}$$

Equation (1.2) and (1.3) shows that maps f and $g$are weak semi compatible.

Also here maps f and g have no common fixed point.

EXAMPLE 1.2. Let $X = [1, 2]$ and d be the usual metric on X. Define $f, g : X \to X$ by

$$fx = x, \quad x \in [1, 2]$$
$$gx = \begin{cases} x^2, & x\in(1,2] \\ 2, & x=1 \end{cases}$$

Taking$\{x_n\} = 1 + {}^1\!/_n$, since

$$\lim fx_n = \lim (x_n) = \lim \left(1 + {}^1\!/_n\right) = 1$$
$$\lim gx_n = [\lim x_n]^2 = \left[\lim \left(1 + {}^1\!/_n\right)\right]^2 = 1$$

Now

$$\lim fgx_n = \lim f [x_n]^2 = \left[\lim f \left(1 + {}^1\!/_n\right)\right]^2 = \left[\lim \left(1 + {}^1\!/_n\right)\right]^2 = 1 \neq g (1) \tag{1.4}$$

$$\lim gfx_n = \lim g (x_n) = \lim g \left(1 + {}^1\!/_n\right) = \left[\lim \left(1 + {}^1\!/_n\right)\right]^2 = 1 = f (1) \tag{1.5}$$

Equation (1.4) and (1.5) shows that maps f and g are weak semi compatible.

Also here maps f and g have no common fixed point.

EXAMPLE 1.3. Let $X = [0, 1]$ and d be the usual metric on$X$. Define $f, g : X \to X$by

$$fx = x, \quad x \in [0, 1]$$
$$gx = \begin{cases} 0, & x=0 \\ 1, & x\in(0,1] \end{cases}$$

Taking $\{x_n\} = {}^1\!/_n$

$$\lim fx_n = \lim (x_n) = \lim \left({}^1\!/_n\right) = 0$$
$$\lim gx_n = 0$$

Now

$$\lim fgx_n = \lim f (0) = 0 = g (0) \tag{1.6}$$

$$\lim gfx_n = \lim g (x_n) = \lim g \left({}^1\!/_n\right) = 1 \neq f (0) \tag{1.7}$$

Equation (1.6) and (1.7) shows that maps f and g are weak semi compatible. Also here maps f&$g$have two common fixed point that are 0 and 1.

Now we see more definitions which will be helpful to improve this result.

DEFINITION 1.2. Let X be a set,f and g are self maps of X. A point x in X is called coincidence point of f and$g$ iff $fx = gx$. We shall call $w = fx = gx$ a point of coincidence of f and g

DEFINITION 1.3. [**11**] Two self maps f and g of a set X are occasionally weakly compatible (owc)iff there is a point x in X which is coincidence point of f and g at which f and g commute.

LEMMA 1.1. [**9**] *Let X be a set, f and g are owc self maps on X. If f and g have a unique point of coincidence,$w := fx = gx$, then w is the unique common fixed point of f and g.*

DEFINITION 1.4. Let X be a set. A symmetric on X is a mapping $d : X \times X \to [0, \infty)$ such that
$d(x, y) = 0\ iff\ x = y$, and $d(x, y) = d(y, x)$ for $x, y \in X$.

THEOREM 1.1. *Let X be a set with a symmetric d.Suppose that f and g are owc self maps of X satisfying*

$$d(fx, fy) \geqslant ad(gx, gy) + bd(gx, fx) + cd(gy, fy) \tag{1.8}$$

Where $a, c > 1$, $b \in R$ such that $a+b+c > 1$.Then f and g have a unique common fixed point.

PROOF. Since the maps are owc, there exist a point x such that $fx = gx$ and $fgx = gfx$. Substituting (1.8) with $y = gx$ gives

$$d(fx, fgx) \geqslant ad(gx, gfx) + bd(gx, fx) + cd(gfx, fgx) = ad(gx, fgx).$$

Thus $0 \geqslant (a-1)\, d(fx, fgx)$, which implies fx is a fixed point of f. But, Since $gfx = fgx$, $fx$is also a fixed point of g.

Suppose that p and q are common fixed point of f and$g$. Substituting in (1.8)

$$d(p, q) = d(fp, fq) \geqslant ad(gp, gq) + bd(gp, fp) + cd(gq, fq) = ad(p, q),$$

which implies that $p = q$. □

LEMMA 1.2. [**11**] *If A and S are weakly compatible ,then they are owc ,but following result shows that converse is not true.*

EXAMPLE 1.4. Let $X = [1, \infty)$ with the usual metric . Define $A, S : X \to X$ by:$Ax = 3x - 2$ *and* $Sx = x^2$. We have $Ax = Sx$ iff $x = 1$ or $x = 2$ and $AS(2) \neq SA(2)$.
Therefore A and S are owc but they are not weak compatible.

LEMMA 1.3. [**4**] *If f and g are compatible of type (A) then they are owc ,but converse is not true in general. He gave following example,*

EXAMPLE 1.5. Let $X = [0, \infty)$with the usual metric. Define $f, g : X \to X$ by,

$$fx = \begin{cases} 4 & if\ x\in[0,1) \\ x^4 & if\ x\in[1,\infty) \end{cases}$$

$$gx = \begin{cases} 3 & if\ x\in[0,1) \\ \frac{1}{x^4} & if\ x\in[1,\infty) \end{cases}$$

We have $f(1) = g(1) = 1$ & $fg(1) = gf(1) = 1$; that is $f$and g are owc. Now consider $x_n = 1 + {}^1\!/_n$ for $n \in \{1, 2, 3...\}$ we have $\lim fx_n = \lim x_n^4 = 1$ *and* $\lim gx_n = \lim {}^1\!/_{x_n^4} = 1$ .But $\lim d(fgx_n, ggx_n) = 1 \neq 0$, Hence f and g are not compatible of type (A).

2. Main Results

THEOREM 2.1. *Let f and g be weakly semi compatible self mappings of a complete metric space (X, d) such that*

$$(a)\ \ f(X) \subseteq g(X)$$
$$(b)\ \ d(fx, fy) \geqslant ad(gx, gy) + bd(fx, gx) + cd(fy, gy)$$

$a > 1$, $c > 1$, $b \in R$ *Such that* $a + b + c > 1$.

If (f, g) *is continuous and weak compatible pair, then* f *and* g *have common fixed point in*X.

PROOF. ***Step1:***
Let x_0 be any point in$X$. Since $f(X) \subseteq g(X)$, there exist $x_1 \in X$ such that $fx_1 = gx_0 = y_0$. Similarly we can have a sequence

$$fx_{n+1} = gx_n = y_n. \tag{2.1}$$

Now by (b)

$$d(fx_n, fx_{n+1}) \geqslant ad(gx_n, gx_{n+1}) + bd(fx_n, gx_n) + cd(fx_{n+1}, gx_{n+1})$$
$$d(y_{n-1}, y_n) \geqslant ad(y_n, y_{n+1}) + bd(y_{n-1}, y_n) + cd(y_n, y_{n+1})$$
$$d(y_n, y_{n+1}) \leqslant \tfrac{1-b}{a+c} d(y_{n-1}, y_n)$$

Since $\frac{1-b}{a+c} = k < 1$ $(a + b + c > 1)$

$$d(y_n, y_{n+1}) \leqslant kd(y_{n-1}, y_n) \tag{2.2}$$

Similarly we can obtain $d(y_{n-1}, y_n) \leqslant kd(y_{n-2}, y_{n-1})$.
Therefore by (2.2) we have $d(y_n, y_{n+1}) \leqslant k^2 d(y_{n-2}, y_{n-1})$. By same way we can have

$$d(y_n, y_{n+1}) \leqslant k^n d(y_0, y_1) \tag{2.3}$$

Now we will show that $\{y_n\}$ is a Cauchy sequence. For any integer $p > 0$, we get

$$d(y_n, y_{n+p}) \leqslant d(y_n, y_{n+1}) + d(y_{n+1}, y_{n+2}) + ... + d(y_{n+p-1}, y_{n+p}),$$

Now by (2.3)

$$d(y_n, y_{n+p}) \leqslant k^n d(y_0, y_1) + k^{n+1} d(y_0, y_1) + ... + k^{n+p-1} d(y_0, y_1)$$
$$d(y_n, y_{n+p}) \leqslant k^n \left[1 + k + k^2 + k^3 + ... + k^{p-1}\right] d(y_0, y_1)$$
$$d(y_n, y_{n+p}) \leqslant \tfrac{k^n}{1-k} d(y_0, y_1)$$

Since $k < 1$ then taking limit $n \to \infty$ we have $d(y_n, y_{n+p}) \to 0$.Therefore $\{y_n\}$ is a Cauchy sequence. Since X is complete, there exist a point t in Xsuch that $\{y_n\} \to t$as $n \to \infty$ moreover $y_n = fx_{n+1} = gx_n \to t$ for$t \in X$.
Step2:
Since f and g are weakly semi compatible mappings therefore
$\lim gfx_n = ft$, or $\lim fgx_n = gt$.
Case 1- when$\lim gfx_n = ft$. Since $f(X) \subseteq g(X)$ then there exist a point $u \in X$ such that $ft = gu$. Since f is continuous mapping then $\lim ffx_n = ft$. Now by (b)

$d(ffx_n, fu) \geqslant ad(gfx_n, gu) + bd(ffx_n, gfx_n) + cd(fu, gu)$.
Now limiting $n \to \infty$ we have, $d(ft, fu) \geqslant ad(ft, gu) + bd(ft, ft) + cd(fu, gu)$
$d(gu, fu) \geqslant cd(fu, gu)$.Since $c > 1$ therefore $fu = gu$.Again since (f, g) is a weak compatible pair therefore$fu = gu \Rightarrow fgu = gfu$, or
$fgu = gfu = ffu = ggu$. Now by using (b) we have

$$d(ffu, fu) \geqslant ad(gfu, gu) + bd(ffu, gfu) + cd(fu, gu)$$
$$d(ffu, fu) \geqslant ad(ffu, fu)$$

Since $a > 1$ therefore $ffu = fu$ or $ffu = gfu = fu$. So fu is common fixed point of f and g.
Case 2 - when $\lim fgx_n = gt$.Since g is continuous mapping therefore $\lim ggx_n = gt$.
Now by using (b) we have

$$d(fgx_n, ft) \geqslant ad(ggx_n, gt) + bd(fgx_n, ggx_n) + cd(ft, gt)$$
$$d(gt, ft) \geqslant ad(gt, gt) + bd(gt, gt) + cd(ft, gt)$$
$$d(gt, ft) \geqslant cd(ft, gt)$$

Since $c > 1$, therefore $ft = gt$. Since (f, g) is weak compatible pair therefore $ft = gt \Rightarrow fgt = gft$ or $fgt = gft = fft = ggt$. Now by using (b) we have,

$$d(fft, ft) \geqslant ad(gft, gt) + bd(fft, gft) + cd(ft, gt)$$

$d(fft, ft) \geqslant ad(fft, ft)$. Since $a > 1$ therefore$fft = ft$ or $fft = gft = ft$ or ft is common fixed point of f and g. □

Example 2.1. Let $x, y \in X$ $(x \geqslant y)$ and $X = [1, 10]$ and d be the usual metric on X. Define $f, g : X \to X$ as follows
$fx = 1$ if $x = 1$ &$x > 5$and $fx = 4$ if $1 < x \leqslant 5$,
$gx = 1$if $x = 1$,$gx = 5$if $1 < x \leqslant 5$,and $gx = x/2$if $x > 5$.
If$x_n = 1$,
$\lim fx_n = \lim gx_n = 1$. Also $\lim fgx_n = \lim f(1) = 1 = g(1)$
And $\lim gfx_n = \lim g(1) = 1 = f(1)$
But when$x_n = 2 + 1/n$,
$\lim fx_n = 1$and $\lim gx_n = \lim \left(\frac{2+1/n}{2}\right) = 1$Or $\lim fx_n = \lim gx_n = 1$
Moreover $\lim fgx_n = \lim f\left(\frac{2+1/n}{2}\right) = \lim f\left(1 + 1/2n\right) = 4 \neq g(1)$
And $\lim gfx_n = \lim g(1) = 1 = f(1)$ therefore (f, g) is weakly semi compatible.
Also pair (f, g) is continuous and weak compatible at $x = 1$.
Moreover pair (f, g) satisfy equal to condition of (b) if $x, y = 1$
Also if $fx = 4$ & $gx = 5$ for $x, y \in (1, 5]$ when $a > 1$, $b = -2$ & $c = 2$, pair (f, g) satisfies again equal to condition of (b). Now finally if $fx = 1$ & $gx = x/2$for $x, y > 5$.The left hand side of given inequality yields $d(fx, fy) = |fx - fy| = 0$and right hand side gives

$$ad(gx, gy) + bd(fx, gx) + cd(fy, gy)$$

$$4\left|\frac{x}{2}-\frac{y}{2}\right|-4\left|1-\frac{x}{2}\right|+3\left|1-\frac{y}{2}\right|$$
$$\Rightarrow 4\left(\frac{x}{2}-\frac{y}{2}\right)-4\left(\frac{x}{2}-1\right)+3\left(\frac{y}{2}-1\right)=1-\frac{y}{2}\,(negative)$$

Therefore pair (f,g) satisfy greater than condition of (b)when $a=4, b=-4\,\&\, c=3$, and 1 is common fixed point of$f$ & g.

COROLLARY 2.1. *Let X be a set, and d be the symmetric on X. Let maps f and g satisfy all the conditions of theorem (2.1). Since pair (f,g) is weak compatible pair then by lemma (1.2) pair (f,g) will be owc and therefore the conclusion of theorem (2.1) follows from the theorem (1.1).*

THEOREM 2.2. *Let f and g be weakly semi compatible self mappings of a complete metric space (X,d) satisfy conditions (a) and (b). If maps f & g are commute at their coincidence point and compatible of type (A). Then f and g have common fixed point in$X$.*

PROOF. Step1 of this theorem is same as theorem (1.1). Now from step 2, since $f$and $g$are weak semi compatible mappings thereforelim $gfx_n = ft$ orlim $fgx_n = gt$.
Case 1- when $\lim gfx_n = ft$. Since $f(X)\subseteq g(X)$ then there exist a point $u\in X$ such that $ft=gu$.
Since pair (f,g) is compatible of type (A), then $\lim d(gfx_n, ffx_n)=0$. This yields $\lim ffx_n = ft$. Now by (b),
$d(ffx_n, fu)\geqslant ad(gfx_n, gu)+bd(ffx_n, gfx_n)+cd(fu,gu)$, since $c>1$and taking limit$n\to\infty$this Yields $fu=gu$.
Now by using (b)

$$\begin{aligned}
d(ffu,fu)&\geqslant ad(gfu,gu)+bd(ffu,gfu)+cd(fu,gu)\\
d(ffu,fu)&\geqslant ad(fgu,fu)+b(ffu,fgu)+c(fu,gu)\\
d(ffu,fu)&\geqslant ad(ffu,fu)
\end{aligned}$$

Since $a>1$ therefore$ffu=fu$. Also$gfu=fgu=ffu=fu$. So fu is common fixed point of f and g.
Case 2 - when $\lim fgx_n = gt$.Since pair of maps (f,g) is a compatible of type (A). This yields $\lim d(fgx_n, ggx_n)=0$ implies $\lim ggx_n = gt$. Now by using (b)

$$d(fgx_n, ft)\geqslant ad(ggx_n, gt)+bd(fgx_n, ggx_n)+cd(ft,gt)$$

Since $c>1$ and Limiting $n\to\infty$ yields $gt=ft$.
Now by (b)

$$\begin{aligned}
d(fft,ft)&\geqslant ad(gft,gt)+bd(fft,gft)+cd(ft,gt)\\
d(fft,ft)&\geqslant ad(fgt,ft)+bd(fft,fgt)+cd(ft,gt)\\
d(fft,ft)&\geqslant ad(fft,ft)
\end{aligned}$$

Since $a>1$ yields $fft=ft$. Also $gft=fgt=fft=ft$ or $fft=gft=ft$, therefore ft is common fixed point of $f\,\&\,g$. □

EXAMPLE 2.2. Let $x,y\in X\,(x\geqslant y)$and $X=[1,10]$ and d be the usual metric on X. Define $f,g: X\to X$ as follows
$fx=1$ if $x=1$ &$x>5$ and $fx=5$ if $1<x\leqslant 5$,

$gx = 1$ if $x = 1$,$gx = 5$ if $1 < x \leqslant 5$,and $gx = {}^{x}\!/_{2}$if $x > 5$.
If$x_n = 1$,
$\lim fx_n = \lim gx_n = 1$ Also $\lim fgx_n = \lim gfx_n = 1$
But when$x_n = 2 + {}^{1}\!/_{n}$,
$\lim fx_n = 1$and $\lim gx_n = \lim \left(\frac{2+{}^{1}\!/_{n}}{2}\right) = 1$Or $\lim fx_n = \lim gx_n = 1$
Moreover $\lim fgx_n = \lim f\left(\frac{2+{}^{1}\!/_{n}}{2}\right) = \lim f\left(1 + {}^{1}\!/_{2n}\right) = 5 \neq g\,(1)$
And $\lim gfx_n = \lim g\,(1) = 1 = f\,(1)$ therefore (f, g) is weakly semi compatible.
Also at $x = 1 \,\&\, x \in (1, 5]$, f and g are commute.
Since $\lim ffx_n = \lim f\,(1) = 1$ & $\lim gfx_n = \lim g\,(1) = 1$
therefore

$$\lim ffx_n = \lim gfx_n \tag{2.4}$$

Also $\lim ggx_n = \lim g\,({}^{x_n}\!/_{2}) = \lim g\left(1 + {}^{1}\!/_{2n}\right) = 5$
and $\lim fgx_n = \lim f\left(1 + {}^{1}\!/_{2n}\right) = 5$.

Therefore

$$\lim ggx_n = \lim fgx_n. \tag{2.5}$$

Equation (2.4) and (2.5) shows that pair $(f, g)$is compatible of type (A).
Moreover pair (f, g) satisfy equal to condition if $x, y = 1$Also if $fx = 4$ & $gx = 5$ for $x, y \in (1, 5]$ when $a > 1$, $b = -2$ & $c = 2$, pair (f, g)satisfies again equal to condition of (b). Now finally if $fx = 1$ & $gx = {}^{x}\!/_{2}$for $x, y > 5$.The left hand side of given inequality yields $d\,(fx, fy) = |fx - fy| = 0$and right hand side gives

$$ad\,(gx, gy) + bd\,(fx, gx) + cd\,(fy, gy)$$
$$4\left|\frac{x}{2} - \frac{y}{2}\right| - 4\left|1 - \frac{x}{2}\right| + 3\left|1 - \frac{y}{2}\right|$$
$$\Rightarrow 4\left(\frac{x}{2} - \frac{y}{2}\right) - 4\left(\frac{x}{2} - 1\right) + 3\left(\frac{y}{2} - 1\right) = 1 - \frac{y}{2}\,(negative)$$

Therefore pair (f, g) satisfy greater than condition of (b) when $a = 4, b = -4 \,\&\, c = 3$, and 1 and 5 are two common fixed point of$f$ & g.

Corollary 2.2. *Let X be a set, and d be the symmetric on X. Let maps f and g satisfy all the conditions of theorem (2.1). Since pair (f, g) is compatible of type (A). then by lemma (1.2) pair (f, g) will be owc and therefore the conclusion of theorem (2.2) follows from the theorem (1.1).*

Theorem 2.3. *Let f and g be weakly semi compatible self mappings of a complete metric space (X, d) satisfy conditions (a) and (b). If mapping f is continuous and pair (f, g) is R-weakly commuting of type (A_f). Then f and g have common fixed point in X.*

Proof. Step1 of this theorem is same as theorem (2.1). Now from step 2, since $f$and $g$are weak semi compatible mappings therefore $\lim gfx_n = ft$ or$\lim fgx_n = gt$.

Case 1- when $\lim gfx_n = ft$. Since $f(X) \subseteq g(X)$ then there exist a point $u \in X$such that $ft = gu$. Since f is continuous mapping then$\lim ffx_n = ft$.
Now by (b)
$d(ffx_n, fu) \geqslant ad(gfx_n, gu) + bd(ffx_n, gfx_n) + cd(fu, gu)$, since $c > 1$ and taking limit$n \to \infty$ this yields $fu = gu$.
Since pair (f, g) is R-weakly compatible map of type (A_f), this yields
$d(fgu, ggu) \leqslant Rd(fu, gu)$ Implies $fgu = ggu$. Or$fgu = gfu = ffu = ggu$.Now by (b)
$d(ffu, fu) \geqslant ad(gfu, gu) + bd(ffu, gfu) + cd(fu, gu)$,since $a > 1$ this yields $ffu = fu$.
Therefore $ffu = gfu = fu$ and so $fu$is common fixed point of f and g.
Case 2 - when $\lim fgx_n = gt$.Since (f, g) is pair of R-weakly commuting of type (A_f).This yields $d(fgx_n, ggx_n) \leqslant Rd(fx_n, gx_n)$.Taking limit $n \to \infty$ we have $\lim d(fgx_n, ggx_n) \leqslant 0$ implies $\lim ggx_n = gt$. Now by (b)
$d(fgx_n, ft) \geqslant ad(ggx_n, gt) + bd(fgx_n, ggx_n) + cd(ft, gt)$ limiting $n \to \infty$ when $c > 1$ this yields $ft = gt$. Since pair (f, g) is R-weakly commuting of type (A_f)this yields
$d(fgt, ggt) \leqslant Rd(ft, gt)$ implies $fgt = ggt$or $fgt = gft = fft = ggt$.Now by using (b)
$d(fft, ft) \geqslant ad(gft, gt) + bd(fft, gft) + cd(ft, gt)$ Since $a > 1$, then it yields $fft = ft$. Or$fft = gft = ft$, and so ft is common fixed point of f and g. □

Example 2.3. Let $x, y \in X\,(x \geqslant y)$ and $X = [1, 10]$ and d be the usual metric on X. Define $f, g : X \to X$ as follows

$$fx = 1 \text{ if } x = 1 \,\&x > 5 \text{ and } fx = 5 \text{ if } 1 < x \leqslant 5,$$
$$gx = 1 \text{ if } x = 1, gx = 5 \text{ if } 1 < x \leqslant 5, \text{and } gx = x \text{if } x > 5.$$

If $x_n = 1$, $\lim fx_n = \lim gx_n = 1$. Also $\lim fgx_n = \lim gfx_n = 1$. But when $x_n = 1 + {}^1\!/_n$,
$\lim fx_n = 1$ and $\lim gx_n = \lim\left(1 + {}^1\!/_n\right) = 1$ or $\lim fx_n = \lim gx_n = 1$. Moreover $\lim fgx_n = \lim f\left(1 + {}^1\!/_n\right) = 5 \neq g(1)$. And $\lim gfx_n = \lim g(1) = 1 = f(1)$ therefore (f, g) are weakly semi compatible maps.
Moreover at $x = 1 \,\&\, 1 < x \leqslant 5$ maps $f\&g$ satisfy R-weakly commuting of type (A_f).
Also when $x > 5$, maps $f\&g$ satisfy R-weakly commuting of type (A_f), that is $d(fgx, ggx) \leqslant Rd(fx, gx)$ when $R = 1$. Moreover pair (f, g) satisfy equal to condition of (b) if $x, y = 1$ and $x, y \in (1, 5]$ when $a > 1, c > 1 \,\&\, b \in R$. Now finally if $fx = 1 \,\&\, gx = x$ for $x, y > 5$. The left hand side of given inequality yields $d(fx, fy) = |fx - fy| = 0$ and right hand side gives

$$ad(gx, gy) + bd(fx, gx) + cd(fy, gy)$$
$$3(x - y) - 3(x - 1) + 2(y - 1) \Rightarrow 1 - y\,(negative)$$

And therefore pair (f, g) satisfies greater than condition of (b) when $a = 3, b = -3\&c = 2$. Also $1 \,\&\, 5$ are two common fixed point of $f\&g$.

THEOREM 2.4. *Let f and g be weakly semi compatible self mappings of a complete metric space(X, d)satisfy conditions (a) and (b).*
If mapping g is continuous and pair$(f, g)$is R-weakly commuting of type(A_g). Then $f$and $g$have common fixed point in$X$.

PROOF. Step 1 of this theorem is same as theorem (2.1). Now from step 2, since f and g are weak semi compatible mappings therefore$\lim gfx_n = ft$ or $\lim fgx_n = gt$.

Case 1- when $\lim gfx_n = ft$. Since $f(X) \subseteq g(X)$ then there exist a point $u \in X$ such that $ft = gu$. Since pair (f, g) is $R-$weakly commuting of type (A_g),this yields $d(ffx_n, gfx_n) \leqslant Rd(fx_n, gx_n)$. Now limiting $n \to \infty$ we have $\lim d(ffx_n, gfx_n) \leqslant 0$ implies $\lim ffx_n = \lim gfx_n \Rightarrow \lim ffx_n = ft$. Now by (b) we have,

$$d(ffx_n, fu) \geqslant ad(gfx_n, gu) + bd(ffx_n, gfx_n) + cd(fu, gu),$$

since $c > 1$ and taking limit $n \to \infty$ this yields $fu = gu$. Again since pair (f, g) is $R-$ weakly commuting of type (A_g), this yields $d(ffu, gfu) \leqslant Rd(fu, gu) \Rightarrow gfu = ffu$ or $fgu = gfu = ffu = ggu$.
Now by (b),$d(ffu, fu) \geqslant ad(gfu, gu) + bd(ffu, gfu) + cd(fu, gu)$, since $a > 1$ this yields $ffu = fu$. Therefore $ffu = gfu = fu$ and so fu is common fixed point of $f\&g$.

Case 2-when $\lim fgx_n = gt$. Since map g is continuous, therefore $\lim ggx_n = gt$. By (b)
$d(fgx_n, ft) \geqslant ad(ggx_n, gt) + bd(fgx_n, ggx_n) + cd(ft, gt)$. limiting $n \to \infty$ when $c > 1$ this yields $ft = gt$. Since pair (f, g) is R-weakly commuting of type (A_g) this yields
$d(fft, gft) \leqslant Rd(ft, gt)$ implies $gft = fft$or $fgt = gft = fft = ggt$. Now by using (b)
$d(fft, ft) \geqslant ad(gft, gt) + bd(fft, gft) + cd(ft, gt)$ Since $a > 1$, then it yields $fft = ft$ or $fft = gft = ft$, and so $ft$is common fixed point of f and g. □

EXAMPLE 2.4. Let $x, y \in X\,(x \geqslant y)$ and $X = [1, 10]$ and d be the usual metric on X. Define $f, g : X \to X$ as follows
$fx = 1$ if $x = 1$ &$x > 5$ and $fx = 5$ if $1 < x \leqslant 5$,
$gx = 1$ if $x = 1$,$gx = 5$ if $1 < x \leqslant 5$,and $gx = x + 1$ if $x > 5$.
If$x_n = 1$,
$\lim fx_n = \lim gx_n = 1$ Also $\lim fgx_n = \lim gfx_n = 1$
But when $x_n = {}^1\!/_n$,
$\lim fx_n = 1$and $\lim gx_n = \lim \left(1 + {}^1\!/_n\right) = 1$Or $\lim fx_n = \lim gx_n = 1$
Moreover $\lim fgx_n = \lim f\left(1 + {}^1\!/_n\right) = 5 \neq g(1)$
And $\lim gfx_n = \lim g(1) = 1 = f(1)$ therefore (f, g) are weakly semi compatible maps.
Moreover at $x = 1\,\&\,1 < x \leqslant 5$ maps $f\&g$ satisfy R-weakly commuting of type (A_g).

Also when $x > 5$, maps $f\&g$ satisfy R-weakly commuting of type (A_g), that is $d(gfx, ffx) \leqslant Rd(fx, gx)$, when $R > 0$. Moreover pair (f, g) satisfy equal to condition of (b) if $x, y = 1$ and $x, y \in (1, 5]$ when $a > 1, c > 1$ & $b \in R$. Now finally if $fx = 1$ & $gx = x + 1$ for $x, y > 5$.
The left hand side of given inequality yields $d(fx, fy) = |fx - fy| = 0$ and right hand side gives

$$ad(gx, gy) + bd(fx, gx) + cd(fy, gy)$$
$$3(x - y) - 4x + 3y = -x\,(negative)$$

Therefore (f, g) satisfy greater than condition of (b)when $a = 3, b = -4, c = 3$. And 1&5 are two common fixed point of$f\&g$.

Theorem 2.5. *Let f and g be weakly semi compatible self mappings of a complete metric space(X, d)satisfy conditions (a) and (b).*

If pair (f, g) is R-weakly commuting maps. Then f and g have common fixed point in$X$.

Proof. Step 1 of this theorem is same as theorem (2.1). Now from step 2, since f and g are weak semi compatible mappings therefore $\lim gfx_n = ft$ or $\lim fgx_n = gt$.
Case 1- when $\lim gfx_n = ft$. Since $f(X) \subseteq g(X)$ then there exist a point $u \in X$ such that $ft = gu$. Since pair (f, g) is $R-$ weakly commuting maps, this yields $d(gfx_n, fgx_n) \leqslant Rd(fx_n, gx_n)$. Now limiting $n \to \infty$ we have

$$\lim d(fgx_n, gfx_n) \leqslant 0$$

implies

$$\lim fgx_n = \lim gfx_n \Rightarrow \lim fgx_n = ft.$$

Also equation (2.1) of theorem (2.1) yields $ffx_{n+1} = fgx_n = ft$as $n \to \infty$. Now, by using (b), we have

$$d(ffx_n, fu) \geqslant ad(gfx_n, gu) + bd(ffx_n, gfx_n) + cd(fu, gu)$$

since $c > 1$ and taking limit $n \to \infty$, we get $fu = gu$. Since pair $(f, g)$is $R-$weakly commuting pair, this yields $d(fgu, gfu) \leqslant Rd(fu, gu)$ this implies $fgu = gfu$or $fgu = gfu = ffu = ggu$. Again, by (b), we have, $d(ffu, fu) \geqslant ad(gfu, gu) + bd(ffu, gfu) + cd(fu, gu)$. Since $a > 1$,this yields $ffu = fu$ or $ffu = gfu = fu$. Therefore fu is common fixed point of f and g.

Case 2-when $\lim fgx_n = gt$. Since pair(f, g) is pair of $R-$weakly commuting map, this yields $d(gfx_n, fgx_n) \leqslant Rd(fx_n, gx_n)$.
Now limiting $n \to \infty$ we have $\lim d(fgx_n, gfx_n) \leqslant 0$ implies

$$\lim fgx_n = \lim gfx_n \Rightarrow \lim gfx_n = gt.$$

Now by equation (2.1) of theorem (2.1), yields $gfx_{n+1} = ggx_n = gt$ as $n \to \infty$.
Now by (b) we have $d(fgx_n, ft) \geqslant ad(ggx_n, gt) + bd(fgx_n, ggx_n) + cd(ft, gt)$ limiting $n \to \infty$ when $c > 1$ this yields $ft = gt$. Since pair (f, g) is $R-$weakly commuting maps, this yields $d(fgt, gft) \leqslant Rd(ft, gt) \Rightarrow d(fgt, gft) \leqslant 0$ or $fgt =$

gft. Then $fgt = gft = fft = ggt$. Again, by (b), $d(fft, ft) \geqslant ad(gft, gt) + bd(fft, gft) + cd(ft, gt)$. Since $a > 1$, then it yields $fft = ft$. Or $fft = gft = ft$, and so ft is common fixed point of f and g. □

EXAMPLE 2.5. Let $x, y \in X$ $(x \geqslant y)$ and $X = [1, 10]$ and d be the usual metric on$X$. Define $f, g : X \to X$ as follows
$fx = 1$ if $x = 1$ &$x > 5$ and $fx = 6$ if $1 < x \leqslant 5$,
$gx = 1$ if $x = 1$,$gx = 5$ if $1 < x \leqslant 5$ and $gx = x$ if $x > 5$.
If $x_n = 1$, then $\lim fx_n = \lim gx_n = 1$. Also $\lim fgx_n = \lim gfx_n = 1$.

But, when $x_n = 1 + {}^1\!/_n$,
$\lim fx_n = 1$and $\lim gx_n = \lim \left(1 + {}^1\!/_n\right) = 1$Or $\lim fx_n = \lim gx_n = 1$
Moreover $\lim fgx_n = \lim f\left(1 + {}^1\!/_n\right) = 6 \neq g(1)$

And $\lim gfx_n = \lim g(1) = 1 = f(1)$, therefore (f, g) are weakly semi compatible maps. Moreover at $x = 1$, $1 < x \leqslant 5$ and $x > 5$ maps f&g satisfy R-weakly commutativity, that is $d(fgx, gfx) \leqslant Rd(fx, gx)$ when $R > 0$. Moreover pair (f, g) satisfy equal to condition of (b)when $x, y = 1$.

Also, if $fx = 6$ & $gx = 5$ for $x, y \in (1, 5]$ when $a > 1$, $b = -2$ & $c = 2$, pair (f, g) satisfies again equal to condition of (b). Now finally if $fx = 1$ & $gx = {}^x\!/_2$ for $x, y > 5$. The left hand side of given inequality yields $d(fx, fy) = |fx - fy| = 0$ and right hand side gives

$$ad(gx, gy) + bd(fx, gx) + cd(fy, gy)$$
$$3(x - y) - 3(x - 1) + (y - 1) \Rightarrow 1 - y \,(negative)$$

And therefore pair (f, g) satisfies greater than condition of (b) when $a = 3, b = -3$&$c = 2$. Also 1 is a common fixed point of$f$ and g.

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References

[1] Abbas, M., Gopal, D. and Radenovic, S., *A note on recently introduced commutative conditions*, Indian journal of mathematics, 2011, in press.
[2] Alghamdi, M.A., Radenovic, S. and Shahzad, N., *On some generalizations of commuting mappings*, Abstract and Applied Analysis, volume 2011, Article ID 952052, 6 pages.
[3] Aliouche, A. and Popa,V., *Common fixed point theorems for occasionally compatible mappings via implicit relations*, Filomat, **22:2**(2008), 99-107.
[4] Bouhadjera,H., *On common fixed point theorems for three and four self mappings satisfying contractive conditions*, Acta Univ. Palacki. Olomuc., Fac. rer. nat., mathematica, **49**(1)(2010), 25-31.
[5] Cho,Y.J., Sharma,B.K. and Sahu,D.R., *Semi compatibility and fixed points*, Math Japonica **42**(1995), 91-98.
[6] Doric, D., Kadelburg, Z. and Radenovic, S., *A note on occasionally weakly compatible mappings and common fixed points*, Fixed Point Theory, 2011, in press.
[7] Jain, M.K., Rhoades, B.E. and Saluja, A.S., *Fixed Point Theorems for Occasionally Weakly Compatible Expansive Mappings*, J. Adv. Math. Stud., Vol.**5**(2012),No. 2, 54-58.

[8] Jungck,G. Murthy, P.P. and Cho,Y.J.,*Compatible mappings of type(A) and common fixed points*, Math.Japon., **38**(1993), No.2, 381-390.
[9] Jungck,G. and Rhoades, B.E., *Fixed point theorems for occasionally weakly compatible mappings*, Fixed Point theory **7**(2006) 287-296.
[10] Jungck,G., *Compatible mappings and common fixed points*, Int. J. Math. Math. Sci.**9** (1986), no.4, 771-779.
[11] M.A.Al-Thagafi and Shahzad, N., *Generalized I-nonexpansive self maps and invariant pproximations*, Acta Mathematica Sinica **24**(2008), 867-876.
[12] Pant,R.P., Bisht,R.K. and Arora,D., *weak reciprocal continuity and fixed point theorems*, Ann. Univ. Ferrara, **57**(2011):181-190.
[13] Pant,R.P., *Common fixed points of non commuting mappings*, J. Math. Anal. Appl. **188**(1994), 436-440.
[14] Pathak,H.K.,Cho,Y.J.and Kang,S.M.,*Remarks of R-weakly commuting mappings and common fixed point theorems*, Bull.Korean Math. Soc.**34**(1997), 247-257.
[15] Sessa,S., *On a weak commutativity condition of mappings in fixed point considerations*, Publ. Inst. Math. **32** (1982), 149-153.
[16] Singh,B., Jain,S. ,*Semi compatibility, compatibility and fixed point theorems in fuzzy metric spaces*, J.Chungcheong Math. Soc. **18**(2005),1-22.

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