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DOMINATION INTEGRITY IN TREES

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ABSTRACT. C.A. Barefoot, et. al. [**6**] introduced the concept of the integrity of a graph. It is an useful measure of vulnerability and it is defined as follows. $I(G) = \min\ \{|S| + m(G - S) : S \subseteq V(G)\}$, where $m(G - S)$ denotes the order of the largest component in $G - S\}$. Unlike the connectivity measures, integrity shows not only the difficulty to break down the network but also the damage that has been caused. A subset S of $V(G)$ is said to be an I-set if $I(G) = |S| + m(G - S)$. We define the concept of Domination Integrity of a graph G is defined as $DI(G) = \min\ \{|S| + m(G - S)$: where S is a dominating set of G and $m(G - S)$ denotes the order of the largest component in $G - S\}$ and is denoted by $DI(G)$. In this paper, we found the Domination Integrity in trees.

1. Introduction

In an administrative set up, decisions are taken by a small group who have effective communication links with other members of the organization. Domination in graphs provides a model for such a concept. A subset D of $V(G)$ of a graph is a dominating set if for every $u \in V - D$, there exists a $v \in D$ such that $uv \in E(G)$.In a network, a minimum dominating set of nodes provides a link with the rest of the nodes. If D is a minimum dominating set and if the order of the largest component of $G - D$ is small, then the removal of D results in a chaos in the network because not only the decision making process is paralyzed but also the communication between the remaining members is minimized. So, we introduce the concept of Domination Integrity of a graph as another measure of vulnerability of a graph.

2. Domination Integrity in Trees

OBSERVATION 2.1. If T is a tree of order at least 3, then $\gamma(T) \geqslant \frac{n+2-n_1}{3}$, where n_1 denotes the number of end vertices of T [**8**]. Therefore, $\frac{n+2-n_1}{3} + 1 \leqslant \gamma(T) + 1 \leqslant DI(T)$.

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Observation 2.2. If T is a tree with l leaves, then $\frac{n-l+2}{3} \leqslant \gamma(G)$ [**8**]. Therefore, $\frac{n-l+2}{3} + 1 \leqslant \gamma(G) + 1 \leqslant DI(G)$.

Observation 2.3. For any tree T, $\alpha_0(T) \leqslant n - \Delta(T)$. For : Let $v \in V(T)$ be a vertex of maximum degree. Then $S = V - N(v)$ is a vertex cover of T. Therefore, $\alpha_0(T) \leqslant |S| = n - \Delta(T)$.

Proposition 2.1. *Let T be a tree. Then $\alpha_0(T) = n - \Delta(T)$ if and only if T is a wounded spider.*

Proof. Let $v \in V(T)$ be a vertex of maximum degree in T.

Let $u_1, u_2, \cdots, u_{\Delta(T)}$ be the neighbours of v. Clearly, they are independent as T is a tree. Let $S = V - N(v)$. Since $|N(v)| = \Delta(T)$, $|S| = n - \Delta(T)$. Since S is a vertex cover of T and since $\alpha_0(T) = n - \Delta(T)$, S is a minimum vertex cover of T. Let $x \in S$. Clearly, x and v are independent for any $x \in S$, $x \neq v$ (since $N(v) = V - S$). If x is not adjacent to any vertex of $N(v)$, then $S - \{x\}$ is a vertex cover of T, since all neighbours of x are in S, a contradiction as S is a minimum vertex cover. Therefore, every vertex in S is adjacent to some vertex of $N(v)$. Let $x, y \in S$. Suppose x, y are adjacent. Then as x, y are adjacent to vertices in $N(v)$. It results a cycle, a contradiction.

If $x, y \in S$ are adjacent to some vertex, say $w \in N(v)$, then $(S - \{x, y\}) \cup \{w\}$ is a vertex cover of cardinality less than $|S|$, a contradiction. Therefore, each vertex in S except v is adjacent to exactly one vertex in $N(v)$. Suppose there are $\Delta(T)$ vertices in $S - \{v\}$. Then we get a subdivided star. $|S| = \Delta(T) + 1$ and $|V - S| = \Delta(T)$. In this case, $V - S$ is a vertex cover of cardinality less than $|S|$, a contradiction. Therefore, there exists at least one vertex in $N(v)$ which is not adjacent to any vertex in S.

Therefore, T is a wounded spider. Therefore, $\gamma(T) = n - \Delta(T) = \alpha_0(T)$. The converse is obvious. □

Theorem 2.1. *For any tree T, $DI(T) = n - \Delta(T) + 1$ if and only if T is a wounded spider.*

Proof.

(i): Suppose T is a wounded spider. Then for a vertex v of maximum degree $\Delta(T)$, $V - N(v)$ is a γ-set of T. Also, $N(v)$ is independent. Therefore, $\gamma(T) + 1 \leqslant DI(T) \leqslant n - \Delta(T) + 1$. Since $\gamma(T) = n - \Delta(T)$, $DI(T) = n - \Delta(T) + 1$.

Conversely, let $DI(T) = n - \Delta(T) + 1$. Then $n - \Delta(T) + 1 = DI(T) \leqslant \alpha_0(T) + 1 \leqslant n - \Delta(T) + 1$ (since $\alpha_0(T) \leqslant n - \Delta(T)$, for any tree T). Therefore, $n - \Delta(T) = \alpha_0(T)$. Hence, T is a wounded spider.

(ii): Suppose T is a wounded spider. Then for a vertex v of maximum degree $\Delta(T)$, $V(T) - N(v)$ is a γ-set of T. Also, $N(v)$ is independent.
Therefore, $\gamma(T) + 1 \leqslant DI(T) \leqslant n - \Delta(T) + 1$. Since $\gamma(T) = n - \Delta(T)$, $DI(T) = n - \Delta(T) + 1$.

Conversely, let $DI(T) = n - \Delta(T) + 1$. Let v be a vertex of maximum degree in T. Then $V(T) - N(v)$ is a dominating set of T. $V(T) - (V(T) - N(v)) = N(v)$

which is independent since T is a tree. Therefore, $(V(T) - N(v)) + m(V(T) - (V(T) - N(v)) = n - \Delta(T) + m(N(v)) = n - \Delta(T) + 1 = DI(T)$.
Therefore, $V - N(v)$ is a DI-set of T.
Suppose S is a γ-set of T such that $|S| + m(V(T) - S) = DI(T)$. Therefore, $\gamma(T) + m(V(T) - S) = n - \Delta(T) + 1$. Since, $m(V(T) - S) \geqslant 1$, $\gamma(T) \leqslant n - \Delta(T)$.

Suppose $\gamma(T) < n - \Delta(T)$. Then $m(V(T) - S) \geqslant 2$. Let M be a maximum order component of $V(T) - S$ of cardinality $\geqslant 2$.

Suppose $|M| \geqslant 3$. Let $y_1, y_2, y_3 \in V(M)$. Let $x_1, x_2, x_3 \in S$ dominate y_1, y_2, y_3 respectively. Without loss of generality, let y_1 be adjacent to y_2 and y_2 be adjacent to y_3. Let $S_1 = S - \{y_2\}$. Then $m(V(T) - S_1) \leqslant m(V(T) - S) - 2$. Therefore, $|S| + m(V(T) - S_1) \leqslant |S| + 1 + m(V(T) - S) - 1 = DI(T) - 1$, a contradiction. Therefore, $|M| \leqslant 2$. But $m(V(T) - S) \geqslant 2$. Therefore, $|M| = 2$. Let $V(M) = \{y_1, y_2\}$. Let $x_1, x_2 \in S$ dominate y_1, y_2 respectively. Let S_1 be a subset of S dominate M. Then clearly, S_1 is independent. Further, any vertex in S cannot dominate more than one vertex in M (otherwise, which results in a cycle). Let $y_1, y_2 \in M$ such that $y_1 y_2 \in E(T)$. Let $x_1, x_2 \in S_1$ dominate y_1, y_2 respectively. If x_1 dominates a vertex in a component M_1 in $V(T) - S$, then x_2 cannot dominate any vertex in M_1. Let every component of $V(T) - S$ other than M is of cardinality 1.

Since $|S| + m(V(T) - S) \geqslant DI(T) = n - \Delta(T) + 1$, $\gamma(T) + 2 \geqslant n - \Delta(T) + 1$. That is, $\gamma(T) \geqslant n - \Delta(T)$. But $\gamma(T) < n - \Delta(T)$. Therefore, $\gamma(T) = n - \Delta(T) - 1$.

Case(1): $\gamma(T) < \Delta(T)$.
If x_1 is not adjacent to any vertex of $V - S$ other than y_1 (similarly if x_2 is not adjacent to any other vertex of $V(T) - S$ other than y_2), then $S_1 = S - \{x_1\} \cup \{y_1\}$ is a γ-set of T, $m(V(T) - S) = 1$ and $DI(T) \leqslant |S_1| + m(V - S_1) = n - \Delta(T) - 1 + 1 = n - \Delta(T)$, a contradiction. Therefore, x_1 is adjacent to $y_3 \neq y_1, y_2$ and x_2 is adjacent to $y_4 \neq y_1, y_2$. If $y_4 = y_3$, then T is a cycle. Therefore, $y_4 \neq y_3$. Let $S = \{x_1, x_2, x_3, \cdots, x_{\gamma(T)}\}$ and $V(T) - S = \{y_1, y_2, y_3, \cdots, y_{\Delta(T)+1}\}$. If $deg(x_i) = \Delta(T)$ for some $i \geqslant 3$, then x_i can be adjacent to exactly one vertex in $\{y_1, y_2, y_3, y_4\}$. Therefore, $|N(x_i) \cap (V(T) - S)| \leqslant \Delta(T) - 2$. x_i can not be adjacent to x_1, x_2, otherwise, T is a cycle. Suppose x_i is adjacent to x_j and x_k, j, $k \geqslant 4$. Then S being a γ-set, x_j and x_k must have private neighbours in $V - S$ which is not possible. Suppose $deg(x_i) = \Delta(T)$. Then x_1 is adjacent with all vertices of $V - S$ except one vertex. In this case, which results a cycle, a contradiction. A similar argument can be given to show that $deg(x_2) \neq \Delta(T)$.

Suppose $deg(y_i) = \Delta(T)$, for some i, $1 \leqslant i \leqslant \Delta(T) + 1$. Since $y_3, y_4, \cdots$, $y_{\Delta(T)+1}$ are all isolates in $< V - S >$ and since $|S| < \Delta(T)$, $deg(y_i) < \Delta(T)$, for every i, $3 \leqslant i \leqslant \Delta(T) + 1$. Suppose $deg(y_1) = \Delta(T)$.
Then $|N(y_1) \cap S| = \Delta(T) + 1$. Since $\gamma(T) \leqslant \Delta(T) - 1$, y_1 is adjacent to every vertex of S and $\gamma(T)$ must be equal to $\Delta(T) - 1$. Therefore, y_1 is adjacent to both x_1 and x_2, which results in a cycle, a contradiction.(A similar argument shows that $deg(y_2) < \Delta(T)$).

Case(2): $\gamma(T) = \Delta(T)$.

$\gamma(T) = n - \Delta(T) - 1 = \Delta(T)$. Therefore, $n = 2\Delta(T) + 1$. As in case (1), $S = \{x_1, x_2, x_3, \cdots, x_{\gamma(T)}\}$ and $V - S = \{y_1, y_2, y_3, \cdots, y_{\gamma(T)+1}\}$, x_1 is adjacent to y_1, y_3 and x_2 is adjacent to y_2, y_4. Suppose $deg(x_i) = \Delta(T)$, $1 \leqslant i \leqslant \Delta(T)$. In this case, as $|V(T) - S| = \Delta(T) + 1$, we get that x_i is adjacent with every vertex of $V - S$ except one vertex. This results in a cycle, a contradiction. Therefore $deg(x_i) < \Delta(T)$, for every i. Let $deg(y_j) = \Delta(T)$, $1 \leqslant j \leqslant \Delta(T) + 1$. Then y_j is adjacent to every vertex in S which is not possible since which results a cycle. Therefore, $deg(y_j) < \Delta(T)$, $1 \leqslant j \leqslant \Delta(T) + 1$, a contradiction.

Case(3): $\gamma(T) > \Delta(T)$.
$S = \{x_1, x_2, x_3, \cdots, x_{\gamma(T)}\}$ and $V(T) - S = \{y_1, y_2, y_3, \cdots, y_{\Delta(T)+1|}\}$. If $deg(x_i) = \Delta(T)$ and x_i is adjacent to k vertices $\{x_{i1}, x_{i2}, \cdots, x_{ik}\}$ in S, then $x_{i1}, x_{i2}, \cdots, x_{ik}$ must have private neighbours say $y_{i1}, y_{i2}, \cdots, y_{ik}$. Therefore, x_i is adjacent to $\Delta(T) - k$ vertices in $(V - S) - \{y_{i1}, y_{i2}, \cdots, y_{ik}\}$ whose cardinality is $\Delta(T) - k + 1$. Therefore, x_i is adjacent to all but one of the vertices y_1, y_2, y_3, and y_4 which results a cycle, a contradiction.
If $deg(y_1) = \Delta(T)$ $(deg(y_2) = \Delta(T))$, then y_1 is adjacent to $x_3, x_4, \cdots, x_{\Delta(T)}$. $\{x_1, x_2, x_3, x_4, \cdots, x_{\Delta(T)}\}$ is an independent set, since otherwise which results a cycle, a contradiction. If x_3 is adjacent to some vertex of S, then x_3 has a private neighbour in $V - S$. Therefore, x_3 is adjacent to some vertex y_j, $5 \leqslant j \leqslant \Delta(T) + 1$. Suppose x_3 is not adjacent to any vertex in S. If x_2 is not adjacent to any y_j, $5 \leqslant j \leqslant \Delta(T) + 1$, then $S_1 = (S - \{x_3\}) \cup \{y_1\}$ is a γ-set of T with $m(V(T) - S_1) = 1$. Therefore, $DI(T) \leqslant |S_1| + m(V - S_1) = \gamma(T) + 1 < n - \Delta(T) + 1$, a contradiction. Therefore, x_3 is adjacent to some vertex y_j, $5 \leqslant j\Delta(T) + 1$. Hence, every vertex x_i, $3 \leqslant i \leqslant \Delta(T)$ is adjacent to at least one vertex y_j, $5 \leqslant j \leqslant \Delta(T) + 1$. If $x_{i1}, x_{i2} \in \{x_3, x_4, \cdots, x_{\Delta(T)}\}$ are adjacent to the same vertex y_j, then we get a cycle, a contradiction. Therefore, each of $x_3, x_4, \cdots, x_{\Delta(T)}$ is adjacent to distinct vertices in$\{y_5, y_6, \cdots, y_{\Delta(T)+1}\}$. Since $|\{x_3, x_4, \cdots, x_{\Delta(T)}\}| = \Delta(T) - 2$ and $|\{y_5, y_6, \cdots, y_{\Delta(T)+1}\}| = \Delta(T) - 3$, a contradiction. Similar argument can be given for any y_i, $2 \leqslant i \leqslant \Delta(T) + 1$ and $deg(y_i) = \Delta(T)$. Therefore, the cardinality of any component in $V - S$ is less than or equal to 2. Suppose there exists at least two components of cardinality 2.

Case(i): $\gamma(T) < \Delta(T)$.
Let x_i, $(1 \leqslant i \leqslant \gamma(T))$ be of degree $\Delta(T)$. Let x_i be adjacent to t vertices in S and $\Delta(T) - t$ vertices in $V - S$. Each of these $\Delta(T) - t$ vertices belong to different components of $V - S$. If t neighbours of x_i in S can not be adjacent to any of these $\Delta(T) - 1$ components, there can be at most $\Delta(T) - 1$ components (since there are at least two components of order 2). The t neighbours of x_i must be adjacent to one vertex in each of $(\Delta(T) - 1) - (\Delta(T) - t) = t - 1$ components (since the neighbours being not isolates of S must have private neighbours in $V - S$). Therefore, these are two neighbours of x_i which are adjacent to vertices in the same components of $V - S$. Since x_i is adjacent to these two neighbours, which results a cycle, a contradiction.

Let y_j, $1 \leqslant j \leqslant \Delta(T)+1$ be of degree $\Delta(T)$. Since y_j is adjacent to at most one vertex in $V-S$, $|N(y_j) \cap S| \geqslant \Delta(T)-1$. If $\gamma(T) < \Delta(T)-1$, then we get a contradiction. Therefore, $\gamma(T) = \Delta(T)-1$. In this case, y_j is adjacent to every vertex in S. Since x_1, x_2 are adjacent to y_1, y_2 respectively and y_1 is adjacent to y_2, which results a cycle, a contradiction.

Case(ii): $\gamma(T) = \Delta(T)$.
As in Case(i), no x_i, $1 \leqslant i \leqslant \gamma(T)$ can be of degree $\Delta(T)$. Suppose y_j, $1 \leqslant j \leqslant \Delta(T)+1$ be of degree $\Delta(T)$. Let y_j belong to a maximum order component in $V(T)-S$. Then $|N(y_j) \cap S| = \Delta(T)-1$. Therefore, y_j is adjacent to the vertices x_{i1}, x_{i2} which are adjacent to vertices in another maximum order component. Therefore, we get a cycle, a contradiction. Suppose y_j belongs to a singleton component of $V(T)-S$. Then y_j is adjacent to both the vertices x_{i1} and x_{i2} which are adjacent to the vertices of a maximum order component. Therefore, which results a cycle, a contradiction.

Case(iii): $\gamma(T) > \Delta(T)$.
Let $S = \{x_1, x_2, x_3, \cdots, x_{\gamma(T)}\}$ and $V-S = \{y_1, y_2, y_3, \cdots, y_{\Delta(T)+1}\}$. Since T has no isolates, $V(T)-S$ is a dominating set of T. Therefore $|V(T)-S| \geqslant \gamma(T)+1$. That is, $\Delta(T)+1 \geqslant \gamma(T) > \Delta(T)$. Therefore, $\gamma(T) = \Delta(T)+1$. Let $\{y_1, y_2\}$ be a maximum order component in $V-S$. Let y_1 be adjacent to x_1 and y_2 be adjacent to x_2. Clearly x_1 and x_2 are independent.
If $deg(x_i) = \Delta(T)$, $1 \leqslant i \leqslant \Delta(T)+1$, proceeding as in case(ii) we get a contradiction.
Let $deg(y_j) = \Delta(T)$ for some $1 \leqslant j \leqslant \Delta(T)+1$. Therefore, y_j is adjacent to every x_i except one vertex, $1 \leqslant i \leqslant \Delta(T)+1$. Suppose $deg(y_1) = \Delta(T)$. Let y_1 be adjacent to $x_3, x_4, \cdots, x_{\Delta(T)}$. There exists a component of cardinality 2 in $V-S$ other than $\{y_1, y_2\}$. Let y_j be adjacent to y_{j+1} $(3 \leqslant j \leqslant \Delta(T)+1)$. Let y_j and y_{j+1} be dominated by x_k and $x_l, (x_k \neq x_l)$. At least one of x_k and x_l does not belong to $\{x_3, x_4, \cdots, x_{\Delta(T)}\}$.

Subcase(i): Both x_k and x_l do not belong to $\{x_3, x_4, \cdots, x_{\Delta(T)}\}$.

Case(A): $x_k = x_1$ and $x_l = x_2$. In this case, we get a cycle, a contradiction.

Case(B): $x_k = x_1$ and $x_l = x_{\Delta(T)+1}$. $x_1, x_2, \cdots, x_{\Delta(T)}$ are all independent. If S is itself independent, then $V(T)-S$ is a minimum dominating set and $m(S) = 1$. Therefore, $DI(T) = \Delta(T)+2$. But $DI(T) = \gamma(T)+2 = \Delta(T)+3$, a contradiction. In this case, which results a cycle, a contradiction.

Case(C): $x_k = x_2$ and $x_l = x_{\Delta(T)+1}$. This case is similar to Case(B). Hence Case(C) results in a contradiction.

Subcase(ii): $x_k \notin \{x_3, x_4, \cdots, x_{\Delta(T)}\}$ and $x_l \in \{x_3, x_4, \cdots, x_{\Delta(T)}\}$.

Suppose $x_k = x_1$ and $x_l = x_i$, $3 \leqslant i \leqslant \Delta(T)$. In this case, we get a cycle, a contradiction. A similar argument when $x_k = x_2$ and $x_l = x_i$, $3 \leqslant i \leqslant \Delta(T)$, which results a cycle, a contradiction. Let $x_k = x_{\Delta(T)+1}$ and $x_l = x_i$, $3 \leqslant i \leqslant \Delta(T)$. In this case, arguing in Subcase(i) in Case(B), which results a cycle, a contradiction. A similar arguments can be given when $deg(y_j) = \Delta(T)$, $2 \leqslant j \leqslant \Delta(T)+1$, leading to a contradiction. Therefore, Case(ii) does not arise. Therefore, $\gamma(T) = n - \Delta(T)$. Therefore, T is a wounded spider. □

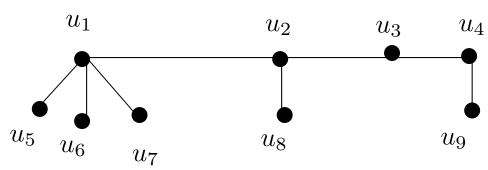
PROPOSITION 2.2. *For any graph G without isolates, $DI(G) = n - m + 1$ if and only if G is a galaxy.*

PROOF. Let G be a galaxy. It is easy to verify that $DI(G) = n - m + 1$. Conversely, suppose $DI(G) = n - m + 1$. For any graph G without isolates, $n - m + 1 \leqslant \gamma(G) + 1 \leqslant DI(G) = n - m + 1$ (since, for any (n, m)-graph G, $n - m - n_0 \leqslant \gamma(G) \leqslant n - \Delta(G)$, where n_0 denotes the number of isolates in G [**9**].) Therefore, $\gamma(G) = n - m$. Therefore, G is a galaxy. □

PROPOSITION 2.3. *Let T be a tree which is not a star satisfying the condition that $d(u, v) \equiv 2$ (mod 3) for any two end vertices $u, v \in V(T)$. Then $DI(T) = \frac{n-n_1+8}{3}$, where n_1 is the number of end vertices in T.*

PROOF. Suppose T is a tree satisfying the hypothesis. Then there exists a dominating set D of T such that $V - D$ contains all the end vertices of T and $d(u, v) \equiv 0$ (mod 3) for any $u, v \in D$ [**8**]. All the end vertices in $V - D$ are independent in $V - D$. Suppose $V - D$ contains a P_3. Let $V(P_3) = \{x_1, x_2, x_3\}$. Then x_1, x_2, x_3 are neither supports nor end vertices. Let y_1, y_2, y_3 belonging to D dominate respectively x_1, x_2, x_3. Clearly, y_1, y_2, y_3 are all independent. Since $d(y_1, y_3) = 4$, a contradiction. Therefore, there exists at least two vertices say y_1, y_2 which dominate x_1, x_2, x_3, a contradiction, since which results a cycle. Therefore, $V - D$ does not contain a path of length 3. Therefore, $m(V - D) \leqslant 2$. Suppose $m(V - D) = 1$. Since, $< D >$ is independent, which results a star, a contradiction. Therefore, $m(V - D) = 2$. Therefore, $DI(T) \leqslant |D| + 2$. That is, $DI(T) \leqslant \frac{n-n_1+8}{3}$. If $DI(T) < \frac{n-n_1+8}{3} = \gamma(T) + 2$, then $DI(T) = \gamma(T) + 1$. Therefore, $\gamma(T) = \alpha_0(T)$. As per condition in theorem 4.1 in [**9**] for $\gamma(T) = \alpha_0(T)$, the subgraph $G^* = G - N[\Omega(G)]$ is bipartite, the components are G^* are bipartite graphs $L_1, L_2, \cdots, L_k$ with $\gamma(L_j) = \alpha_0(L_j)$ and $\delta(L_j) \geqslant 1$ for every j, $1 \leqslant j \leqslant k$. Further L_j's are either stars with at least three vertices or nor stars in which the removal of pendent vertices result in a connected graph with minimum degree 2. Here G^* has components $K_2's$ which does not satisfy the condition for $\gamma(T) = \alpha_0(T)$. Therefore, $DI(T) \neq \gamma(T) + 1$.
Therefore, $DI(T) = \gamma(T) + 2 = \frac{n-n_1+8}{3}$. □

REMARK 2.1. The converse is not true. Consider the tree T.



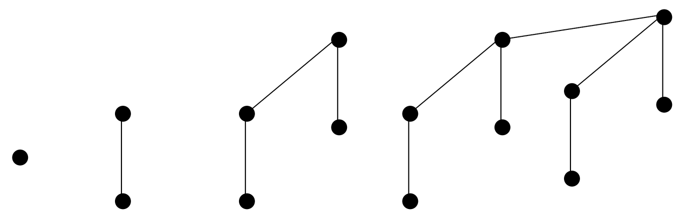
$DI(T) = 4$; $n = 9$; $n_1 = 5$. $DI(T) = \frac{n-n_1+8}{3} = 4$. Here u_5 and u_8 are end vertices such that $d(u_5, u_8) \not\equiv 2 \pmod 3$. $\frac{n-n_1+2}{3} = 2 < \gamma(T)$.

Remark 2.2. There are trees for which $DI(T) = \gamma(T)+2$ but $\gamma(T) > \frac{n-n_1+2}{3}$. For example, $DI(P_{14}) = \lceil \frac{14}{3} \rceil + 2 = 7 = \gamma(T) + 2$. $\gamma(T) = \lceil \frac{14}{3} \rceil = 5$. $\frac{n-n_1+2}{3} = \frac{14-2+2}{3} = \frac{14}{3}$ and $\gamma(P_{14}) > \frac{n-n_1+2}{3}$.

Remark 2.3. For a star $K_{1,n}$, $d(u, v) \equiv 2 \pmod 3$ for any two end vertices $u, v \in V(T)$ and $\gamma(K_{1,n}) = 1$; $DI(K_{1,n}) = \gamma(K_{1,n}) + 1 = 2$.

Remark 2.4. If $DI(T) = \gamma(T) + 2 = \frac{n-n_1+8}{3}$, then $T \in \mathcal{R}$,where $\mathcal{R}$ is the collection of trees which satisfy the condition that $d(u, v) \equiv 2 \pmod 3$ for every two end vertices $u, v \in V(T)$.

Definition 2.1. The Binomial tree B_n is an ordered tree defined recursively. The binomial tree B_0 consists of a single vertex. The binomial tree B_n consists of two binomial trees B_{n-1} that are linked together: the root of one is the leftmost child of the root of the other. In the following figure, we call the vertex u top vertex of B_n.



B_0 B_1 B_2 B_3

Theorem 2.2. *Let $n \geqslant 1$ be a positive integer. Then $DI(B_n) = 2^{n-1} + 1$.*

Proof. Since $B_n = B^+_{n-1}, n \geqslant 1$, $\gamma(B_n) = |V(B_{n-1})| = 2^{n-1} (n \geqslant 1)$. The removal of $V(B_{n-1})$ from B_n, results in totally disconnected graph and B_{n-1} is a minimum dominating set for B_n. Therefore $DI(B_n) = 2^{n-1} + 1$. □

Definition 2.2. Using the notation of [**7**], define H^k_n as the rooted complete k-ary tree of height $n - 1$, each vertex except the leaves has k children, and all leaves are distance $n - 1$ from the root. Thus H^n_k has order $(k^n - 1)/(k - 1)$.

Theorem 2.3. *[7] For $k \geqslant 2$, the integrity of the complete k-ary tree of height $n - 1$ is given by* $I(H^k_n) = \begin{cases} \frac{k^{(n+1)}/2-1}{k-1} & \text{if } n \text{ is odd} \\ \frac{(2k-1)k^{n/2-1}-1}{k-1} & \text{if } n \text{ is even} \end{cases}$

Theorem 2.4. *[7] The integrity of the complete binary tree of height $n - 1$ is given by* $I(H^2_n) = \begin{cases} 2^{(n+1)/2} - 1 & \text{if } n \text{ is odd} \\ 3.2^{n/2-1} - 1 & \text{if } n \text{ is even} \end{cases}$

Theorem 2.5.

$$\gamma(H_n^2) = \begin{cases} \frac{2(2^{(n/3)}-1)}{7} & \text{if } n \equiv 0 \pmod 3 \\ 1 + \frac{2^2(2^{(\frac{n-1}{3})}-1)}{7} & \text{if } n \equiv 1 \pmod 3 \\ 1 + \frac{2^3(2^{(\frac{n-2}{3})}-1)}{7} & \text{if } n \equiv 2 \pmod 3 \end{cases}$$

Proof. Consider H_{3n}^2. The vertices at the levels $3n-1, 3n-4, \cdots, 2$ are to be taken to get a minimum dominating set. Since the number of vertices at level t is 2^{t-1}. Therefore, $\gamma(H_{3n}^2) = 2^1 + 2^4 + \cdots + 2^{3n-2} = \frac{2(2^{(n/3)}-1)}{7}, n \geqslant 1$.
Consider H_{3n-1}^2. The vertices at the levels $3n-2, 3n-5, \cdots, 4, 1$ are to be taken to get a minimum dominating set. $\gamma(H_{3n-1}^2) = 2^0 + 2^3 + \cdots + 2^{3n-3} = 1 + \frac{2^2(2^{(\frac{n-1}{3})}-1)}{7}, n \geqslant 1$.
Consider H_{3n-2}^2. The vertices at the levels $3n-3, 3n-6, \cdots, 3, 1$ are to be taken to get a minimum dominating set.
$\gamma(H_{3n-2}^2) = 2^0 + 2^2 + \cdots + 2^{3n-4} = 1 + \frac{2^3(2^{(\frac{n-2}{3})}-1)}{7}$, $n \geqslant 1$. □

Corollary 2.1. *In H_n^2, the removal of a γ-set, results in a disconnected graph in which the maximum order of the component is 3.*
Therefore, $DI(H_n^2) = \gamma(H_n^2) + 3$.

Observation 2.4. A similar argument leads to the calculation of $\gamma(H_n^k)$ and $DI((H_n^k)$. Observe that the number of vertices at level t is k^{t-1}.

Therefore, $\gamma(H_n^k) = \begin{cases} \frac{k(k^{(n/3)}-1)}{7} & \text{if } n \equiv 0 \pmod 3 \\ 1 + \frac{k^2(k^{(\frac{n-1}{3})}-1}{7} & \text{if } n \equiv 1 \pmod 3 \\ 1 + \frac{k^3(k^{(\frac{n-2}{3})}-1)}{7} & \text{if } n \equiv 2 \pmod 3 \end{cases}$ and

$DI(H_n^k) = \gamma(H_n^2) + (k+1)$.

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