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# STRONGLY REGULAR INTEGRAL CIRCULANT GRAPHS AND THEIR ENERGIES

**Thirugnanam Tamizh Chelvam[1], Sekar Raja[2] and Ivan Gutman[3]**

Abstract. Let $S \subset \mathbb{Z}_n$ be a finite cyclic group of order $n \geqslant 1$. Assume that $0 \notin S$ and $-S = \{-s : s \in S\} = S$. The circulant graph $G = Cir(n, S)$ is the undirected graph having the vertex set $V(G) = \mathbb{Z}_n$ and edge set $E(G) = \{ab \ : \ a, b \in \mathbb{Z}_n,\ a - b \in S\}$. Let $\mathcal{D}$ be a set of positive, proper divisors of the integer $n$. We characterize certain strongly regular integral circulant graphs with energy $2n(1 - 1/d)$ for a fixed $d \in \mathcal{D}$ , $d > 1$.

## 1. Introduction

In this paper, we characterize integral circulant graphs with three eigenvalues. Graphs with few distinct eigenvalues form an interesting class of graphs. Clearly if all the eigenvalues of a graph coincide, then the graph is trivial. Connected graphs with only two distinct eigenvalues are easily proven to be complete graphs. The first non-trivial graphs with three distinct eigenvalues are the strongly regular graphs [**10, 11**]. Graphs with exactly three distinct eigenvalues are generalizations of strongly regular graphs by dropping the regularity requirement [**9**].

Let $G$ be an undirected finite simple graph with $n$ vertices and adjacency matrix $A(G)$. Since $A(G)$ is a real symmetric matrix, its eigenvalues are real numbers. Without loss of generality one can assume that the eigenvalue set of $A(G)$ is $\{\lambda_1, \lambda_2, \lambda_3, \ldots, \lambda_n\}$, such that $\lambda_1 \geqslant \lambda_2 \geqslant \lambda_3 \geqslant \cdots \geqslant \lambda_n$. The energy $En(G)$ of a graph $G$ is defined as the sum of the absolute values of its eigenvalues

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[1] Department of Mathematics, Manonmaniam Sundaranar University, Tirunelveli 627 012, Tamil Nadu, India, e-mail: tamche_59@yahoo.co.in

[2] Department of Mathematics, Manonmaniam Sundaranar University, Tirunelveli 627 012, Tamil Nadu, India, e-mail: nellairajaa@yahoo.com

[3] Faculty of Science, University of Kragujevac, P. O. Box 60, 34000 Kragujevac, Serbia, e-mail: gutman@kg.ac.rs

of $A(G)$. The concept of graph energy has a chemical origin but is nowadays much studied in mathematics; for details see the reviews [**12, 14**].

A graph is integral if all the eigenvalues of its adjacency matrix are integers. The notion of integral graphs dates back to Harary and Schwenk [**15**]. Since then, many integral graphs have been discovered; for a survey see [**3**]. The problem of characterizing integral graphs seems to be very difficult and so it is wise to restrict ourselves to certain families of graphs. In this paper we proceed towards a characterization of divisors and corresponding integral circulant graphs.

The concept of Cayley graphs was introduced by Arthur Cayley, aimed at explaining the concept of abstract groups that are described by a set of generators. The Cayley graph $Cay(\Gamma, \Omega)$ of a group $\Gamma$ with identity 1 and a set $\Omega \subset \Gamma$ is defined to have vertex set $\Gamma$ and edge set $\{a, b \in \Gamma, ab^{-1} \in \Omega\}$. The set $\Omega$ is usually assumed to satisfy $1 \notin \Omega$ and $\Omega = \Omega^{-1} = \{a^{-1} : a \in \Omega\}$ which implies that $Cay(\Gamma, \Omega)$ is loop-free and undirected. For general properties of Cayley graphs we refer to Godsil and Royle [**11**].

Circulant graphs are Cayley graphs on finite cyclic groups, and found applications in telecommunication networks and distributed computation [**5**]. Recall that for a positive integer $n$ and a subset $S \subseteq \{0, 1, 2, \ldots, n-1\}$, the circulant graph $G(n, S)$ is the graph with $n$ vertices, labeled with integers modulo $n$, such that each vertex $i$ is adjacent to $|S|$ other vertices $\{i + s(mod\ n) | s \subset S\}$.

Wasin So [**26**] studied integral circulant graphs. Actually he proved that there are exactly $2^{\tau(n)-1}$ non-isomorphic integral circulant graphs on $n$ vertices, where $\tau(n)$ is the number of divisors of $n$.

Integral circulant graphs can be characterized as follows: Given an integer $n$ and a set $D$ of positive divisors of $n$, define the integral circulant graph $ICG(n, S)$ to have vertex set $\mathbb{Z}_n = \{0, 1, 2, \ldots, n-1\}$ and edge set $\{\{a, b\} : a, b \in \mathbb{Z}_n, \gcd(a - b, n) \in D\}$. The necessary and sufficient condition for the circulant graph to be integral is $S = \bigcup_{d \in \mathcal{D}} G_n(d)$ for some set of divisors $D \in D_n$, where $G_n(d) = \{k \mid \gcd(k, n) = d,\ 1 \leqslant k < n\}$ is the set of all positive integers less than $n$, having the same greatest common divisor $d$ with $n$. In addition, $D_n$ is the set of positive divisors $d$ of $n$, such that $d \leqslant n/2$. By this characterization, it is easy to see that the integral circulant graphs arise as a natural generalization of the unitary Cayley graphs, that are exactly the integral circulant graphs $ICG(n, S)$.

Our motivation for this research came from the constructions of the circulant graphs $G = Cir(n, S)$ which are not unitary Cayley graphs, except for $n/d = 2$, where $d$ is a divisor of $n$. There has been some recent work on the energy of unitary Cayley graphs [**22, 17, 20, 23, 24, 18, 19**]. In particular, Ramaswamy and Veena [**22**] calculated the energy of arbitrary unitary Cayley graphs with prime power, by showing that $En(Cay(\mathbb{Z}_n, U_n)) = 2^k\, \phi(n)$ for $n = p_1^{\alpha_1} p_2^{\alpha_2} \ldots p_k^{\alpha_k}$ with distinct primes $p_i$ and positive integer $\alpha_i$'s. The same result was independently obtained also by Ilić [**17**].

It should be noted that the energy of strongly regular graphs was not much studied so far (with two noteworthy exceptions [**16, 21**]). This is interesting, bearing in mind that general expressions for the eigenvalues of strongly regular graphs are known, and thus the calculation of their energy looks as a simple task. In reality, this task is not at all easy. The present work may be viewed as an attempt to partially fill this gap in the theory of graph energy.

In this paper our intention is to move a step forward in the investigation of properties of integral circulant graphs. Our paper is organized as follows. In Section 2 we present some preliminary definitions and results on integral circulant graphs, while in Section 3 we obtain the energy of certain circulant graphs. In Section 4 we characterize some strongly regular integral circulant graphs.

## 2. Definitions and preliminaries

In this section, we outline a few useful definitions and results.

THEOREM 2.1. [**26**] *A circulant graph $G = Cir(n, S)$ is integral if and only if $S = \bigcup_{d \in D} G_n(d)$ for some set of divisors $D$ and $0 \notin S, -S = \{-s : s \in S\} = S$.*

DEFINITION 2.1. The tensor product $A \otimes B$ of an $r \times s$ matrix $A = (a_{ij})$ and a $t \times u$ matrix $B = (b_{ij})$ is defined as the $rt \times su$ matrix obtained by replacing each entry $a_{ij}$ of $A$ by the double array $a_{ij}\, B$.

DEFINITION 2.2. [**2**] The tensor product of two graphs $G_1$ and $G_2$ is the graph $G_1 \otimes G_2$ with vertex set $V(G_1) \times V(G_2)$, in which the vertices $(u_1, u_2)$ and $(v_1, v_2)$ are adjacent if and only if $u_1v_1 \in E(G_1)$ and $u_2v_2 \in E(G_2)$.

It may be noted that if $G_1$ and $G_2$ are finite graphs with adjacency matrices $A$ and $A'$, respectively, then $G_1 \otimes G_2$, is graph whose adjacency matrix is $A \otimes A'$.

LEMMA 2.1. [**10**] *If $A$ is a matrix of order $r$ with spectrum $\{\lambda_1, \lambda_2, \ldots, \lambda_r\}$, and $B$, a matrix of order $s$ with spectrum $\{\mu_1, \mu_2, \ldots, \mu_s\}$, then the spectrum of $A \otimes B$ is $\{\lambda_i\, \mu_j \;:\; 1 \leqslant i \leqslant r\,;\, 1 \leqslant j \leqslant s\}$.*

DEFINITION 2.3. Two graphs $G = (V, E)$ and $H = (V', E')$ are said to be isomorphic if there is a bijective mapping $\phi$ from the vertex set $V$ to the vertex set $V'$ such that $(u, v) \in E(G)$ if and only if $(\phi(u), \phi(v)) \in E'(H)$. The mapping $\phi$ is called an isomorphism. We denote the fact that $G$ and $H$ are isomorphic by $G \cong H$.

That is, an isomorphism between two graphs is a bijection on the vertices that preserves edges and nonedges. This definition has the following special case:

DEFINITION 2.4. An automorphism of a graph is an isomorphism from the graph to itself.

DEFINITION 2.5. The set of all automorphisms of a graph $G$ forms a group, denoted by $Aut(G)$, the automorphism group of $G$.

## 3. A class of integral circulant graphs

Let $n > 1$ be a composite integer and $\mathcal{D}$ the set of all its positive, proper integer divisors. Let $d \in \mathcal{D}$.

Throughout this paper we denote by $M_n(d)$ the set of multiples of $d$ less than $n$, i. e.,

$$M_n(d) = \{d, 2d, 3d, \ldots, n-d\} \ .$$

In this section, we consider circulant graphs $G = Cir(n, S)$, where $S = \mathbb{Z}_n^* \setminus M_n(d)$ for a fixed $d \in \mathcal{D}$ , $d > 1$.

THEOREM 3.1. *Let $d$ be a proper divisor of a positive composite integer $n$ and $M_n(d) = \{d, 2d, \ldots, n-d\}$. If $S = \mathbb{Z}_n^* \setminus M_n(d)$ and $G = Cir(n, S)$, then $G$ is an integral circulant graph.*

PROOF. Clearly $\mathbb{Z}_n^* = \bigcup_{d \in \mathcal{D}} G_n(d)$ for all $d$, $1 \leqslant d \leqslant n-1$. Let $G = Cir(n, S)$, where $S = \mathbb{Z}_n^* \setminus M_n(d)$. It means that $S = \bigcup_{d \in \mathcal{D}'} G_n(d)$, where $\mathcal{D}' = \mathcal{D} \setminus \{d, 2d, \ldots, n-d\}$. Then by Theorem 2.1, $G = Cir(n, S)$ is integral. □

THEOREM 3.2. *Let $d$ be a proper divisor of a positive composite integer $n$. If $S = \mathbb{Z}_n^* \setminus M_n(d)$ and $G = Cir(n, S)$, then the energy of $G$ is equal to $2n(1 - 1/d)$.*

PROOF. Note that $|S| = |\mathbb{Z}_n^*| - |M_n(d)| = (n-1) - (n/d - 1) = n - n/d$. The adjacency matrix of $G$ is

$$\begin{array}{c} \\ 1 \\ 2 \\ \cdots \\ d-1 \\ d \\ d+1 \\ \cdots \\ 2d \\ \cdots \\ \cdots \\ n-1 \end{array}
\begin{array}{c} \begin{array}{ccccccccccc} 1 & 2 & \ldots & d-1 & d & d+1 & \ldots & 2d-1 & 2d & \ldots & n-1 \end{array} \\
\begin{pmatrix}
0 & 1 & \ldots & 1 & 0 & 1 & \ldots & 1 & 0 & \ldots & 1 \\
1 & 0 & \ldots & 1 & 1 & 0 & \ldots & 1 & 1 & \ldots & 1 \\
\ldots & \ldots & \ldots & \ldots & \ldots & \ldots & \ldots & \ldots & \ldots & \ldots & \ldots \\
1 & 1 & \ldots & 0 & 1 & 1 & \ldots & 0 & 1 & \ldots & 0 \\
0 & 1 & \ldots & 1 & 0 & 1 & \ldots & 1 & 0 & \ldots & 1 \\
1 & 0 & \ldots & 1 & 1 & 0 & \ldots & 1 & 1 & \ldots & 1 \\
\ldots & \ldots & \ldots & \ldots & \ldots & \ldots & \ldots & \ldots & \ldots & \ldots & \ldots \\
0 & 1 & \ldots & 1 & 0 & 1 & \ldots & 1 & 0 & \ldots & 1 \\
\ldots & \ldots & \ldots & \ldots & \ldots & \ldots & \ldots & \ldots & \ldots & \ldots & \ldots \\
\ldots & \ldots & \ldots & \ldots & \ldots & \ldots & \ldots & \ldots & \ldots & \ldots & \ldots \\
1 & 1 & \ldots & 0 & 1 & 1 & \ldots & 0 & 1 & \ldots & 0
\end{pmatrix} \end{array}$$

$$= \begin{pmatrix}
B & B & \ldots & B & B \\
B & B & \ldots & B & B \\
.. & .. & \ldots & .. & \ldots \\
B & B & \ldots & B & B \\
B & B & \ldots & B & B
\end{pmatrix}_{\frac{n}{d} \times \frac{n}{d}} \qquad \text{where } B = [B_{ij}]_{d \times d} = A(K_d).$$

From the above and by the definition of tensor product,

$$A(G) = C \otimes B, \text{ where } C = \begin{pmatrix}
1 & 1 & \ldots & 1 & 1 \\
1 & 1 & \ldots & 1 & 1 \\
.. & .. & \ldots & .. & \ldots \\
1 & 1 & \ldots & 1 & 1 \\
1 & 1 & \ldots & 1 & 1
\end{pmatrix}_{(n/d) \times (n/d)} .$$

Note that

$$Spec(B) = \{d-1\ ,\ -1^{d-1}\} \qquad \text{and} \qquad Spec(C) = \left\{\frac{n}{d}\ ,\ 0^{n/d-1}\right\}.$$

From Lemma 2.1,

$$Spec(A) = Spec(B)\, Spec(C) = \left\{(d-1)\frac{n}{d}\ ,\ -\frac{n}{d}^{d-1}\ ,\ 0^{n-d}\right\}$$

implying that the energy of $G$ is equal to

$$n - \frac{n}{d} + (d-1)\frac{n}{d} = 2n\left(1 - \frac{1}{d}\right)\ .$$

□

REMARK 3.1. For any proper divisor $d$ of $n$, $n - n/d < n - 1$ and therefore $En(G) < 2(n-1)$. Thus the circulant graphs $G = Cir(n, S)$ are not hypeenergetic (cf. [**13**]).

THEOREM 3.3. *Let $d$ be a proper divisor of a positive composite integer $n$. If $S = \mathbb{Z}_n^* \smallsetminus M_n(d)$, then $G = Cir(n, S)$ is a complete regular $d$-partite graph.*

PROOF. Consider $V_i = M_n(d) \cup \{i\}$ for $i = 0, 1, \ldots, d-1$. Clearly $V(G) = \cup_{i=0}^{i=d-1} V_i$. Since $d \notin S$, no two elements in $V_i$ are adjacent and so $V_i$ for $i = 0, 1, \ldots, n-1$ are independent sets in $G = Cir(n, S)$. For $i \neq j$, the difference between an element in $V_i$ and $V_j$ is an element of $S$ and therefore the respective vertices are adjacent in $G$. Hence the circulant graph $G = Cir(n, S)$ is a complete regular $d$-partite graph. □

Since any regular $d$-partite graph is a circulant graph, we have:

THEOREM 3.4. *A graph $G$ with $n$ vertices is a regular complete $d$-partite graph if and only if $G = Cir(n, S)$ where $S = \mathbb{Z}_n^* \smallsetminus M_n(d)$ and $d$ is a proper divisor of $n$.*

REMARK 3.2. Theorems 3.3 and 3.4 are relatively easy, and previously known results. We stated their proofs in order to make the paper self-contained. The same is the case with Theorem 4.1 in the subsequent section.

## 4. Strongly regular circulant graphs

A $k$-regular graph $G$ with $n$ vertices is said to be strongly regular with parameters $(n, k, a, c)$ if the following conditions are obeyed [**10**].

(1) $G$ is neither complete, nor empty;
(2) any two adjacency vertices of $G$ have $a$ common neighbors;
(3) any two nonadjacent vertices of $G$ have $c$ common neighbors.

We assume throughout that the considered strongly regular graph $G$ is connected. Consequently, $k$ is an eigenvalue of the adjacency matrix of $G$ with unit multiplicity, and

$$n - 1 > k \geqslant c > 0 \qquad \text{and} \qquad k - 1 > a \geqslant 0\ .$$

Counting the number of edges in $G$ connecting the vertices adjacent to a vertex $x$ and the vertices not adjacent to $x$ in two ways, we obtain

$$k(k-a-1) = (n-k-1)c \; .$$

Thus, if three of the parameters $(n, k, a, c)$ are given, then the fourth is uniquely determined [**10**]. If $A$ is the adjacency matrix of a strongly regular graph with parameters $n, k, a, c$, then

$$A^2 = kI + aA + c(J - A - I) \; . \tag{4.1}$$

Since eigenvectors with eigenvalue $\lambda = k$ are orthogonal to the all-one vector, by Eq. (4.1) the remaining eigenvalues must satisfy the equation

$$\lambda^2 - (a-c)\lambda - (k-c) = 0 \; . \tag{4.2}$$

Thus the eigenvalues of G are [**10**]

$$k \qquad \text{and} \qquad \lambda_1,\, \lambda_2 = \frac{(a-c) \pm \sqrt{\Delta}}{2} \tag{4.3}$$

where $\Delta = (a-c)^2 + 4(k-c) > 0$. Because the sum of the eigenvalues equals $trace(A) = 0$, it easily follows that the corresponding multiplicities are [**10**]

$$m_1 = 1 \qquad \text{and} \qquad m_2, m_3 = \frac{1}{2}\left(n - 1 \pm \frac{(n-1)(c-a) - 2k}{\sqrt{\Delta}}\right) \; .$$

Lemma 4.1. [**4**] *Let $G$ be a connected regular graph with exactly three distinct eigenvalues. Then $G$ is strongly regular.*

Lemma 4.2. [**8**] *A strongly regular graph with parameters $(n, k, a, c)$ is connected if and only if $c \neq 0$.*

Lemma 4.3. [**25**] *A graph has exactly one positive eigenvalue if and only if its non-isolated vertices form a complete multipartite graph.*

We are now prepared to prove our main result.

Theorem 4.1. *Let $d$ be a proper divisor of a positive composite integer $n$, $S = \mathbb{Z}_n^* \setminus \{d, 2d, \ldots, n-d\}$ and $G = Cir(n, S)$. Then $G$ is a strongly regular graph with parameters*

$$\left(n \, , \, n - \frac{n}{d} \, , \, n - \frac{2n}{d} \, , \, n - \frac{n}{d}\right) .$$

Proof. As specified in the proof of Theorem 3.2, $G$ has exactly three distinct eigenvalues and so by Lemma 4.1, $G$ is strongly regular. Clearly, $G$ is $|S| = n - n/d$ regular. By Theorem 3.3, $G$ is a $d$-partite graph with $n/d$ vertices in each partition. Hence any two adjacent vertices in $G$ have $n - 2n/d$ common neighbors. Also any two non-adjacent vertices $G$ have $n - n/d$ common neighbors. □

Remark 4.1. Theorem 4.1 has the following immediate consequences: Because $n - n/d \neq (n-1)/2$, the graph $G$ specified in Theorem 4.1 is not a conference graphs. Then by Theorem 1.3.1(ii) from [**7**], the eigenvalues of $G$ must be integers. Furthermore, by Theorems 1.3.1(v) and 1.3.1(vi) from [**7**], these eigenvalues are

$n - n/d$ with multiplicity 1, 0 with multiplicity $n - d$, and $-n/d$ with multiplicity $d - 1$.

The same conclusion could be obtained also by the following reasoning. Any strongly regular circulant graph is either a Paley graph on $p$ vertices, where $p$ is a prime congruent 1 modulo 4, or a complete multipartite graph or the complement of a complete multipartite graph, see Corollary 2.10.6 in [**7**]. Since the Paley graph on $p$ vertices is not integral, every strongly regular integral circulant graph is either a complete multipartite graph or its complement. The spectra of these graphs are well understood (see, for instance, [**10, 11**]) and it is elementary to establish when these consist of integers.

THEOREM 4.2. *Let $n$ be a positive composite integer and $G = Cir(n, S)$ a (connected) circulant graph with three distinct integer eigenvalues. Then $S = \mathbb{Z}_n^* \setminus M_n(d)$ for some proper divisor $d$ of $n$.*

PROOF. Let $G = Cir(n, S)$ be a circulant graph with three distinct eigenvalues $\lambda_1$, $\lambda_2$, and $\lambda_3$. Without loss of generality, let us take $\lambda_1 > \lambda_2 > \lambda_3$ with respective multiplicities $1, m_1, m_2$. Since $\delta(G) \leqslant \lambda_1 \leqslant \Delta(G)$, it is $\lambda_1 = |S|$. Since $G$ is connected, the multiplicity of $\lambda_1$ is unity [**10, 6**]. By Lemma 4.1, $G$ is strongly regular with parameters $(n, k, a, c)$.

Suppose that $\lambda_2 < 0$. Note that always $\lambda_3 < 0$ and hence $\lambda_2 \lambda_3 = a - k > 0$. Hence $a > k$, which is a contradiction.

Suppose that $\lambda_2 > 0$. By Eq. 4.2, $\lambda_2 \lambda_3 = c - k \neq 0$. By a result of Ahmadi [**1**], $G$ is not isomorphic to $\overline{mK_{k+1}}$. Hence $G$ is not a complete regular $d$-partite graphs and so by Theorem 3.4, $G$ is not a circulant graph, a contradiction to our assumption. Hence $\lambda_2 = 0$. By Eq. 4.2, $\lambda_2 \lambda_3 = 0 = c - k$. By [**1**], $c = k$ if and only if $G \cong \overline{mK_{k+1}}$, for some $m > 1$.

**Case (i)** Suppose that $a = 0$. Then $G$ must be a complete bipartite graph. By Theorem 3.4, $S = \mathbb{Z}_n^* \setminus M_n(2)$.

**Case (ii)** Suppose that $a \neq 0$. Then by Theorem 3.4, $S = \mathbb{Z}_n^* \setminus M_n(d)$ for some $d \in \mathcal{D}$ , $d > 2$. □


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