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A COMMON FIXED POINT THEOREM FOR SIX SELF MAPS IN A MENGER SPACE

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Abstract. In this paper, we claim that the comment of Sastry et.al[2] on a result of Servet Kutukcu and Sushil Sharma[5] is not true by pointing out their non observation of a condition in the hypothesis of the result. Further, the results of [5] are generalized and supported by examples.

1. Introduction

Survet Kutukcu and Sushil Sharma ([**5**]) established results in complete Menger space using compatibility of type(P-1) or of type (P-2). Recently Sastry et.al ([**2**]) claimed that the main result of the above authors is not valid by means of an example. We observed that the space considered in their example is not a Menger space. Further, we generalized the results of [5] and exhibited supporting examples to our claims.

2. Preliminaries

We follow the standard definitions and results given in [**3**]. In fact, we mainly use the following results in the subsequent sections.

Result 2.1 ([3]). Let $\{x_n\}(n = 0, 1, 2, \dots)$ be a sequence in a Menger space $(X, F, *)$, where $*$ is continuous and $x*x \geqslant x$ for all $x \in [0, 1]$. If there is a $k \in (0, 1)$ such that

$$F_{x_n,x_{n+1}}(ku) \geqslant F_{x_{n-1},x_n}(u)$$

for all $u > 0$ and $n \in \mathbb{N}$, then $\{x_n\}$ is a Cauchy sequence in X.

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Result 2.2 ([4]). Let $(X, F, *)$ be a Menger space. If there is a $k \in (0,1)$ such that

$$F_{x,y}(ku) \geqslant F_{x,y}(u)$$

for all $x, y \in X$ and $u > 0$, then $y = x$.

3. Main Results

Now, we state below the Theorem of [**5**].

THEOREM 3.1 ([**5**], Theorem 1). *Let A, B, P, Q, S and T be self mappings on a complete Menger space $(X, F, *)$ with continuous t-norm $*$ such that $u * u \geqslant u$ for all $u \in [0,1]$(i.e, $*$ is the min t-norm), satisfying:*

(3.1.1) *$P(X) \subseteq ST(X)$ and $Q(X) \subseteq AB(X)$,*

(3.1.2) *there is a $k \in (0,1)$ such that*

$$F_{Px,Qy}(ku) \geqslant$$
$$F_{ABx,STy}(u) * F_{Px,ABx}(u) * F_{Qy,STy}(u) * F_{Px,STy}(\alpha u) * F_{Qy,ABx}((2-\alpha)u)$$

for all $x, y \in X$, $u > 0$ and $\alpha \in (0,2)$;

(3.1.3) *either P or AB is continuous;*

(3.1.4) *the pairs $\{P, AB\}$ and $\{Q, ST\}$ are both compatibility of type(P-1) or of type(P-2);*

(3.1.5) *$AB = BA$, $ST = TS$, $PB = BP$, $QT = TQ$.*

Then A, B, P, Q, S and T have a unique common fixed point in X. □

Further, they deduced the following, by taking $A = B = S = T = I_X$(the identity mapping on X) in the above Theorem.

COROLLARY 3.1 ([**5**], Corollary 1). *Let P and Q be self mappings on a complete Menger space $(X, F, *)$ with continuous t-norm $*$ such that $u * u \geqslant u$ for all $u \in [0,1]$(i.e, $*$ is the min t-norm). There is a $k \in (0,1)$ such that*

$$F_{Px,Qy}(ku) \geqslant F_{x,y}(u) * F_{Px,x}(u) * F_{Qy,y}(u) * F_{Px,y}(\alpha u) * F_{Qy,x}((2-\alpha)u)$$

for all $x, y \in X$, $u > 0$ and $\alpha \in (0,2)$.
Then P and Q have a unique common fixed point in X. □

Sastry et.al ([**2**]) claimed that the above Corollary and hence the corresponding above Theorem are not valid in view of the following:

EXAMPLE 3.1. ([**2**], Example 2.3) Let $X = \mathbb{Z}^+$(the set of all positive integers) and the function F be defined as follows; for any $m, n \in \mathbb{Z}^+$,

$$F_{m,n}(u) = \begin{cases} 0 & \text{if } u \leqslant max\{m,n\}, \\ 1 & \text{if } u > max\{m,n\}. \end{cases}$$

P and Q are defined on $\mathbb{Z}^+$ by $P(n) = n + 1$ and $Q(n) = 1$ for all $n \in \mathbb{Z}^+$. And $*$ is as in the statement of the Theorem.

They stated that, even though the conditions given in the Corollary are all satisfied, P and Q have no common fixed point. So they imposed the condition that the Menger space has to be strict(i.e. $F_{x,y}(u)$ is strictly increasing in u). They forgot that primarily $(X, F, *)$ is a Menger space. For that

$$F_{m,m}(u) = 1, \forall u > 0 \ \ and \ \ \forall m \in \mathbb{Z}^{+};$$

but in their example

$$F_{1,1}(u) = 1 \Leftrightarrow u > 1 (but \ \ not \ \ for \ \ all \ \ u > 0).$$

So their $(X, F, *)$ is not a Menger space.

Hence imposing the condition that the Menger space is to be strict need not be necessary for the validity of the Theorem and Corollary under consideration.

REMARK 3.1. In fact, if we take

$$F_{m,n}(u) = \begin{cases} 0 & \text{if } u \leqslant \mid m-n \mid \\ 1 & \text{if } u > \mid m-n \mid, \end{cases}$$

then $(X, F, *)$ is a Menger space.
For the same P and Q, the inequality stated in the Theorem is not satisfied. So, having a common fixed point for P and Q does not arise.

Now, we prove the following generalization of that Theorem & Corollary and consequently our results generalized that of [5] and as well as [2].

THEOREM 3.2. *Let A, B, P, Q, S and T be self mappings on a complete Menger space $(X, F, *)$ where $*$ is the min t-norm and satisfying:*

(3.2.1) $P(X) \subseteq ST(X)$ *and* $Q(X) \subseteq AB(X)$,

(3.2.2) *there is a* $k \in (0, 1)$ *such that*

$$F^{m}_{Px,Qy}(ku) \geqslant$$
$$F^{m}_{ABx,STy}(u) * F^{m}_{Px,ABx}(u) * F^{m}_{Qy,STy}(u) * F_{Px,STy}(\alpha u) * F_{Qy,ABx}((2-\alpha)u)$$

for all $x, y \in X$*, for all* $u > 0$*, for all* $\alpha \in (0, 2)$ *and for some positive integer* m;

(3.2.3) *either* $P(X)$ *or* $Q(X)$ *is a complete subspace of* X;

(3.2.4) *The pairs* $\{Q, ST\}$ *and* $\{P, AB\}$ *are weakly compatible;*

(3.2.5) $ST = TS$ *and* $AB = BA$;

(3.2.6) *"ether* $QT = TQ$ *or* $QS = SQ$*" and "either* $AP = PA$ *or* $BP = PB$*".*

Then A, B, P, Q, S and T have a unique common fixed point in X.

Proof: Let $x_0 \in X$. By (3.2.1) there exist sequences $\{x_n\}$ and $\{y_n\}$ in X such that

$$Px_{2n} = STx_{2n+1} = y_{2n} (say)$$

$$and \ \ Qx_{2n+1} = ABx_{2n+2} = y_{2n+1} (say), for \quad n = 0, 1, 2, \ldots$$

Taking $x = x_{2n}(n \geqslant 1), y = x_{2n+1}, \alpha = 1 - q$ with $q \in (0,1)$ in (3.2.2) and using the properties

$$F_{y_{2n-1},y_{2n+1}}((1+q)u) \geqslant F_{y_{2n-1},y_{2n}}(u) * F_{y_{2n},y_{2n+1}}(qu),$$

$$F_{y_{2n},y_{2n}}(\alpha u) = 1 \ \ and \ \ F_{y_{2n-1},y_{2n}}(u) \geqslant F^m_{y_{2n-1},y_{2n}}(u),$$

we get that

$$F^m_{y_{2n},y_{2n+1}}(ku) \geqslant F^m_{y_{2n-1},y_{2n}}(u) * F^m_{y_{2n},y_{2n+1}}(u) * F_{y_{2n},y_{2n+1}}(qu).$$

As t-norm is continuous and F is left continuous, as $q \to 1 - 0$, we get that

$$F^m_{y_{2n},y_{2n+1}}(ku) \geqslant F^m_{y_{2n-1},y_{2n}}(u) * F^m_{y_{2n},y_{2n+1}}(u)$$

$$\Rightarrow F_{y_{2n},y_{2n+1}}(ku) \geqslant F_{y_{2n-1},y_{2n}}(u) * F_{y_{2n},y_{2n+1}}(u).$$

Similarly, taking $x = x_{2n+2}, y = x_{2n+1}, \alpha = 1 + q$ with $q \in (0,1)$ in (3.2.2), we get that

$$F_{y_{2n+1},y_{2n+2}}(ku) \geqslant F_{y_{2n},y_{2n+1}}(u) * F_{y_{2n+1},y_{2n+2}}(u).$$

Thus for all positive integers n, we have

$$F_{y_n,y_{n+1}}(ku) \geqslant F_{y_{n-1},y_n}(u) * F_{y_n,y_{n+1}}(u).$$

Consequently, $F_{y_n,y_{n+1}}(u) \geqslant F_{y_{n-1},y_n}(k^{-1}u) * F_{y_n,y_{n+1}}(k^{-1}u)$.
By repeated application of the above inequality and using the associative property of the t-norm, we get that

$$F_{y_n,y_{n+1}}(u) \geqslant F_{y_{n-1},y_n}(k^{-1}u) * F_{y_n,y_{n+1}}(k^{-l}u)$$

for any positive integer l.
Since $F_{y_n,y_{n+1}}(k^{-l}u) \to 1$ as $l \to \infty$ (as $k^{-l}u \to \infty$),
we get that

$$F_{y_n,y_{n+1}}(u) \geqslant F_{y_{n-1},y_n}(k^{-1}u)$$

that is $F_{y_n,y_{n+1}}(ku) \geqslant F_{y_{n-1},y_n}(u)$, for all positive integer n.
Now, by Result (2.1), follows that $\{y_n\}$ is a Cauchy sequence in X. So, there is a $z \in X$ such that $y_n \to z$ as $n \to \infty$. Further, the subsequences $\{y_{2n}\}$ and $\{y_{2n+1}\}$ of $\{y_n\}$ are such that

$$y_{2n} = Px_{2n} = STx_{2n+1} \to z$$

$$and \quad y_{2n+1} = Qx_{2n+1} = ABx_{2n+2} \to z, \quad as \quad n \to \infty$$

Case 1: Let $P(X)$ be a complete subspace of X. So, follows that $z \in P(X)$. Since $P(X) \subseteq ST(X)$, there is a $v \in X$ such that $STv = z$.
Taking $x = x_{2n}, y = v$ and $\alpha = 1$ in (3.2.2), we get that

$$F^m_{y_{2n},Qv}(ku) \geqslant F^m_{y_{2n-1},y_{2n}}(u) * F^m_{y_{2n-1},Qv}(u) * F^m_{z,Qv}(u) * F_{y_{2n-1},Qv}(u) * F_{y_{2n-1},z}(u).$$

Now, as $n \to \infty$,we get that

$$F^m_{z,Qv}(ku) \geqslant F^m_{z,z}(u) * F^m_{z,Qv}(u) * F^m_{z,Qv}(u) * F_{z,Qv}(u) * F_{z,Qv}(u) \geqslant F^m_{z,Qv}(u).$$

By Result(2.2), we get that $Qv = z = STv$.
Since $\{Q, ST\}$ is weakly compatible, follows that $QSTv = STQv$; i.e, $Qz = STz$.
Taking $x = x_{2n}, y = z$ and $\alpha = 1$ in (3.2.2), we get that

$$F^m_{y_{2n},Qz}(ku) \geqslant F^m_{y_{2n-1},y_{2n}}(u) * F^m_{y_{2n-1},Qz}(u) * F^m_{Qz,Qz}(u) * F_{y_{2n-1},Qz}(u) * F_{y_{2n-1},Qz}(u).$$

Now, as $n \to \infty$,we get that

$$F^m_{z,Qz}(ku) \geqslant F^m_{z,z}(u) * F^m_{z,Qz}(u) * 1 * F_{z,Qz}(u) * F_{z,Qz}(u) \geqslant F^m_{z,Qz}(u).$$

By the Result(2.2), we get that $Qz = z = STz$.

Suppose $QT = TQ$. So, we have $QTz = TQz = Tz$ and since $ST = TS$, $STTz = TSTz = Tz$.

Taking $x = x_{2n}, y = Tz$ and $\alpha = 1$ in (3.2.2), we get that

$$F^m_{y_{2n},QTz}(ku) \geqslant$$
$$F^m_{y_{2n-1},STTz}(u) * F^m_{y_{2n},y_{2n-1}}(u) * F^m_{QTz,STTz}(u) * F_{y_{2n},STTz}(u) * F_{QTz,y_{2n-1}}(u)$$

i.e,

$$F^m_{y_{2n},Tz}(ku) \geqslant F^m_{y_{2n-1},Tz}(u) * F^m_{y_{2n},y_{2n-1}}(u) * F^m_{Tz,Tz}(u) * F_{y_{2n},Tz}(u) * F_{Tz,y_{2n-1}}(u).$$

Now, as $n \to \infty$,we get that

$$F^m_{z,Tz}(ku) \geqslant F^m_{z,Tz}(u) * F^m_{z,z}(u) * F^m_{Tz,Tz}(u) * F_{z,Tz}(u) * F_{Tz,z}(u) \geqslant F^m_{z,Tz}(u).$$

By the Result(2.2), we get that $Tz = z$. $\Rightarrow Sz = z$. Thus $Qz = Sz = Tz = z$.

Similarly, if $QS = SQ$; taking $x = x_{2n}, y = Sz$ and $\alpha = 1$ in (3.2.2) and using the property $ST = TS$, we first get that $Sz = z$ and then $Tz = z$. Thus $Qz = Sz = Tz = z$.

Since $Q(X) \subseteq AB(X)$, there is a $w \in X$ such that $ABw = z$.

Taking $x = w, y = x_{2n+1}$ and $\alpha = 1$ in (3.2.2), we get that

$$F^m_{Pw,y_{2n+1}}(ku) \geqslant F^m_{z,y_{2n}}(u) * F^m_{Pw,z}(u) * F^m_{y_{2n+1},y_{2n}}(u) * F_{Pw,y_{2n}}(u) * F_{y_{2n+1},z}(u).$$

Now, as $n \to \infty$,we get that

$$F^m_{Pw,z}(ku) \geqslant F^m_{z,z}(u) * F^m_{Pw,z}(u) * F^m_{z,z}(u) * F_{Pw,z}(u) * F_{z,z}(u) \geqslant F^m_{Pw,z}(u).$$

Follows $F_{Pw,z}(ku) \geqslant F_{Pw,z}(u)$ and this is true for all $u > 0$.
Hence, by Result (2.2), $Pw = z$. Thus $ABw = Pw = z$.
Since $\{P, AB\}$ is weakly compatible, $ABPw = PABw$; i.e,$ABz = Pz$.

Similarly, by taking $x = z, y = x_{2n+1}$ and $\alpha = 1$ in (3.2.2),we get that $F_{Pz,z}(ku) \geqslant F_{Pz,z}(u)$ and this is true for all $u > 0$. $\Rightarrow Pz = z$. Thus $ABz = Pz = z$.

Suppose that $AP = PA$. Since $AB = BA$, we have $PAz = APz = Az$ and $ABAz = AABz = Az$.

Taking $x = Az, y = x_{2n+1}$ and $\alpha = 1$ in (3.2.2), we get that

$$F^m_{Az,y_{2n+1}}(ku) \geqslant F^m_{Az,y_{2n}}(u) * F^m_{Az,Az}(u) * F^m_{y_{2n+1},y_{2n}}(u) * F_{Az,y_{2n}}(u) * F_{y_{2n+1},Az}(u).$$

Now, as $n \to \infty$,we get that

$$F^m_{Az,z}(ku) \geqslant F^m_{Az,z}(u) * F^m_{Az,Az}(u) * F^m_{z,z}(u) * F_{Az,z}(u) * F_{z,Az}(u) \geqslant F^m_{Az,z}(u).$$

$\Rightarrow F_{Az,z}(ku) \geqslant F_{Az,z}(u)$ and this is true for all $u > 0$. So $Az = z$.
Since $AB = BA$ and $ABz = z$, follow that $Bz = z$. Thus $Az = Bz = Pz = z$.

Similarly, if $BP = PB$; taking $x = Bz, y = x_{2n+1}$ and $\alpha = 1$ in (3.2.2) and using the property $AB = BA$, we first get that $Bz = z$ then $Az = z$. Thus $Az = Bz = Pz = z$. Hence, $Az = Bz = Pz = Qz = Sz = Tz = z$.

Case 2: Suppose $Q(X)$ is complete subspace of X.

As in case 1, we first get that $Az = Bz = Pz = z$ and then $Qz = Sz = Tz = z$. Hence,$Az = Bz = Pz = Qz = Sz = Tz = z$.

Uniqueness follows from the fact that, if $z^{'}$ is another common fixed point, then we get that $F_{z,z'}(ku) \geqslant F_{z,z'}(u)$ is true for all $u > 0$ and this is a contradiction.

Now, taking $A = B = S = T = I_X$ in the above Theorem, we have the following:

Corollary 3.2. *Let P, Q be self mappings on a complete Menger space $(X, F, *)$ where $*$ is a continuous triangular norm with $u * u \geqslant u$ for all $u \in [0, 1]$, satisfying:*

(3.2.i) *there is a constant $k \in (0, 1)$ such that*

$$F^m_{Px,Qy}(ku) \geqslant F^m_{x,y}(u) * F^m_{Px,x}(u) * F^m_{Qy,y}(u) * F_{Px,y}(\alpha u) * F_{Qy,x}((2 - \alpha)u)$$

for all $x, y \in X$, for all $u > 0$, for all $\alpha \in (0, 2)$ and for some positive integer m, and

(3.2.ii) *either $P(X)$ or $Q(X)$ is a complete subspace of X.*

Then P and Q have a unique common fixed point in X.

K.P.R.Sastry et.al ([**2**]) proved the following:

Theorem 3.3 ([**2**], Theorem 2.6). *Let P, Q, R and C be self mappings on a complete Menger space $(X, F, *)$ where $*$ is the min t-norm, satisfying:*

(3.3.1) *$P(X) \subseteq R(X)$ and $Q(X) \subseteq C(X)$,*

(3.3.2) *there is a $k \in (0, 1)$ such that*

$$F_{Px,Qy}(ku) \geqslant F_{Cx,Ry}(u) * F_{Px,Cx}(u) * F_{Qy,Ry}(u) * F_{Px,Ry}(2u) * F_{Qy,Cx}(2u)$$

for all $x, y \in X$, for all $u > 0$, for all $\alpha \in (0, 2)$.

(3.3.3) *either P or C is continuous;*

(3.3.4) *The pairs $\{P, C\}$ and $\{Q, R\}$ are both compatible of type (P_1) or type (P_2)*

Then P, Q, R and C have a unique common fixed point in X.

Corollary 3.3 ([**2**], Corollary 2.7). *Let A, B, P, Q, S and T be self mappings of a complete strict Menger space $(X, F, *)$ where $*$ is the min t-norm, satisfying:*

(3.3.i) *$P(X) \subseteq ST(X)$ and $Q(X) \subseteq AB(X)$,*

(3.3.ii) *there is a $k \in (0, 1)$ such that*

$$F_{Px,Qy}(ku) \geqslant F_{ABx,STy}(u) * F_{Px,ABx}(u) * F_{Qy,STy}(u) * F_{Px,STy}(2u) * F_{Qy,ABx}(2u)$$

for all $x, y \in X$, $u > 0$

(3.3.iii) *either P or AB is continuous;*

(3.3.iv) *The pairs $\{P, ST\}$ and $\{Q, AB\}$ are both compatible of type (P_1) or type (P_2)*

(3.3.v) *$AB = BA$, $ST = TS$,$PB = BP$ and $TQ = QT$.*

Then A, B,P, Q, S and T have a unique common fixed point in X.

REMARK 3.2. (1.a) The inequality (3.1.2) of Theorem 3.1 is obtained by taking $m = 1$ in our Theorem 3.2.

(1.b) The condition (3.1.3) of Theorem 3.1 is replaced by a realistic one in our Theorem 3.2. In fact, the result can also be obtained if our condition is replaced by their condition.

(1.c) The condition (3.1.4) of Theorem 3.1 is replaced by weaker conditions given in (3.2.5) and (3.2.6) in our Theorem 3.2.
Hence, we can regard that our Theorem 3.2 is a generalization of their Theorem [2].

(2) As the R.H.S in the inequality given in Theorem 3.2 of (3.2.2)(with m=1) $\leqslant$ the R.H.S of the inequality given in Corollary 3.3(Corollary 2.7 of [2]), follow that our Theorem is generalization of the above Corollary 3.3(Corollary 2.7 of [2]).

(3) Denoting $R = ST$ and $C = AB$ in our Theorem 3.2., we can conclude that our Theorem 3.2 is a generalization of Theorem 3.3(Theorem 2.6 of [2]).

(4) In Theorem 3.3(Theorem 2.6 of [2]), they assumed that either P or C is continuous. In example 3.2, we show that the result in valid even if they are not continuous.

(5) As our Theorem is valid in a complete Menger space, it solves the open problem 2.11 of [2]. "Is the Theorem valid if (X, F, *) is not necessary strict?". (We proved in the affirmative).

We conclude our paper, with the following examples in support of our Theorem.

EXAMPLE 3.2. $(X, F, *)$ is a Menger space, where $X = [0, 10)$ with the usual metric and $F : \mathbb{R} \to [0, 1]$ is defined by

$$F_{x,y}(u) = \frac{u}{u + |x - y|}$$

for all $x, y \in \mathbb{R}$, $u > 0$ and $*$ is the min t-norm, i.e, $a * b = min\{a, b\}$ for all $a, b \in [0, 1]$.

Let A, B, P, Q, S and T be the self maps on X, defined by

$$A(x) = \begin{cases} 0 & \text{if } x \leqslant 9, \\ x^{\frac{1}{2}} & \text{if } x > 9. \end{cases}$$

$$P(x) = \begin{cases} 0 & \text{if } x \leqslant 9, \\ 1 & \text{if } x > 9. \end{cases}$$

$Bx = Sx = Tx = x$ and $Qx = 0$, for all $x \in X$.

Then, clearly A, B, P, Q, S and T satisfy the hypothesis of Theorem(3.2) with $k \in [\frac{1}{2}, 1) \subset (0, 1)$.

For, when $x > 9$,

$$F^m_{Px,Qy}(ku) = \Big(\frac{ku}{ku + 1}\Big)^m = \Big(\frac{u}{u + \frac{1}{k}}\Big)^m$$

and

$$F^m_{ABx,Px}(u) = \Big(\frac{u}{u + (x^{\frac{1}{2}} - 1)}\Big)^m < \Big(\frac{u}{u+2}\Big)^m.$$

So, in (3.2.2), L.H.S $\geqslant$ R.H.S when $\frac{1}{k} \leqslant 2$, that is $k \geqslant \frac{1}{2}$.
Clearly 0 is the unique common fixed point of A, B, P, Q, S and T.

EXAMPLE 3.3. $(X, F, *)$ is a Menger space, where $X = [0, 10)$ with the usual metric and $F : \mathbb{R} \to [0, 1]$ is defined by

$$F_{x,y}(u) = \frac{u}{u + |x - y|}$$

for all $x, y \in \mathbb{R}$, $u > 0$ and $*$ is the min t-norm, i.e, $a * b = min\{a, b\}$ for all $a, b \in [0, 1]$.
Let A, B, P, Q, S and T be the self maps on X, defined by

$$P(x) = \begin{cases} 0 & \text{if } x \leqslant 9, \\ 1 & \text{if } x > 9. \end{cases}$$

$Ax = x^{\frac{1}{2}}$, $Bx = Sx = Tx = x$ and $Qx = 0$ for all $x \in X$.
Then, clearly A, B, P, Q, S and T satisfy the hypothesis of Theorem(3.2) with $k \in [\frac{1}{2}, 1) \subset (0, 1)$.
For, when $x > 9$,

$$F^m_{ABx,STy}(ku) = \Big(\frac{ku}{ku + 1}\Big)^m = \Big(\frac{u}{u + \frac{1}{k}}\Big)^m$$

and

$$F^m_{ABx,Lx}(u) = \Big(\frac{u}{u + (x^{\frac{1}{2}} - 1)}\Big)^m < \Big(\frac{u}{u+2}\Big)^m.$$

So, in (3.2.2), L.H.S $\geqslant$ R.H.S when $\frac{1}{k} \leqslant 2$, that is $k \geqslant \frac{1}{2}$.
Clearly 0 is the unique common fixed point of A, B, P, Q, S and T.

EXAMPLE 3.4. $(\mathbb{R}, F, *)$ is a Menger space, where $\mathbb{R}$ is the set of all reals with usual metric and F is as in example (3.1). Let A, B, P, Q, S and T be self maps on $\mathbb{R}$ defined by

$$P(x) = \begin{cases} 0 & \text{if } x \leqslant 2, \\ 1 & \text{if } x > 2. \end{cases}$$

$$A(x) = \begin{cases} 0 & \text{if } x \leqslant 2, \\ x^2 & \text{if } x > 2. \end{cases}$$

$Qx = 0$, $Bx = Tx = x$ and $Sx = x^3$ for all $x \in \mathbb{R}$. It can be shows that $k \in [\frac{1}{2}, 1) \subset (0, 1)$ serves the purpose and 0 is the unique common fixed point.

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