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A new Extension of Gegenbauer Matrix Polynomials and Their Properties

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Abstract. The aim of this paper is to define and study of the Gegenbauer matrix polynomials of two variables. An explicit representation, a three-term matrix recurrence relations, differential recurrence relations and hypergeometric matrix representation for the Gegenbauer matrix polynomials of two variables are given. The Gegenbauer matrix polynomials are solutions of the matrix differential equations and expansion of the Gegenbauer matrix polynomials as series of Hermite and Laguerre matrix polynomials of two variables are established.

1. Introduction

An extension to the matrix framework of the classical families of Hermite, Jacobi, Gegenbauer, Laguerre and Chebyshev matrix polynomials was introduced and studied in a number of previous papers [**1, 2, 3, 4, 5, 8, 10, 15, 17**] for matrix in $\mathbb{C}^{N\times N}$. Moreover, some properties of the Hermite matrix polynomials are given [**12, 18**] and a generalized form of the Hermite matrix polynomials has been introduced and studied in [**12, 13, 14, 19**]. Jódar and Cortés introduced and studied the hypergeometric matrix function and the hypergeometric matrix differential equation in [**6**] and the explicit closed form general solution of it has been given in [**7**]. Sayyed et al. introduced and studied the Gegenbauer matrix polynomials and second order matrix differential equation in [**9, 11, 19**].

The main goal of this paper is to consider a new system of matrix polynomials, namely the Gegenbauer matrix polynomials of two variables. The paper is organized as follows: In Section 2 a definition of Gegenbauer matrix polynomials of two variables are given. Some differential recurrence relations, in particular

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Gegenbauer's matrix differential equation are established in Section 3. Moreover, hypergeometric matrix representations of these polynomials are given in Section 4. Finally, in Section 5 an expansion of the Gegenbauer's as series of Hermite and Laguerre matrix polynomials are obtained.

Throughout this paper D_0 denotes the complex plane cut along the negative real axis and its spectrum $\sigma(A)$ denotes the set of all the eigenvalues of A. If A is a matrix in $\mathbb{C}^{N\times N}$, its two-norm denoted by $||A||_2$ is defined by

$$||A||_2 = \sup_{x\neq 0} \frac{||Ax||_2}{||x||_2}$$

where for a vector y in $\mathbb{C}^N$, $||y||_2$ denotes the Euclidean norm of y, $||y||_2 = (y^T y)^{\frac{1}{2}}$. The set of all the eigenvalues of A is denoted by $\sigma(A)$. If $f(z)$ and $g(z)$ are holomorphic functions of the complex variable z, which are defined in an open set Ω of the complex plane, and if A is a matrix in $\mathbb{C}^{N\times N}$ such that $\sigma(A)\subset\Omega$, then from the properties of the matrix functional calculus [**2, 8**], it follows that

$$f(A)g(A) = g(A)f(A).$$

If A is a matrix in $\mathbb{C}^{N\times N}$ with $\sigma(A)\subset D_0$, then $A^{\frac{1}{2}} = \sqrt{A} = \exp(\frac{1}{2}\log(A))$ denotes the image by $z^{\frac{1}{2}} = \sqrt{z} = \exp(\frac{1}{2}\log(z))$ of the matrix functional calculus acting on the matrix A. Let A is a matrix in $\mathbb{C}^{N\times N}$ such that

$$Re(z) > 0, \quad \text{for all } z\in\sigma(A). \tag{1.1}$$

The reciprocal gamma function denoted by $\Gamma^{-1}(z) = \frac{1}{\Gamma(z)}$ is an entire function of the complex variable z. Then for any matrix A in $\mathbb{C}^{N\times N}$, the image of $\Gamma^{-1}(z)$ acting on A denoted by $\Gamma^{-1}(A)$ is a well-defined matrix. Then $\Gamma(A)$ is invertible, its inverse coincides with $\Gamma^{-1}(A)$ and one gets the formula [**16**]

$$\begin{aligned}(A)_n = A(A+I)...(A+(n-1)I) = \Gamma(A+nI)\Gamma^{-1}(A);\\ n\geqslant 1;\quad (A)_0 = I.\end{aligned} \tag{1.2}$$

From (1.2), it is easy to find that

$$(A)_{n-k} = (-1)^k (A)_n [(I-A-nI)_k]^{-1};\quad 0\leqslant k\leqslant n. \tag{1.3}$$

From the relation (1.3) of [**9, 19, 15**], one obtains

$$\frac{(-1)^k}{(n-k)!} I = \frac{(-n)_k}{n!} I = \frac{(-nI)_k}{n!};\quad 0\leqslant k\leqslant n. \tag{1.4}$$

The hypergeometric function $F(A,B;C;z)$ has been given in the form [**16**]

$${}_2F_1(A,B;C;z) = \sum_{k=0}^{\infty} \frac{(A)_k (B)_k [(C)_k]^{-1}}{n!} z^k \tag{1.5}$$

for matrices A, B and C in $\mathbb{C}^{N\times N}$ such that $C+nI$ is invertible for all integer $n\geqslant 0$. We will exploit the following relation due to [**6**]

$$(1-z)^{-A} = {}_1F_0(A;-;z) = \sum_{n=0}^{\infty} \frac{1}{n!}(A)_n z^n;\quad |z|<1. \tag{1.6}$$

It has been seen by Defez and Jódar [**2**] that, for matrices $A(k,n)$ and $B(k,n)$ are matrices in $\mathbb{C}^{N\times N}$ for $n \geqslant 0$, $k \geqslant 0$, the following relations are satisfied

$$\sum_{n=0}^{\infty}\sum_{k=0}^{\infty} A(k,n) = \sum_{n=0}^{\infty}\sum_{k=0}^{[\frac{1}{2}n]} A(k,n-2k) \tag{1.7}$$

and

$$\sum_{n=0}^{\infty}\sum_{k=0}^{\infty} B(k,n) = \sum_{n=0}^{\infty}\sum_{k=0}^{n} B(k,n-k). \tag{1.8}$$

Similarly, we can write

$$\sum_{n=0}^{\infty}\sum_{k=0}^{[\frac{1}{2}n]} A(k,n) = \sum_{n=0}^{\infty}\sum_{k=0}^{\infty} A(k,n+2k), \tag{1.9}$$

$$\sum_{n=0}^{\infty}\sum_{k=0}^{n} A(k,n) = \sum_{n=0}^{\infty}\sum_{k=0}^{[\frac{1}{2}n]} A(k,n-k) \tag{1.10}$$

and

$$\sum_{n=0}^{\infty}\sum_{k=0}^{n} B(k,n) = \sum_{n=0}^{\infty}\sum_{k=0}^{\infty} B(k,n+k). \tag{1.11}$$

2. Gegenbauer matrix polynomials of two variables

Let A be a positive stable matrix in $\mathbb{C}^{N\times N}$. We define the Gegenbauer matrix polynomials of two variables by the relation

$$\begin{aligned} F(x,y,t,A) = (1-2xt+yt^2)^{-A} &= \sum_{n=0}^{\infty} C_n^A(x,y)t^n; \\ &|x| \leqslant 1, |y| \leqslant 1, |t| < 1. \end{aligned} \tag{2.1}$$

By using (1.6) and (1.10), we have

$$\begin{aligned} (1-2xt+yt^2)^{-A} &= \sum_{n=0}^{\infty} \frac{(A)_n(2x-yt)^n t^n}{n!} \\ &= \sum_{n=0}^{\infty}\sum_{k=0}^{n} \frac{(A)_n(-1)^k y^k (2x)^{n-k}}{k!(n-k)!}\, t^{n+k} \\ &= \sum_{n=0}^{\infty}\sum_{k=0}^{[\frac{1}{2}n]} \frac{(-1)^k y^k (2x)^{n-2k}(A)_{n-k}}{k!(n-2k)!}\, t^n. \end{aligned} \tag{2.2}$$

By equating the coefficients of t^n in (2.1) and (2.2), we obtain an explicit representation of the Gegenbauer matrix polynomials in the form

$$C_n^A(x,y) = \sum_{k=0}^{[\frac{1}{2}n]} \frac{(-1)^k y^k (2x)^{n-2k}}{k!(n-2k)!}(A)_{n-k}. \tag{2.3}$$

It is clear that

$$C_{-1}^A(x,y) = 0,\ \ C_0^A(x,y) = I,\ \ C_1^A(x,y) = 2xA,$$
$$C_2^A(x,y) = 2x^2A(A+I) - yA \ \text{ and } \ C_n^A(x,0) = \frac{(2x)^n}{n!}(A)_n.$$

It has already been shown that most of the properties of the $C_n^A(x,y)$ matrix polynomials, linked to the ordinary case by

$$C_n^A(x,y) = \quad y^{\frac{n}{2}} C_n^A(\frac{x}{\sqrt{y}}). \tag{2.4}$$

Clearly, $C_n^A(x,y)$ is a matrix polynomial of degree n in x. Replacing x by $-x$ and t by $-t$ in (2.1), the left side remains unchanged, we obtain

$$C_n^A(-x,y) = (-1)^n C_n^A(x,y). \tag{2.5}$$

For $x = 1$ and $y = 1$, we have

$$(1-t)^{-2A} = \sum_{n=0}^{\infty} t^n C_n^A(1,1); |t| < 1.$$

By (1.6) to obtain

$$C_n^A(1,1) = \frac{1}{n!}(2A)_n. \tag{2.6}$$

For $x = 0$, it follows

$$(1+yt^2)^{-A} = \sum_{n=0}^{\infty} t^n C_n^A(0,y).$$

Also, by (1.6) one gets

$$(1+yt^2)^{-A} = \sum_{n=0}^{\infty} \frac{(-1)^n}{n!} y^n t^{2n} (A)_n; |yt^2| < 1.$$

Therefore, we have

$$C_{2n}^A(0,y) = \frac{(-1)^n}{n!} y^n (A)_n, C_{2n+1}^A(0,y) = 0. \tag{2.7}$$

3. Matrix Differential recurrence relations

By differentiating (2.1) with respect to x, y and t yields respectively

$$\frac{\partial}{\partial x}F(x,y,t,A)=\frac{t}{1-2xt+yt^2}2A\;F(x,y,t,A), \tag{3.1}$$

$$\frac{\partial}{\partial y}F(x,y,t,A)=\frac{-t^2}{1-2xt+yt^2}A\;F(x,y,t,A) \tag{3.2}$$

and

$$\frac{\partial}{\partial t}F(x,y,t,A)=\frac{x-yt}{1-2xt+yt^2}2A\;F(x,y,t,A). \tag{3.3}$$

So that the matrix function F satisfies the partial matrix differential equation

$$(x-yt)\frac{\partial}{\partial x}F(x,y,t,A)-t\frac{\partial}{\partial t}F(x,y,t,A)=0.$$

Therefore, by (2.1), we get

$$\sum_{n=0}^{\infty}x\frac{\partial}{\partial x}C_n^A(x,y)t^n-\sum_{n=0}^{\infty}nC_n^A(x,y)t^n=\sum_{n=1}^{\infty}y\frac{\partial}{\partial x}C_{n-1}^A(x,y)t^n.$$

Since $\frac{\partial}{\partial x}C_0^A(x,y)=0$, we obtain the matrix differential recurrence relation

$$x\frac{\partial}{\partial x}C_n^A(x,y)-nC_n^A(x,y)=y\frac{\partial}{\partial x}C_{n-1}^A(x,y);n\geqslant 1. \tag{3.4}$$

From (3.1) and (3.3) with the aid of (2.1), we get respectively the following

$$\frac{2A}{1-2xt+yt^2}(1-2xt+yt^2)^{-A}=\sum_{n=1}^{\infty}\frac{\partial}{\partial x}C_n^A(x,y)t^{n-1} \tag{3.5}$$

and

$$\frac{2(x-yt)A}{1-2xt+yt^2}(1-2xt+yt^2)^{-A}=\sum_{n=1}^{\infty}nC_n^A(x,y)t^{n-1}. \tag{3.6}$$

Note that $1-yt^2-2t(x-yt)=1-2xt+yt^2$. Thus by multiplying (3.5) by $1-yt^2$ and (3.6) by $2t$ and subtracting (3.6) from (3.4), we obtain

$$2(A+nI)C_n^A(x,y)=\frac{\partial}{\partial x}C_{n+1}^A(x,y)-y\frac{\partial}{\partial x}C_{n-1}^A(x,y). \tag{3.7}$$

From (3.4) and (3.7), one gets

$$x\frac{\partial}{\partial x}C_n^A(x,y)=\frac{\partial}{\partial x}C_{n+1}^A(x,y)-(2A+nI)C_n^A(x,y). \tag{3.8}$$

Substituting $n-1$ for n in (3.8) and putting the resulting expression for

$$\frac{\partial}{\partial x}C_{n-1}^A(x,y)$$

into (3.4), gives

$$(x^2-y)\frac{\partial}{\partial x}C_n^A(x,y)=nxC_n^A(x,y)-(2A+(n-1)I)yC_{n-1}^A(x,y). \tag{3.9}$$

Now, by multiplying (3.4) by (x^2-y) and substituting for $(x^2-y)\frac{\partial}{\partial x}C_n^A(x,y)$ and $(x^2-y)\frac{\partial}{\partial x}C_{n-1}^A(x,y)$ from (3.9) to obtain the three terms matrix recurrence relation in the form

$$\begin{aligned} nC_n^A(x,y) &= \\ 2x(A+(n-1)I)C_{n-1}^A(x,y) &- y(2A+(n-2)I)C_{n-2}^A(x,y). \end{aligned} \tag{3.10}$$

By the same way, we get

$$\begin{aligned} (n-1)C_{n-1}^A(x,y) &= 2x\frac{\partial}{\partial y}C_n^A(x,y) - 2y\frac{\partial}{\partial y}C_{n-1}^A(x,y), \\ 2(A+nI)C_n^A(x,y) &= y\frac{\partial}{\partial y}C_n^A(x,y) - \frac{\partial}{\partial y}C_{n+2}^A(x,y), \\ (4A+5(n-1)I)C_{n-1}^A(x,y) &= 2x\frac{\partial}{\partial y}C_n^A(x,y) - 2\frac{\partial}{\partial y}C_{n+1}^A(x,y). \end{aligned} \tag{3.11}$$

Formulas (3.4), (3.7), (3.8), (3.9) (3.10) and (3.11) are called the matrix recurrence formulas for Gegenbauer matrix polynomials.

In the following theorem, we obtain the properties Gegenbauer matrix polynomials as follows.

THEOREM 3.1. *The Gegenbauer matrix polynomials satisfying the following relations*

$$\frac{\partial^r}{\partial x^r}C_n^A(x,y) + (-1)^{r-1}2^r\frac{\partial^r}{\partial y^r}C_{n+r}^A(x,y) = 0. \tag{3.12}$$

Proof. Differentiating the identity (2.1) with respect to x and y, we get

$$2tA(1-2xt+yt^2)^{-(A+I)} = \sum_{n=0}^{\infty}\frac{\partial}{\partial x}C_n^A(x,y)t^n \tag{3.13}$$

and

$$-t^2A(1-2xt+yt^2)^{-(A+I)} = \sum_{n=0}^{\infty}\frac{\partial}{\partial y}C_n^A(x,y)t^n. \tag{3.14}$$

Iteration (3.13) and (3.14), for $0 \leqslant r \leqslant n$, implies (3.12) and the proof of Theorem **3.1** is completed. □

We can write (3.13) and (3.14) in the form

$$\begin{aligned} 2A(1-2xt+yt^2)^{-(A+I)} &= \sum_{n=1}^{\infty}\frac{\partial}{\partial x}C_n^A(x,y)t^{n-1} = \sum_{n=0}^{\infty}\frac{\partial}{\partial x}C_{n+1}^A(x,y)t^n, \\ -A(1-2xt+yt^2)^{-(A+I)} &= \sum_{n=2}^{\infty}\frac{\partial}{\partial y}C_n^A(x,y)t^{n-2} = \sum_{n=0}^{\infty}\frac{\partial}{\partial y}C_{n+2}^A(x,y)t^n. \end{aligned} \tag{3.15}$$

By applying (2.1), it follows

$$\begin{aligned} 2A(1-2xt+yt^2)^{-(A+I)} &= \sum_{n=0}^{\infty} 2A\, C_n^{A+I}(x,y)t^n, \\ -A(1-2xt+yt^2)^{-(A+I)} &= -\sum_{n=0}^{\infty} A\, C_n^{A+I}(x,y)t^n. \end{aligned} \tag{3.16}$$

Identification of the coefficients of t^n in (3.15) and (3.16) yields

$$\begin{aligned} \frac{\partial}{\partial x} C_{n+1}^{A}(x,y) &= 2A\, C_n^{A+I}(x,y), \\ \frac{\partial}{\partial y} C_{n+2}^{A}(x,y) &= -A\, C_n^{A+I}(x,y). \end{aligned}$$

which gives

$$\begin{aligned} \frac{\partial}{\partial x} C_{n}^{A}(x,y) &= 2A\, C_{n-1}^{A+I}(x,y), \\ \frac{\partial}{\partial y} C_{n}^{A}(x,y) &= -A\, C_{n-2}^{A+I}(x,y). \end{aligned} \tag{3.17}$$

Iteration (3.17) yields, for $0 \leqslant r \leqslant n$;

$$\begin{aligned} \frac{\partial^r}{\partial x^r} C_{n}^{A}(x,y) &= 2^r (A)_r\, C_{n-r}^{A+rI}(x,y), \\ \frac{\partial^r}{\partial y^r} C_{n}^{A}(x,y) &= (-1)^r (A)_r\, C_{n-2r}^{A+rI}(x,y). \end{aligned} \tag{3.18}$$

Now, we can state and prove the following theorem.

Theorem 3.2. *The Gegenbauer's matrix polynomials are solutions of the matrix partial differential equations in the form*

$$\begin{aligned} (y-x^2)\frac{\partial^2}{\partial x^2} C_n^A(x,y) - x(2A+I)\frac{\partial}{\partial x} C_n^A(x,y) + \\ n(2A+nI)C_n^A(x,y) = 0. \end{aligned} \tag{3.19}$$

Proof. In (3.8), replace n by $n-1$ and differentiate with respect to x to find

$$x\frac{\partial^2}{\partial x^2} C_{n-1}^A(x,y) = \frac{\partial^2}{\partial x^2} C_n^A(x,y) - (2A+nI)\frac{\partial}{\partial x} C_{n-1}^A(x,y). \tag{3.20}$$

Also, by differentiating (3.4) with respect to x, we have

$$x\frac{\partial^2}{\partial x^2} C_n^A(x,y) - (n-1)\frac{\partial}{\partial x} C_n^A(x,y) = y\frac{\partial^2}{\partial x^2} C_{n-1}^A(x,y). \tag{3.21}$$

From (3.4) and (3.21) by putting $\frac{\partial}{\partial x} C_{n-1}^A(x,y)$ and $\frac{\partial^2}{\partial x^2} C_n^A(x,y)$ into (3.20) and rearrangement of terms in the above equation gives us Gegenbauer's matrix differential equation for Gegenbauer's matrix polynomials in the form (3.19) and hence the proof of Theorem. $\square$

4. Hypergeometric matrix representations of $C_n^A(x,y)$

From the relation (1.4) of [**9, 19**], one obtains

$$\frac{1}{(n-2k)!}I = \frac{(-n)_{2k}}{n!}I = \frac{(-nI)_{2k}}{n!}; \quad 0 \leqslant 2k \leqslant n. \tag{4.1}$$

By using (1.3) and taking into account that

$$(-nI)_{2k} = 2^{2k}(-\frac{1}{2}nI)_k(-\frac{1}{2}(n-1)I)_k \tag{4.2}$$

the explicit representation (2.3) becomes

$$\begin{aligned} C_n^A(x,y) &= \sum_{k=0}^{[\frac{1}{2}n]} \frac{(-1)^k (A)_{n-k} y^k (2x)^{n-2k}}{k!(n-2k)!} \\ &= \sum_{k=0}^{[\frac{1}{2}n]} \frac{(-1)^k (A)_{n-k} (-nI)_{2k} y^k (2x)^{n-2k}}{k!n!} \\ &= \sum_{k=0}^{[\frac{1}{2}n]} \frac{(A)_n[(I-A-nI)_k]^{-1}(-nI)_{2k} y^k (2x)^{n-2k}}{k!n!} \\ &= \sum_{k=0}^{[\frac{1}{2}n]} \frac{2^{2k}(A)_n(-\frac{1}{2}nI)_k(-\frac{1}{2}(n-1)I)_k[(I-A-nI)_k]^{-1} y^k (2x)^{n-2k}}{k!n!} \\ &= \frac{(2x)^n}{n!}(A)_n \sum_{k=0}^{[\frac{1}{2}n]} \frac{2^{+2k}(-\frac{1}{2}nI)_k(-\frac{1}{2}(n-1)I)_k[(I-A-nI)_k]^{-1} y^k (2x)^{-2k}}{k!} \\ &= \frac{(2x)^n}{n!}(A)_n \; {}_2F_1(-\frac{1}{2}nI, -\frac{1}{2}(1-n)I; I-A-nI; \frac{y}{x^2}) \end{aligned} \tag{4.3}$$

which gives another hypergeometric matrix representation in the form:

$$C_n^A(x,y) = \frac{(2x)^n}{n!}(A)_n \; {}_2F_1(-\frac{1}{2}nI, -\frac{1}{2}(1-n)I; I-A-nI; \frac{y}{x^2}) \tag{4.4}$$

where the hypergeometric matrix function ${}_2F_1(\ldots,\ldots;\ldots;\ldots)$ is given as

$$\begin{aligned} &{}_2F_1(-\frac{1}{2}nI, -\frac{1}{2}(1-n)I; I-A-nI; \frac{y}{x^2}) = \\ &\sum_{k=0}^{[\frac{1}{2}n]} \frac{(-\frac{1}{2}nI)_k(-\frac{1}{2}(n-1)I)_k[(I-A-nI)_k]^{-1} y^k x^{-2k}}{k!} \end{aligned}$$

such that $I - A - nI + kI$ is invertible for all $n - k \leqslant 1$.

Note that, we can write

$$(1-2xt+yt^2)^{-A} = \left(1 - \frac{(x^2-y)t^2}{(1-xt)^2}\right)^{-A}(1-xt)^{-2A}. \tag{4.5}$$

Therefore, by using (1.6) and (1.7) we have that

$$\begin{aligned}\sum_{n=0}^{\infty} C_n^A(x,y)t^n &= \sum_{k=0}^{\infty} \frac{(A)_k(x^2-y)^k t^{2k}}{k!}(1-xt)^{-2A-2kI}\\ &= \sum_{n=0}^{\infty}\sum_{k=0}^{\infty} \frac{(A)_k(A)_{n+2k}[(2A)_{2k}]^{-1}x^n(x^2-y)^k t^{n+2k}}{k!n!}\\ &= \sum_{n=0}^{\infty}\sum_{k=0}^{\infty} \frac{(2A)_{n+2k}[(A+\frac{1}{2}I)_k]^{-1}x^n(x^2-y)^k t^{n+2k}}{k!n!2^{2k}}\\ &= \sum_{n=0}^{\infty}\sum_{k=0}^{[\frac{1}{2}n]} \frac{(2A)_n[(A+\frac{1}{2}I)_k]^{-1}x^{n-2k}(x^2-y)^k t^{n}}{k!(n-2k)!2^{2k}}.\end{aligned} \tag{4.6}$$

By identification of the coefficients of t^n, another form for the Gegenbauer matrix polynomials follows

$$C_n^A(x,y) = \sum_{k=0}^{[\frac{1}{2}n]} \frac{(2A)_n[(A+\frac{1}{2}I)_k]^{-1}x^{n-2k}(x^2-y)^k}{k!(n-2k)!2^{2k}}. \tag{4.7}$$

Equation (4.7) yields

$$\begin{aligned}\sum_{n=0}^{\infty}[(2A)_n]^{-1}C_n^A(x,y)t^n &= \sum_{n=0}^{\infty}\sum_{k=0}^{[\frac{1}{2}n]} \frac{[(A+\frac{1}{2}I)_k]^{-1}x^{n-2k}(x^2-y)^k t^n}{k!(n-2k)!2^{2k}}\\ &= \sum_{n=0}^{\infty}\sum_{k=0}^{\infty} \frac{[(A+\frac{1}{2}I)_k]^{-1}x^{n}(x^2-y)^k t^{n+2k}}{k!n!2^{2k}}\end{aligned} \tag{4.8}$$

By identification of the coefficients of t^n, we obtain a generating relation for the Gegenbauer matrix polynomials in the form:

$$\begin{aligned}\sum_{n=0}^{\infty}[(2A)_n]^{-1}C_n^A(x,y)t^n &= \sum_{n=0}^{\infty}\frac{x^n t^n}{n!}\sum_{k=0}^{\infty} \frac{[(A+\frac{1}{2}I)_k]^{-1}(x^2-y)^k t^{2k}}{k!2^{2k}}\\ &= \exp(xt)\ {}_0F_1(-;A+\frac{1}{2}I;\frac{t^2(x^2-y)}{4})\end{aligned} \tag{4.9}$$

where ${}_0F_1(-;...;...)$ is given as

$${}_0F_1(-;A+\frac{1}{2}I;\frac{t^2(x^2-y)}{4}) = \sum_{k=0}^{\infty} \frac{[(A+\frac{1}{2}I)_k]^{-1}(x^2-y)^k t^{2k}}{k!2^{2k}}$$

and $A+\frac{1}{2}I+kI$ is invertible for all $k \geqslant -\frac{1}{2}$. The expansion of x^nI indeed easy in a series of Gegenbauer matrix polynomials of two variables as follows

$$(2x)^nI = n!\sum_{k=0}^{[\frac{1}{2}n]} \frac{(A+(n-2k)I)[(A)_{n-k+1}]^{-1}}{k!}y^k C_{n-2k}^A(x,y). \tag{4.10}$$

Finally, we will expand the Gegenbauer matrix polynomials of two variables in series of Hermite and Laguerre matrix polynomials of two variables.

5. Expanding of Gegenbauer matrix polynomials in series of Hermite and Laguerre matrix polynomials of two variables

If A is a positive stable matrix in $\mathbb{C}^{N\times N}$, then the n^{th} Hermite matrix polynomials of two variables was defined by [**1, 12**]

$$H_n(x,y,A) = n!\sum_{k=0}^{[\frac{1}{2}n]}\frac{(-1)^k y^k}{k!(n-2k)!}(x\sqrt{2A})^{n-2k} \tag{5.1}$$

and the expansion of $x^n I$ in a series of Hermite matrix polynomials of two variables has been given in [**1, 12**]

$$(x\sqrt{2A})^n = n!\sum_{k=0}^{[\frac{1}{2}n]}\frac{y^k}{k!(n-2k)!}H_{n-2k}(x,y,A). \tag{5.2}$$

Now, the Gegenbauer matrix polynomials of two variables are expanded in series of Hermite matrix polynomials of two variables. Employing (2.1) and (1.9) with the aid of (5.2) and taking into account that each matrix commutes with itself, one gets

$$\begin{aligned}\sum_{n=0}^{\infty}C_n^A(x,y)t^n &= \sum_{n=0}^{\infty}\sum_{k=0}^{[\frac{1}{2}n]}\frac{(-1)^k(A)_{n-k}y^k(2x)^{n-2k}}{k!(n-2k)!}\,t^n\\ &= \sum_{n=0}^{\infty}\sum_{k=0}^{\infty}\frac{(-1)^k(A)_{n+k}y^k(2x)^n}{k!n!}\,t^{n+2k}\\ &= \sum_{n=0}^{\infty}\sum_{k=0}^{\infty}\sum_{s=0}^{[\frac{1}{2}n]}\frac{(-1)^k 2^n(A)_{n+k}(\sqrt{2A})^{-n}y^{k+s}}{k!s!(n-2s)!}H_{n-2s}(x,y,A)\,t^{n+2k}.\end{aligned} \tag{5.3}$$

Hence, we can write (5.3) in the form

$$\sum_{n=0}^{\infty}C_n^A(x,y)t^n = \sum_{n=0}^{\infty}\sum_{k=0}^{\infty}\sum_{s=0}^{[\frac{1}{2}n]}\frac{(-1)^k 2^n(A)_{n+k}(\sqrt{2A})^{-n}y^{k+s}}{k!s!(n-2s)!}H_{n-2s}(x,y,A)t^{n+2k}.$$

Thus

$$\begin{aligned}&\sum_{n=0}^{\infty}2^{-n}(\sqrt{2A})^n C_n^A(x,y)t^n =\\ &\sum_{n=0}^{\infty}\sum_{k=0}^{\infty}\sum_{s=0}^{[\frac{1}{2}n]}\frac{(-1)^k(A)_{n+k}y^{k+s}}{k!s!(n-2s)!}H_{n-2s}(x,y,A)t^{n+2k}.\end{aligned} \tag{5.4}$$

By using (1.9) the expression (5.4) becomes

$$\sum_{n=0}^{\infty}2^{-n}(\sqrt{2A})^n C_n^A(x,y)t^n = \sum_{n=0}^{\infty}\sum_{k=0}^{\infty}\sum_{s=0}^{\infty}\frac{(-1)^k(A)_{n+k+2s}y^{k+s}}{k!s!n!}H_n(x,y,A)t^{n+2k+2s}$$

and using (1.9), yields,

$$\sum_{n=0}^{\infty} 2^{-n}(\sqrt{2A})^n C_n^A(x,y)t^n = \sum_{n=0}^{\infty}\sum_{k=0}^{\infty}\sum_{s=0}^{k} \frac{(-1)^{k-s}(A)_{n+k+s}y^k}{(k-s)!s!n!} H_n(x,y,A)t^{n+2k}.$$

Since

$$(A)_{n+k+s} = (A+(n+k)I)_s(A)_{n+k}$$

then by using (1.5) and (1.8), we get

$$\sum_{n=0}^{\infty} 2^{-n}(\sqrt{2A})^n C_n^A(x,y)t^n =$$

$$\sum_{n=0}^{\infty}\sum_{k=0}^{\infty}\sum_{s=0}^{k} \frac{(-1)^k(-k)_s(A+(n+k)I)_s(A)_{n+k}}{k!s!n!} y^k H_n(x,y,A)t^{n+2k}$$

$$= \sum_{n=0}^{\infty}\sum_{k=0}^{\infty} \frac{(-1)^k}{k!n!}\ {}_2F_0(-kI, A+(n+k)I; -; 1)(A)_{n+k}y^k H_n(x,y,A)t^{n+2k}$$

$$= \sum_{n=0}^{\infty}\sum_{k=0}^{[\frac{1}{2}n]} \frac{(-1)^k}{k!(n-2k)!}\ {}_2F_0(-kI, A+(n-k)I; -; 1)(A)_{n-k}y^k H_{n-2k}(x,y,A)t^n.$$

Therefore, by identification of coefficient of t^n, we obtain an expansion of Gegenbauer matrix polynomials as a series of Hermite matrix polynomials in the form

$$C_n^A(x,y) = \tag{5.5}$$
$$\sum_{k=0}^{[\frac{1}{2}n]} \frac{(-1)^k(A)_{n-k}}{k!(n-2k)!}\ {}_2F_0(-kI, A+(n-k)I; -; 1)$$
$$\cdot 2^n(\sqrt{2A})^{-n}y^k H_{n-2k}(x,y,A).$$

Furthermore, the n^{th} Laguerre matrix polynomials $L_n^{(A,\lambda)}(x,y)$ of two variables is defined by

$$L_n^{(A,\lambda)}(x,y) = \sum_{k=0}^{n} \frac{(-1)^k\lambda^k x^k y^{n-k}}{k!(n-k)!}(A+I)_n[(A+I)_k]^{-1} \tag{5.6}$$

where A is a matrix in $\mathbb{C}^{N\times N}$ such that $-k$ is not an eigenvalue of A, for every integer $k > 0$ and λ is a complex number such that $Re(\lambda) > 0$.

In (5.6), putting $\lambda = 1$ gives

$$L_n^{(A)}(x,y) = \sum_{k=0}^{n} \frac{(-1)^k x^k y^{n-k}}{k!(n-k)!}(A+I)_n[(A+I)_k]^{-1}. \tag{5.7}$$

The expansion of x^nI in a series of Laguerre matrix polynomials of two variables has been given in [**4**] in the form

$$x^nI = n!\sum_{k=0}^{n} \frac{(-1)^k}{k!(n-k)!}(A+I)_n[(A+I)_k]^{-1}y^{n-k}L_k^{(A)}(x,y). \tag{5.8}$$

We use (5.8) to expand the Gegenbauer matrix polynomials of two variables in series of Laguerre matrix polynomials of two variables. We consider the series

$$\sum_{n=0}^{\infty} C_n^A(x,y)t^n = \sum_{n=0}^{\infty}\sum_{s=0}^{[\frac{1}{2}n]} \frac{(-1)^s (A)_{n-s} y^k (2x)^{n-2s}}{s!(n-2s)!} t^n$$

$$(5.9) \sum_{n=0}^{\infty}\sum_{s=0}^{\infty} \frac{(-1)^s (A)_{n+s} y^k (2x)^n}{s!n!} t^{n+2s}$$

$$= \sum_{n=0}^{\infty}\sum_{s=0}^{\infty}\sum_{k=0}^{n} \frac{(-1)^{k+s} 2^n (A)_{n+s} y^k}{s!(n-k)!} (A+I)_n [(A+I)_k]^{-1} y^{n-k} L_k^{(A)}(x,y) t^{n+2s}.$$

which, by using (1.11), becomes

$$\sum_{n=0}^{\infty} C_n^A(x,y)t^n =$$

$$\sum_{n=0}^{\infty}\sum_{k=0}^{\infty}\sum_{s=0}^{\infty} \frac{(-1)^{k+s}2^{n+k}}{n!s!} (A)_{n+k+s}(A+I)_{n+k}$$

$$[(A+I)_k]^{-1} y^{n+k} L_k^{(A)}(x,y) t^{n+k+2s}.$$

From (1.7), we have

$$\sum_{n=0}^{\infty} C_n^A(x,y)t^n$$

$$(5.10) \quad = \sum_{n=0}^{\infty}\sum_{k=0}^{\infty}\sum_{s=0}^{[\frac{1}{2}n]} \frac{(-1)^{k+s} 2^{n+k-2s}}{s!(n-2s)!} (A)_{n+k-s}(A+I)_{n+k-2s}$$

$$[(A+I)_k]^{-1} y^{n+k-2s} L_k^{(A)}(x,y) t^{n+k}.$$

Form (1.2), it is easy to find that

$$(A)_{2n} = 2^{2n} (\frac{1}{2}(A+I))_n (\frac{1}{2}(A))_n$$

and

$$(A)_{n+k} = (A)_n (A+nI)_k.$$

In accordance with (1.3), one gets

$$(A)_{n+k-s} = (-1)^s (A)_{n+k} [((1-n-k)I-A)_s]^{-1}$$

and

$$(A+I)_{n+k-2s} = 2^{-2s}(A+I)_{n+k} [(\frac{1}{2}((1-n-k)I-A))_s]^{-1} [(-\frac{1}{2}((n+k)I+A))_s]^{-1}.$$

Therefore

$$\sum_{n=0}^{\infty} C_n^A(x,y)t^n =$$

$$\sum_{n=0}^{\infty}\sum_{k=0}^{\infty}\sum_{s=0}^{[\frac{1}{2}n]} \frac{(-1)^{k+s}2^{n+k-2s}}{s!(n-2s)!} (-1)^s (A)_{n+k} [((1-n-k)I-A)_s]^{-1} 2^{-2s}(A+I)_{n+k}$$

$$[(\tfrac{1}{2}((1-n-k)I-A))_s]^{-1} [(-\tfrac{1}{2}((n+k)I+A))_s]^{-1} [(A+I)_k]^{-1} y^{n+k-2s} L_k^{(A)}(x,y) t^{n+k} =$$

$$\sum_{n=0}^{\infty}\sum_{k=0}^{\infty}\sum_{s=0}^{[\frac{1}{2}n]}\frac{1}{s!}(-\tfrac{1}{2}nI)_s(-\tfrac{1}{2}(n-\tfrac{1}{2})I)_s$$

$$[((1-n-k)I-A)_s]^{-1}[(\tfrac{1}{2}((1-n-k)I-A))_s]^{-1}[(-\tfrac{1}{2}((n+k)I+A))_s]^{-1}$$

$$(\tfrac{1}{4})^s\frac{(-1)^k 2^{n+k}}{n!}(A)_{n+k}(A+I)_{n+k}[(A+I)_k]^{-1}y^{n+k-2s}L_k^{(A)}(x,y)t^{n+k}=$$

$$\sum_{n=0}^{\infty}\sum_{k=0}^{\infty}{}_2F_3(-\tfrac{1}{2}nI,-\tfrac{1}{2}(n-\tfrac{1}{2})I;(1-n-k)I-A,\tfrac{1}{2}((1-n-k)I-A)$$

$$,-\tfrac{1}{2}((n+k)I+A);\tfrac{1}{4y^2})\frac{(-1)^k 2^{n+k}}{n!}(A)_{n+k}(A+I)_{n+k}[(A+I)_k]^{-1}y^{n+k}L_k^{(A)}(x,y)t^{n+k}=$$

$$\sum_{n=0}^{\infty}\sum_{k=0}^{n}{}_2F_3(-\tfrac{1}{2}(n-k)I,-\tfrac{1}{2}((n-k)-\tfrac{1}{2})I;(1-n)I-A,\tfrac{1}{2}((1-n)I-A)$$

$$,-\tfrac{1}{2}(A+nI);\tfrac{1}{4y^2})\frac{(-1)^k 2^{n}}{(n-k)!}(A)_{n}(A+I)_{n}[(A+I)_k]^{-1}y^{n}L_k^{(A)}(x,y)t^{n}.$$

where $(1-n)I-A+sI$, $\frac{1}{2}((1-n)I-A+sI$ and $-\frac{1}{2}(A+nI)+sI$ are invertible.

Equation the coefficients of t^n gives an expansion of as a series of Gegenbauer matrix polynomials in the form

$$C_n^A(x,y)=\sum_{k=0}^{n}{}_2F_3(-\frac{1}{2}(n-k)I,-\frac{1}{2}((n-k)-\frac{1}{2})I;(1-n)I-A,\frac{1}{2}((1-n)I-A)$$
$$,-\frac{1}{2}(A+nI);\frac{1}{4y^2})\frac{(-1)^k 2^n}{(n-k)!}(A)_n(A+I)_n[(A+I)_k]^{-1}y^nL_k^{(A)}(x,y) \tag{5.11}$$

which can be written in a convenient form as follows

$$C_n^A(x,y)=\frac{2^n}{n!}(A)_n(A+I)_n\sum_{k=0}^{n}{}_2F_3(-\tfrac{1}{2}(n-k)I,-\tfrac{1}{2}((n-k)-\tfrac{1}{2})I;$$

$$(1-n)I-A,\tfrac{1}{2}((1-n)I-A),-\tfrac{1}{2}(A+nI);\tfrac{1}{4y^2})$$

$$\cdot(-nI)_k[(A+I)_k]^{-1}y^nL_k^{(A)}(x,y).$$

The results of this paper are variant, significant and so it is interesting and capable to develop its study in the future.

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References

[1] R.S. Batahan, A new extension of Hermite matrix polynomials and its applications, *Linear Algebra Appl.*, **419**(1) (2006), 82-92.

[2] E. Defez and L. Jódar, Some applications of the Hermite matrix polynomials series expansions, *J. Comp. Appl. Math.*, **99** (1998), 105-117.

[3] E. Defez and L. Jódar, Chebyshev matrix polynomials and second order matrix differential equations, *Utilitas Math.*, **61** (2002), 107-123.

[4] L. Jódar and R. Company, Hermite matrix polynomials and second order matrix differential equations, *J. Approx. Theory Appl.*, **12**(2) (1996), 20-30.
[5] L. Jódar, R. Company and E. Navarro, Laguerre matrix polynomials and system of second-order differential equations, *Appl. Num. Math.*, **15**(1) (1994), 53-63.
[6] L. Jódar and J.C. Cortés, On the hypergeometric matrix function, *J. Comp. Appl. Math.*, **99**(1-2) (1998), 205-217.
[7] L. Jódar and J.C. Cortés, Closed form general solution of the hypergeometric matrix differential equation, *Math. Computer Modell.*, **32** (9)(2000), 1017-1028.
[8] L. Jódar and E. Defez, On Hermite matrix polynomials and Hermite matrix function, *J. Approx. Theory Appl.*, **14**(1) (1998), 36-48.
[9] G.S. Kahmmash, A study of a two variables Gegenbauer matrix polynomials and second order matrix partial differential equations, *Int. J. Math. Analysis*, **2**(17) (2008), 807-821.
[10] G.S. Khammash, On Hermite matrix polynomials of two variables, *J. Appl. Sciences*, **8**(7) (2008), 1221-1227.
[11] M.A. Khan and G.S. Khammash, A study of a two variables Gegenbauer polynomials, *Appl. Math. Sciences*, **2**(13) (2008), 639-657.
[12] M.S. Metwally, M.T. Mohamed and A. Shehata, On Hermite-Hermite matrix polynomials, *Math. Bohemica*, **133**(4) (2008), 421-434.
[13] M.S. Metwally, M.T. Mohamed and A. Shehata, Generalizations of two-index two-variable Hermite matrix polynomials, *Demonstratio Mathematica*, **42**(4) (2009), 687-701.
[14] M.S. Metwally, M.T. Mohamed and A. Shehata, On pseudo Hermite matrix polynomials of two variables, *Banach J. Math. Anal.*, **4**(2) (2010),169-178.
[15] M.T. Mohamed and A. Shehata, A study of Appell's matrix functions of two complex variables and some properties, *Advan. Appl. Math. Sci.*, **9**(1) (2011), 23-33.
[16] E.D. Rainville, *Special Functions*, The Macmillan Company, New York, 1960.
[17] S.Z. Rida, M. Abul-Dahab, M.A. Saleem and M.T.Mohamed, On Humbert matrix function $\Psi_1(A, B; C, C'; z, w)$ of two complex variables under differential operator, Int. J. Industrial Mathematics, **2**(3) (2010), 167-179.
[18] K.A.M. Sayyed, M.S. Metwally and R.S. Batahan, On generalized Hermite matrix polynomials, *Electron. J. Linear Algebra*, **10** (2003), 272-279.
[19] K.A.M. Sayyed, M.S. Metwally and R.S. Batahan, Gegenbauer matrix polynomials and second order matrix differential equations, *Divulgaciones Matemáticas*, **12**(2) (2004), 101-115.

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