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A COMMON FIXED POINT THEOREM ON TWO PAIRS OF WEAKLY COMPATIBLE MAPPINGS IN MENGER SPACES

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Abstract. The aim of this paper is to show that the comment of Rashwan and Maustafa [5] on a result of Sharma and Deshpande [8] is not true by pointing out the inconsistency in their claimed example. Further the results of the former authors are generalised and are supported by examples.

1. Introduction

Jungck [2] proved a common fixed point theorem for commuting maps generalizing the Banach's fixed point theorem. Sessa [7] introduced weak commutativity and exhibited a common fixed point theorem for weakly commuting mappings. Consequently, Jungck[3] introduced the notion of compatibility and established various fixed point theorems. Jungck and Rhoades [4] introduced the notion of weak compatibility which is more general than compatibility and considered the corresponding fixed point results. Sharma and Deshpande [8] established results in Menger space corresponding to weak compatibility. Recently, Rashwan and Maustafa[5] claimed that the main result of Sharma and Deshpande [8] is not valid by means of an example. We observed that the example does not satisfy the condition stated in the theorem. Further, we generalized the results and exhibited supporting examples.

Very recently a worthy observation is made by Dragan Dorić, Zoran Kadelburg, and Stojan Radenović [1] on occasionally weakly compatible mappings and common fixed points.

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2. Preliminaries

We take the standard definitions given in [6].

We mainly use the following results, in the subsequent section.

Result 2.1. [6]. Let $\{x_n\}(n = 0, 1, 2, \dots)$ be a sequence in a Menger space $(X, F, *)$, where $*$ is continuous and $x*x \geqslant x$ for all $x \in [0, 1]$. If there is a $k \in (0, 1)$ such that

$$F_{x_n,x_{n+1}}(kt) \geqslant F_{x_{n-1},x_n}(t)$$

for all $t > 0$ and $n \in \mathbb{N}$, then $\{x_n\}$ is a Cauchy sequence in X.

Result 2.2. [9]. Let $(X, F, *)$ be a Menger space. If there is a $k \in (0, 1)$ such that

$$F_{x,y}(kt) \geqslant F_{x,y}(t)$$

for all $x, y \in X$ and $t > 0$, then $y = x$.

3. Common Fixed Point Theorems

We state the Theorem(3.2) of [5].

Theorem 3.1. *Let A, B, S and T be self mappings on a Menger space (X, F, t), where t is continuous and $t(x, x) \geqslant x$ for all $x \in [0, 1]$, satisfying:*

(3.1.1) $A(X) \subseteq T(X)$ *and* $B(X) \subseteq S(X)$,

(3.1.2) *there is a* $k \in (0, 1)$ *such that*

$$F_{Ax,By}(ku) \geqslant t(F_{Ax,Sx}(u), t(F_{By,Ty}(u), t(F_{Ax,Ty}(\alpha u), F_{By,Sx}((2-\alpha)u))))$$

for all $x, y \in X$, $u > 0$ *and* $\alpha \in (0, 2)$.

If

(3.1.3) *One of $A(X)$, $B(X)$, $S(X)$ and $T(X)$ is a complete subspace of X, then*

(i.) *A and S have a coincidence point, and*

(ii.) *B and T have a coincidence point.*

Further, if

(3.1.4) *the pairs $\{A, S\}$ and $\{B, T\}$ are weakly compatible, then*

(iii.) *A, B, S and T have a unique common fixed point in X.*

In [5], Rashwan and Maustafa claimed that this result is false. They provided an example, in which the mappings have two common fixed points. This controversy arose due to the vague statement of the inequality (3.1.2). The inequality must hold good for all $x, y \in X$, for all $u > 0$ and for all $\alpha \in (0, 2)$.

Their example 3.1 of [5] is the following:

EXAMPLE 3.1. Let $X = (0,3]$ with Euclidean metric d and F be defined by

$$F_{x,y}(u) = \begin{cases} G(\frac{u}{d(x,y)}) & \text{if } x \neq y, \\ H(u) & \text{if } x = y. \end{cases}$$

for all $x, y \in X$, where G is any distribution function with $G(0) = 0$ $(u \in \mathbb{R})$ and d is the usual metric on $\mathbb{R}$.

Define the mappings A, B, S and T as

$$A(x) = \begin{cases} \frac{1}{2} & \text{if } x \in (0,1), \\ 3 & \text{if } x \in [1,3]. \end{cases}$$

$$B(x) = S(x) = \begin{cases} 1 - x & \text{if } x \in (0,1), \\ 3 & \text{if } x \in [1,3]. \end{cases}$$

$$T(x) = \begin{cases} x & \text{if } x \in (0,1), \\ 3 & \text{if } x \in [1,3]. \end{cases}$$

They have taken $k = \frac{3}{4}$. Let $u > 0$
Now, for $x \in [1,3], y = \frac{3}{4}$,

$$F_{Ax,By}(ku) = F_{Ax,B\frac{3}{4}}(\frac{3}{4}u) = F_{3,\frac{1}{4}}(\frac{3}{4}u) = G\Big(\frac{\frac{3}{4}u}{d(3,\frac{1}{4})}\Big) = G\Big(\frac{\frac{3}{4}u}{\frac{11}{4}}\Big) = G\Big(\frac{3u}{11}\Big).$$

$F_{Ax,Sx}(u) = F_{3,3}(u) = 1$. $F_{By,Ty}(u) = F_{B\frac{3}{4},T\frac{3}{4}}(u) = F_{\frac{1}{4},\frac{3}{4}}(u) = G\Big(\frac{u}{\frac{1}{2}}\Big) = G(2u)$.
$F_{Ax,Ty}(\alpha u) = F_{3,T\frac{3}{4}}(\alpha u) = F_{3,\frac{3}{4}}(\alpha u) = G\Big(\frac{\alpha u}{\frac{9}{4}}\Big) = G\Big(\frac{4\alpha u}{9}\Big)$.
$F_{By,Sx}((2-\alpha)u) = F_{\frac{1}{4},3}((2-\alpha)u) = G\Big(\frac{(2-\alpha)u}{\frac{11}{4}}\Big) = G\Big(\frac{4(2-\alpha)u}{11}\Big)$.

If $\frac{3}{11} < \frac{4\alpha}{9}$ then $\alpha > \frac{27}{44}$, and if $\frac{3}{11} < \frac{4(2-\alpha)}{11}$ then $\alpha < \frac{5}{4}$

$$i.e, \quad \alpha \in \Big(\frac{27}{44}, \frac{5}{4}\Big) \quad then$$

$$F_{Ax,B\frac{3}{4}}\Big(\frac{3}{4}u\Big) < t\Big(F_{Ax,Sx}(u), t\Big(F_{B\frac{3}{4},T\frac{3}{4}}(u), t\Big(F_{Ax,T\frac{3}{4}}(\alpha u), t\Big(F_{B\frac{3}{4},Sx}((2-\alpha)u\Big)\Big)\Big)\Big).$$

Thus (3.1.2) is not satisfied for all $x, y \in X$, $u > 0$ and for every $\alpha \in (0,2)$. So, this example is not a counter one for the validity of the Theorem(3.1).

REMARK 3.1. In fact, if we take $\alpha = 1$ and $x \in [1,3], y = \frac{1}{2}$ then, for no $k \in (0,1)$ the inequality (3.1.2) holds. Now,
$F_{Ax,By}(ku) = F_{3,\frac{1}{2}} = G\Big(\frac{2ku}{5}\Big)$,
$F_{Ax,Sx}(u) = F_{3,3}(u) = 1$,
$F_{By,Ty}(u) = F_{\frac{1}{2},\frac{1}{2}}(u) = 1$,
$F_{Ax,Ty}(\alpha u) = F_{3,\frac{1}{2}}(u) = G\Big(\frac{2u}{5}\Big)$, and
$F_{By,Sx}((2-\alpha)u) = F_{\frac{1}{2},3}(u) = G\Big(\frac{2u}{5}\Big)$.

Now, we modify the inequality (3.1.2) and establish our result:

THEOREM 3.2. *The same Theorem (3.1), with t is replaced by $*$ and (3.1.2) replaced by* $(3.1.2)^1$:

$(3.1.2)^1$ *There is a $k \in (0,1)$ such that*

$$F_{Ax,By}(ku) \geqslant F_{Ax,Sx}(u) * F_{By,Ty}(u) * F_{Sx,Ty}(u) * F_{Ax,Ty}(\alpha u) * F_{By,Sx}((2-\alpha)u)$$

for all $x, y \in X$, for all $u > 0$ and for all $\alpha \in (0,2)$.

Proof: Let $x_0 \in X$. By (3.1.1) there exist sequences $\{x_n\}$ and $\{y_n\}$ in X such that

$$Ax_{2n} = Tx_{2n+1} = y_{2n}$$

and

$$Bx_{2n+1} = Sx_{2n+2} = y_{2n+1}, for \quad n = 0, 1, 2, \ldots$$

Taking $x = x_{2n}, y = x_{2n+1}, \alpha = 1 - q$ with $q \in (0,1)$ in $(3.1.2)^1$ and using the properties

$$F_{y_{2n-1},y_{2n+1}}((1+q)u) \geqslant F_{y_{2n-1},y_{2n}}(u) * F_{y_{2n},y_{2n+1}}(qu) \quad and \quad F_{y_{2n},y_{2n}}(\alpha u) = 1,$$

we get that

$$F_{y_{2n},y_{2n+1}}(ku) \geqslant F_{y_{2n-1},y_{2n}}(u) * F_{y_{2n},y_{2n+1}}(u) * F_{y_{2n},y_{2n+1}}(qu).$$

As t-norm is continuous and F is left continuous, $q \to 1 - 0$

$$\Longrightarrow F_{y_{2n},y_{2n+1}}(ku) \geqslant F_{y_{2n-1},y_{2n}}(u) * F_{y_{2n},y_{2n+1}}(u).$$

Similarly, taking $x = x_{2n+2}, y = x_{2n+1}, \alpha = 1 + q$ with $q \in (0,1)$ in $(3.1.2)^1$, we get that

$$F_{y_{2n+1},y_{2n+2}}(ku) \geqslant F_{y_{2n},y_{2n+1}}(u) * F_{y_{2n+1},y_{2n+2}}(u).$$

Thus for all positive integers n, we have

$$F_{y_n,y_{n+1}}(ku) \geqslant F_{y_{n-1},y_n}(u) * F_{y_n,y_{n+1}}(u).$$

Consequently, $F_{y_n,y_{n+1}}(u) \geqslant F_{y_{n-1},y_n}(k^{-1}u) * F_{y_n,y_{n+1}}(k^{-1}u)$.
By repeated application of the above inequality and the associative property of the t-norm, we get that

$$F_{y_n,y_{n+1}}(u) \geqslant F_{y_{n-1},y_n}(k^{-1}u) * F_{y_n,y_{n+1}}(k^{-l}u)$$

for any positive integer l,
Since $F_{y_n,y_{n+1}}(k^{-l}u) \to 1$ as $l \to \infty$ (since $k^{-l}u \to \infty$),
we get that

$$F_{y_n,y_{n+1}}(u) \geqslant F_{y_{n-1},y_n}(k^{-1}u)$$

that is $F_{y_n,y_{n+1}}(ku) \geqslant F_{y_{n-1},y_n}(u)$ for all positive integer n.
Now, by the Result (2.1), follows that $\{y_n\}$ is a Cauchy sequence in X.

We consider the case, when $A(X)$ is complete.
some label - optional Now $\{y_{2n}\}$ is a Cauchy sequence in $A(X)$. So, there is a $z \in A(X)$ such that $y_{2n} \to z$ as $n \to \infty$. So follows that $y_n \to z$ as $n \to \infty$ $(since \ \{y_n\}$ is a Cauchy sequence). Also $y_{2n+1} \to z$ as $n \to \infty$.
Since $A(X) \subseteq T(X)$, there is a $v \in X$ such that $z = Tv$.
Taking $x = x_{2n}, y = v$ and $\alpha = 1$ in $(3.1.2)^1$, we get that

$$F_{y_{2n},Bv}(ku) \geqslant F_{y_{2n},y_{2n-1}}(u) * F_{Bv,z}(u) * F_{y_{2n-1},z}(u) * F_{y_{2n},z}(u) * F_{Bv,y_{2n-1}}(u).$$

Now, as $n \to \infty$, we get that

$$F_{z,Bv}(ku) \geqslant F_{z,z}(u) * F_{Bv,z}(u) * F_{z,z}(u) * F_{z,z}(u) * F_{Bv,z}(u) \geqslant F_{z,Bv}(u).$$

By the Result(2.2), we get that $Bv = z = Tv$.
Thus B and T have a coincidence point v. Since $\{B, T\}$ is weakly compatible, follows that $BTv = TBv$. i.e, $Bz = Tz$.
Taking $x = x_{2n}, y = z$ and $\alpha = 1$ in $(3.2.2)^1$, we get that

$$F_{y_{2n},Bz}(ku) \geqslant F_{y_{2n},y_{2n-1}}(u) * F_{Bz,z}(u) * F_{y_{2n-1},z}(u) * F_{y_{2n},z}(u) * F_{Bz,y_{2n-1}}(u).$$

As $n \to \infty$, we get that

$$F_{z,Bz}(ku) \geqslant F_{z,z}(u) * F_{Bz,z}(u) * F_{z,Bz}(u) * F_{z,z}(u) * F_{Bz,z}(u) \geqslant F_{z,Bz}(u).$$

So, we get that $Bz = z \Longrightarrow Bz = Tz = z$.
Since $B(X) \subseteq S(X)$, there is a $w \in X$ such that $z = Sw$.
Taking $x = w, y = z$ and $\alpha = 1$ in $(3.1.2)^1$, we get that

$$F_{Aw,Bz}(ku) \geqslant F_{Aw,Sw}(u) * F_{Bz,Tz}(u) * F_{Sw,Tz}(u) * F_{Aw,Tz}(u) * F_{Bz,Sw}(u).$$

Now, as $n \to \infty$, we get that

$$F_{Aw,z}(ku) \geqslant F_{Aw,z}(u) * F_{z,z}(u) * F_{z,z}(u) * F_{Aw,z}(u) * F_{z,z}(u) \geqslant F_{Aw,z}(u).$$

So $Aw = z = Sw \Longrightarrow A$ and S have a coincidence point w. Since $\{A, S\}$ is weakly compatible, follows that $ASw = SAw$. i.e, $Az = Sz$.
Taking $x = z, y = x_{2n+1}$ and $\alpha = 1$ in $(3.2.2)^1$, we get that

$$F_{Az,y_{2n+1}}(ku) \geqslant F_{Az,Sz}(u) * F_{y_{2n+1},y_{2n}}(u) * F_{Sz,y_{2n}}(u) * F_{Az,y_{2n}}(u) * F_{y_{2n+1},Sz}(u).$$

Now, as $n \to \infty$, we get that

$$F_{Az,z}(ku) \geqslant F_{Az,Az}(u) * F_{z,z}(u) * F_{Az,z}(u) * F_{Az,z}(u) * F_{z,Az}(u) \geqslant F_{Az,z}(u)$$

that is $Az = z$. So, $Az = Sz = z$.
Thus $Az = Bz = Sz = Tz = z$.

Similar is the case when $T(X)$ is complete.

When $B(X)$ or $S(X)$ is complete, we first get $Az = Sz = z$ and then $Bz = Tz = z$.

Uniqueness follows trivially by taking $\alpha = 1$.

Remark 3.2. As the R.H.S. in the inequality given in (3.1.2)is $\geqslant$ the R.H.S. of the inequality given in $(3.1.2)^1$, it follows that our Theorem is a generalization of the Theorem (3.2) of [8]. Further, our Theorem is a generalization of the remaining Theorems given in [8].

We conclude our paper with the following examples in support of our Theorem (3.2).

Example 3.2. $(\mathbb{R}, F, *)$ is a Menger space, where $\mathbb{R}$ is the real line with the usual metric and $F : \mathbb{R} \to [0,1]$ is defined by

$$F_{x,y}(u) = \frac{u}{u + |x - y|}$$

for all $x, y \in \mathbb{R}$ and $*$ is the min t-norm, i.e, $a * b = min\{a, b\}$ for all $a, b \in [0, 1]$. Let A, B, S and T be the self maps on $\mathbb{R}$, defined by

$$A(x) = \begin{cases} 0 & \text{if } x \leqslant 2, \\ 1 & \text{if } x > 2. \end{cases}$$

$Bx = 0, Sx = x^3$ and $Tx = x$ for all $x \in \mathbb{R}$.
Then, clearly A, B, S and T satisfy the hypothesis of Theorem(3.2) with $k \in [\frac{1}{7}, 1) \subset (0, 1)$.
For, when $x > 2$,

$$F_{Ax,By}(ku) = \frac{ku}{ku + 1} = \frac{u}{u + \frac{1}{k}}$$

and

$$F_{Ax,Sx}(u) = \frac{u}{u + (x^3 - 1)} < \frac{u}{u + 7}.$$

So, in $(3.1.2)^1$, L.H.S $\geqslant$ R.H.S when $\frac{1}{k} \leqslant 7$, that is $k \geqslant \frac{1}{7}$.
Clearly 0 is the unique common fixed point of A, B, S and T.

Example 3.3. The same Menger space and with same A, B of the above example. Further $Sx = x^2$ and $Tx = x$ for all $x \in \mathbb{R}$. It can be show that $k \in [\frac{1}{3}, 1)$ serves the purpose and 0 is the unique common fixed point of A, B, S and T.

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