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EQUITABLE ASSOCIATE GRAPH OF A GRAPH

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Abstract. Let $G = (V, E)$ be a simple graph. Let H be the graph constructed from G as follows: $V(H) = V(G)$, two points u and v are adjacent in H if and only if u and v are adjacent and degree equitable in G. H is called the **adjacency inherent equitable graph of** G or **equitable associate of** G and is denoted by $e(G)$. This Paper aims at the study of a new concept called equitable associate graph of a graph. In this paper we show that there is relation between $e(G)$ and $e^{-}(G)$. Further results on the new parameter $e(G)$,$e^{-}(G)$ and complexity of equitable dominating set are discussed.

1. Introduction

New concepts of domination arise from practical considerations. In a network, nodes with nearly equal capacity may interact with each other in a better way. In the society, persons with nearly equal status, tend to be friendly. In an industry, employees with nearly equal powers form association and move closely. Equitability among citizens in terms of wealth, health, status etc is the goal of a democratic nation. In order to study this practical concept, a graph model is to be created. Prof. E. Sampathkumar is the first person to recognize the spirit and power of this concept and introduced various types of equitability in graphs like degree equitability, outward equitability, inward equitability, equitability in terms of number of equal degree neighbours, or in terms of number of strong degree neighbours etc. In general, if $G = (V, E)$ is a simple graph and $\phi : V(G) \to N$ is a function, we may define equitability of vertices in terms of ϕ - values of the vertices. A subset D of V is called an equitable dominating set if for every $v \in V - D$ there exists a vertex $u \in D$ such that $uv \in E(G)$ and $|d(u) - d(v)| \leqslant 1$, where $d(u)$ denotes the degree of vertex u and $d(v)$ denotes the degree of vertex v. The minimum cardinality of such a dominating set is denoted by γ^{e} and is called the equitable domination number

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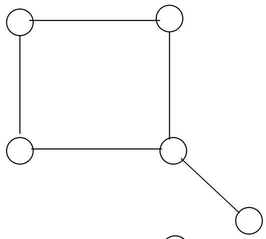
of G. Degree Equitable domination on Graphs were introduced in [10].

Definition 1.1. Let $G = (V, E)$ be a simple graph. Let H be the graph constructed from G as follows: $V(H) = V(G)$, two points u and v are adjacent in H if and only if u and v are adjacent and degree equitable in G. H is called the **adjacency inherent equitable graph of** G or **equitable associate of** G and is denoted by $e(G)$.

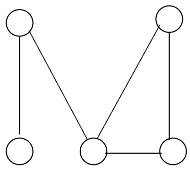
Remark 1.1. (i) $e = uv \in E(e(G))$. Then u and v are adjacent and degree equitable in G. Therefore $e \in E(G)$. $E(e(G)) \subseteq E(G)$.
(ii) An edge $e = uv \in E(G)$ is said to be equitable if $|d(v) - d(u)| \leqslant 1$.
Let $E^e(G)$ be the set of all equitable edges of G. Then clearly,
$E^e(G) = E(e(G))$.

Remark 1.2. $\overline{e(G)}$ need not be equal to $e(\overline{G})$.

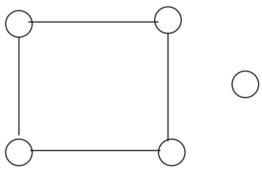
For consider G :



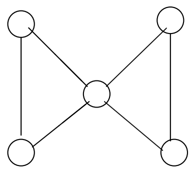
$\overline{G}$:



$e(G)$:



$\overline{e(G)}$:



$e(\overline{G})$:

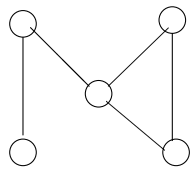


Fig. 1.1

Therefore $\overline{e(G)} \neq e(\overline{G})$.

Remark 1.3. Since $E(e(G)) \subseteq E(G)$ we have $E(\overline{e(G)}) \supseteq E(\overline{G}) \supseteq E(e(\overline{G}))$

Theorem 1.1. *$\overline{e(G)} = e(\overline{G})$ if and only if every edge of G is equitable.*

Proof. If every edge of G is equitable then $G = e(G)$ and hence $\overline{G} = \overline{e(G)}$. Further, since every edge of G is equitable, every edge of $\overline{G}$ is equitable. Therefore $e(\overline{G}) = \overline{G}$. Therefore $e(\overline{G}) = \overline{e(G)}$.

Conversely, let $\overline{e(G)} = e(\overline{G})$. Suppose G has a non equitable edge say $e = uv$. Then u and v are not adjacent in $e(G)$. Therefore u and v are adjacent in $\overline{e(G)}$. Since $e = uv$ in G, u and v are not adjacent in $\overline{G}$ and hence u and v are not adjacent in $e(\overline{G})$. Therefore $\overline{e(G)} \neq e(\overline{G})$, a contradiction. Hence every edge of G is equitable. $\square$

Definition 1.2. Let $u \in V$. The equitable neighborhood of u denoted by $N^e(u)$ is defined as $N^e(u) = \{v \in V | v \in N(u), |d(u) - d(v)| \leqslant 1\}$ and $u \in I_e \Leftrightarrow N^e(u) = \phi$.
The cardinality of $N^e(u)$ is denoted by $d^e_G(u)$.

Definition 1.3. The maximum and minimum equitable degree of a point in G are denoted respectively by $\Delta^e(G)$ and $\delta^e(G)$. That is $\Delta^e(G) = \max_{u \in V(G)} |N^e(u)|$, $\delta^e(G) = \min_{u \in V(G)} |N^e(u)|$

Definition 1.4. A subset S of V is called an equitable independent set, if for any $u \in S$, $v \notin N^e(u)$ for all $v \in S - \{u\}$.

Definition 1.5. Let G be a graph and let H be a graph such that $e(H) = G$. Then H is called a **pre-e-graph** of G. Pre-e-graph of G is not unique. The set of all pre-e-graph of G is denoted by $e^{-}(G)$.

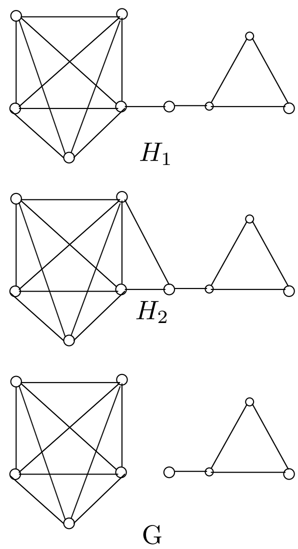


Fig. 1.2

$e(H_1) = e(H_2) = G$.
That is $H_1, H_2 \in e^{-}(G)$.

Remark 1.4. The cardinality of the set of all non isomorphic graphs H such that $e(H) = G$ is denoted by $O(e^{-}(G))$. Given any positive integer $k \geqslant 1$, there exists a graph G such that $O(e^{-}(G)) \geqslant k$.

For: Let G :

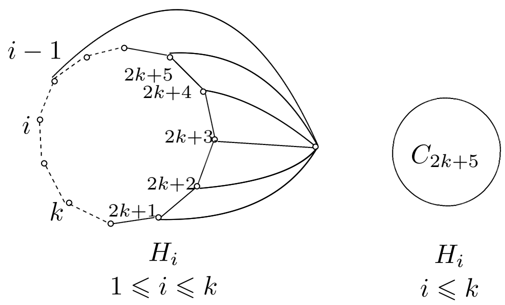


Fig. 1.3

$H_1, H_2, \ldots, H_k$ are the non isomorphic graphs in $e^{-}(G)$.
Therefore $O(e^{-}(G)) \geqslant k$.

ILLUSTRATION 1.1. There exists a graph G in which every edge is equitable and $e^{-}(G)$ has a non-equitable edge.

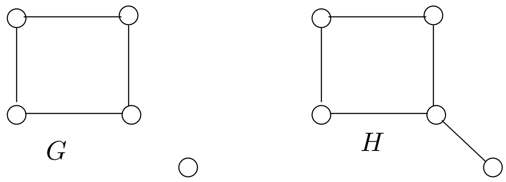


Fig. 1.4

Here $H \in e^{-}(G)$, every edge of G is equitable and H has a non-equitable edge.

ILLUSTRATION 1.2.

G :

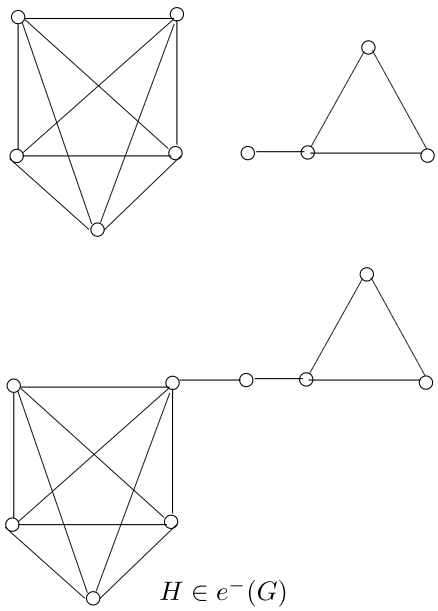
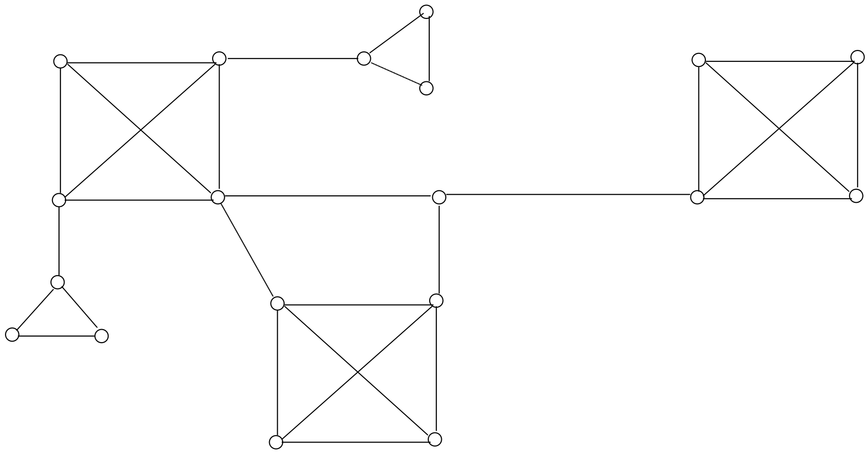


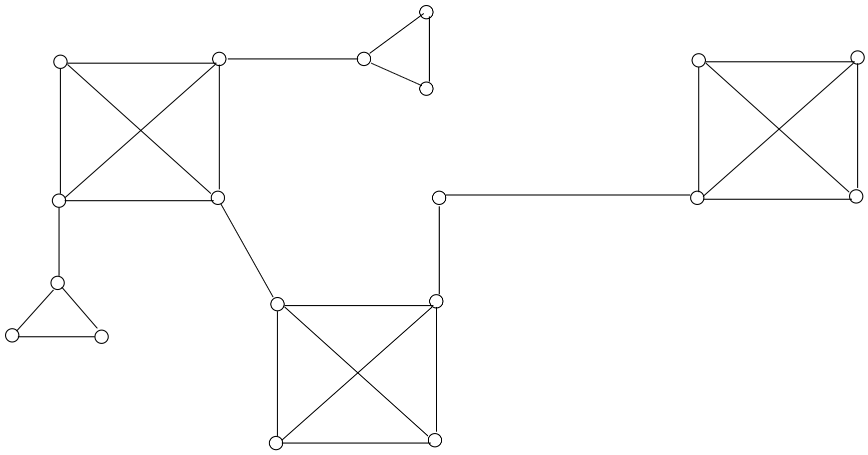
Fig. 1.5

$e(H) = G$. That is $H \in e^{-}(G)$. Here G has no isolates and $H \in e^{-}(G)$. H has a non equitable edge.

Illustration 1.3.



H



G

Fig. 1.6

$e(H) = G$. That is $H \in e^{-}(G)$.
Here G is connected and $H \in e^{-}(G)$. H is connected and H has a non-equitable edge.

Theorem 1.2. *Let G be a graph. Let $H = e(G)$. Then every edge of H is degree equitable if and only if the following holds: If u and v are degree equitable and adjacent in G then $d_G^e(u)$ and $d_G^e(v)$ are equitable.*

Proof. Let $e(G) = H$. Therefore $E^e(G) = E(H)$. Assume that if u and v are degree equitable and adjacent in G, then $d_G^e(u)$ and $d_G^e(v)$ are equitable. Let $e = uv \in E(H)$. Then u and v are degree equitable in G. As $d_H(u) = d_G^e(u)$

and $d_H(v) = d^e_G(v)$ and as $d^e_G(u)$ and $d^e_G(v)$ are equitable, we get that u and v are degree equitable in H. That is e is degree equitable in H.

Suppose u and v are equitably adjacent points in G such that $d^e_G(u)$ and $d^e_G(v)$ are not equitable. Then clearly $e = uv \in E(H)$ is not degree equitable in H. Hence the theorem. □

THEOREM 1.3. *If G and H are any two graphs, then $e(G\Box H) = e(G)\Box e(H)$.*

PROOF. Let $(u, v), (x, y) \in V(G\Box H)$ such that $|d_{G\Box H}(u, v) - d_{G\Box H}(x, y)| \leqslant 1$.
$\Rightarrow |d_G(u) + d_H(v) - d_G(x) - d_H(y)| \leqslant 1$
$\Rightarrow |[d_G(u) - d_G(x)] - [d_H(v) - d_H(y)| \leqslant 1.$
$\Rightarrow ||d_G(u) - d_G(x)| - |d_H(v) - d_H(y)|| \leqslant 1.$ ....(1)

If (u, v) and (x, y) are adjacent in $e(G\Box H)$, then (u, v) and (x, y) are adjacent and degree equitable in $G\Box H$.

Since (u, v) and (x, y) are adjacent in $G\Box H$, either $u = x$ in $V(G)$ and $vy \in E(H)$ or $ux \in E(G)$ and $v = y$ in $V(H)$. Using (1) we get that either $u = x$ in $V(G)$and v and y are adjacent and equitable in H or u and x are adjacent and equitable in G and $v = y$ in $V(H)$. That is ,either $u = x$ in $V(e(G))$ and $uy \in E(e(H))$ or $ux \in E(e(G))$ and $v = y$ in $V(e(H))$. That is (u, v) and (x, y) are adjacent $e(G)\Box e(H)$.

Conversely, let (u, v) and (x, y) be adjacent in $e(G)\Box e(H)$. Then $u = x$ in $V(e(G))$ and $vy \in E(e(H))$ or $ux \in E(e(G))$ and $v = y$ in $V(e(H))$. That is , $u = x$ in $V(G)$ and v and y are adjacent and equitable in H or u and x are adjacent and equitable in G or $v = y$ in $V(H)$. This implies that (u, v), (x, y) are adjacent in $G\Box H$ and $|d_{G\Box H}(u, v) - d_{G\Box H}(x, y)| = |[d_G(u) - d_G(x)] - [d_H(y) - d_H(v)]| \leqslant 1$, since u and x are equitable in G and v and y are equitable in H. Hence $(u, v), (x, y)$ are adjacent and degree equitable in $G\Box H$. That is, $(u, v), (x, y)$ are adjacent in $e(G\Box H)$. Hence the theorem. □

2. Complexity

EQUITABLE DOMINATING SET
INSTANCE
A graph $G = (V, E)$ and a positive integer k.
Question
Does G have a dominating set of cardinality $\leqslant k$?.

THEOREM 2.1. *EQUITABLE DOMINATING SET is NP- Complete.*

PROOF. Let $G = (V, E)$ be an arbitrary graph. Let $e(G)$ be the equitable associate of G. G has an equitable dominating set of size $\leqslant k$ if and only if $e(G)$ has a dominating set of size $\leqslant k$. Since DOMINATING SET is NP- Complete, EQUITABLE DOMINATING SET is NP- Complete. □

THEOREM 2.2. *EQUITABLE DOMINATING SET is NP- Complete even in the class of bipartite graphs.*

Proof. Let $G = (V, E)$ be an arbitrary graph. Let $V = \{u_1, u_2, \ldots, u_n\}, V' = \{u'_1, u'_2, \ldots, u'_n\}, X = \{x_1, x_2, \ldots, x_n\}$. Let H be the graph with vertex set $V(H) = V \cup V' \cup X$. $E(H) = \{u'_i u_j, u'_i x_j : u_j \in N[u_i]\}_{1 \leqslant i \leqslant n}$. H is a bipartite graph in which $d_H(u_i) = d_H(u_{i'}) = d_H(x_i) = d_G(u_i) + 1, 1 \leqslant i \leqslant n$. Let D be a dominating set of G of cardinality $\leqslant n$.

Let $D = \{u_1, u_2, \ldots, u_r\}$, $r \leqslant n$. Let $D' = \{u'_1, u'_2, ..., u'_r, x_1, x_2, ..., x_n\}$. Then D' is an equitable dominating set of H of cardinality $r + n$. Let S be an equitable dominating set of cardinality $r + n$ where $r \leqslant n$. Replace u'_i, x_j in S by u_i, u_j respectively. Let the resulting set be S_1. Clearly $|S_1| \leqslant n$. Let $u_k \in V(G) - S_1$. If $u_k \in S$ then $u_k \in S_1$, a contradiction. Therefore $u_k \in V(H) - S$. Since S is an equitable dominating set, there exists u_j or $u'_j \in S$ such that u_k is dominated by u_j or u'_j. By construction of $S_1, u_j \in S_1$ and u_k is adjacent to u_j. Therefore S_1 is a dominating set of G and $|S_1| \leqslant n$. Therefore EQUITABLE DOMINATING SET remains NP- Complete even in the class of bipartite graphs. □

References

[1] R. B. Allan and R. C. Laskar,*On domination and independent domination numbers of a graph*, Discrete Math. 23(2)(1978), 73-76

[2] D. W. Bange, A. E. Barkankas, L. H. Host and P. J. Slater, *Generalised domination and domination in graphs*, Discrete Math. 159(1-3)(1996), 1-11

[3] C. Berge, *Theory of graphs and its applications*, Dunod, Paris, 1958

[4] C. Berge, *Graphs and Hypergraphs*, North-Holland, Amsterdam, 1973

[5] C. Berge, and P. Duchet, *Recent problem and results about kernels in directed graphs*, Discrete Math., 86(1-3)(1990), 27-31

[6] A. P. Burger, E. J. Cockayne and C. M. Mynhardt, *Domination Numbers for the Queen's graph*, Bull. Inst. Comb. Appl. 10(1994), 73-82

[7] E. J. Cockayne, S. T. Hedetniemi, *Towards a theory of domination in Graphs*, Networks, 7(3)(1977), 247-261

[8] E. J. Cockayne, R. M. Dawes and S. T. Hedetniemi, *Total domination in Graphs*, Networks, 10(3)(1980), 211-219

[9] E. J. Cockayne, C. W. Ko, and F. B. Shepherd, *Inequalities Concerning dominating sets in graphs*, Technical Report DM-370-IR, Dept. Math., Univ. Victoria, 1985

[10] Venkatasubramanian Swaminathan and Kuppusamy Markandan Dharmalingam *Degree Equitable Domination on Graphs*, Kragujevac J. Math., 35 (1)(2011), 191 - 197

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