BULLETIN OF THE INTERNATIONAL MATHEMATICAL VIRTUAL INSTITUTE
ISSN (p) 2303-4874, ISSN (o) 2303-4955
www.imvibl.org /JOURNALS/BULLETIN
Bull. Int. Math. Virtual Inst., **15**(1)(2025), 9–23
DOI: 10.7251/BIMVI25010090

Former
BULLETIN OF THE SOCIETY OF MATHEMATICIANS BANJA LUKA
ISSN 0354-5792 (o), ISSN 1986-521X (p)

ALGEBRAIC FOUNDATIONS OF HESITANT FUZZY SETS IN SHEFFER STROKE BG-ALGEBRAS

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Abstract. This paper introduces and develops the concept of hesitant fuzzy structures within the framework of Sheffer stroke BG-algebras. We focus on defining and analyzing hesitant fuzzy SBG-subalgebras, SBG-ideals, and hesitant fuzzy implicative SBG-ideals, all of which extend classical algebraic structures by incorporating hesitation and uncertainty. By leveraging hesitant fuzzy sets, we explore the algebraic properties and interactions between these structures. Several theorems and propositions are provided to demonstrate the relationships between hesitant fuzzy structures and Sheffer stroke BG-algebras. Our results contribute to the broader understanding of how algebraic systems can model uncertainty, particularly in decision-making and artificial intelligence contexts, where hesitant fuzzy sets offer a more flexible and nuanced approach to handling imprecision.

1. Introduction

The Sheffer stroke, often referred to as the NAND operator, was first introduced by H. M. Sheffer [**15**]. This operation has played a critical role in logic due to its ability to function as the sole operator needed to define an entire logical system, eliminating the necessity for additional logical operators. More specifically, any axiom or theorem within a logical system can be reformulated solely in terms of the Sheffer stroke. This characteristic provides significant advantages, particularly in the simplification of logical systems, offering a streamlined approach to manipulating and understanding various logical properties. Notably, even the axioms of

2020 *Mathematics Subject Classification.* Primary 06F05; Secondary 03G25, 03G10.
Key words and phrases. Sheffer stroke BG-algebra, SBG-ideal, hesitant fuzzy SBG-subalgebra, hesitant fuzzy SBG-ideal.
Communicated by Dusko Bogdanic.

Boolean algebra, which forms the algebraic counterpart of classical propositional logic, can be completely expressed through the Sheffer stroke. This deep connection highlights the Sheffer strokes foundational importance in both logic and algebra [**13**, **14**].

Parallel to these developments in logic, the study of algebraic structures also saw significant advancements with the introduction of BCK-algebras by Imai and Iseki in 1966 [**3**]. Later, in 1980, K. Iseki generalized this concept into BCI-algebras, further enriching the theoretical landscape of algebraic systems [**2**]. Building on this work, J. Neggers and H. S. Kim extended the theory in 2002 by introducing B-algebras, a structure that shares key properties with BCK and BCI-algebras while introducing new elements [**5**]. This evolving line of research culminated in the introduction of BG-algebras by C. B. Kim and H. S. Kim in 2008 [**4**], which generalized the framework of B-algebras. In a BG-algebra, a non-empty set is equipped with a binary operation and a constant, and the structure is defined by axioms that govern its algebraic behavior. The generalization of these algebraic structures provided a versatile framework to study logical systems in new and meaningful ways.

Simultaneously, the emergence of fuzzy set theory, introduced by Zadeh, revolutionized the mathematical treatment of uncertainty by offering a framework that models vague and imprecise information [**21**]. Fuzzy set theory has since evolved, leading to the introduction of hesitant fuzzy sets. These hesitant fuzzy sets extend the traditional fuzzy set theory by allowing multiple possible membership degrees for a single element, making them particularly useful for modeling complex uncertainty. Hesitant fuzzy sets have been successfully applied across various algebraic structures, further expanding the tools available to deal with imprecise information in logical systems. For in-depth studies on hesitant fuzzy sets and their applications, we refer the reader to the following works [**1**, **8**–**12**, **17**–**20**]. The integration of hesitant fuzzy structures into Sheffer stroke BG-algebras represents a promising avenue for new insights into the interplay between algebra and fuzziness.

This manuscript is motivated by the increasing need for robust mathematical frameworks that can model uncertainty, which is prevalent in areas such as decision-making processes, artificial intelligence, and logical systems. Addressing this need, we aim to bridge the gap between hesitant fuzzy theory and algebraic structures, specifically focusing on Sheffer stroke BG-algebras. The main contribution of this paper lies in the introduction and formalization of hesitant fuzzy SBG-subalgebras, SBG-ideals, and implicative SBG-ideals. By extending the theory of hesitant fuzzy sets into Sheffer stroke BG-algebras, we propose novel algebraic structures that effectively incorporate hesitation and uncertainty.

In this manuscript, we will first introduce the necessary background on Sheffer stroke BG-algebras and hesitant fuzzy sets. We will then define and explore the novel concepts of hesitant fuzzy SBG-subalgebras and hesitant fuzzy SBG-ideals, establishing the foundational properties and conditions under which these structures operate. Furthermore, we will extend these ideas to implicative SBG-ideals, offering a detailed examination of their role within the broader algebraic framework. By providing a rigorous theoretical framework for the integration of hesitation into algebraic systems, we aim to contribute not only to the advancement of algebraic logic but also to offer practical tools for applications where managing uncertainty is of paramount importance. Through this exploration, we anticipate that our results will open up new avenues for further research in both mathematical logic and applied fields where uncertainty plays a crucial role.

2. Preliminaries

The concepts introduced are crucial for integrating uncertainty into algebraic systems, particularly through the Sheffer stroke operation and Sheffer stroke BG-algebras, which model logical negation and binary operations. Hesitant fuzzy sets extend these algebraic structures by allowing elements to possess multiple degrees of membership, reflecting uncertainty. These structures are essential for applications in decision-making, artificial intelligence, and fuzzy logic, where precise membership values are often unknown or variable. Thus, these foundational concepts provide the necessary tools to explore the interaction between algebraic operations and hesitation in uncertain environments.

Definition 2.1. *[**15**] Let $\Gamma = \langle \Gamma, | \rangle$ be a groupoid. The operation $|$ is said to be a Sheffer stroke operation if it satisfies the following conditions:*

(S1) $\alpha|\beta = \beta|\alpha$,
(S2) $(\alpha|\alpha)|(\alpha|\beta) = \alpha$,
(S3) $\alpha|((\beta|\gamma)|(\beta|\gamma)) = ((\alpha|\beta)|(\alpha|\beta))|\gamma$
(S4) $(\alpha|((\alpha|\alpha)|(\beta|\beta)))|(\alpha|((\alpha|\alpha)|(\beta|\beta))) = \alpha$.

Definition 2.2. *[**7**] A Sheffer stroke BG-algebra (briefly, SBG-algebra) is a structure $\langle A, | \rangle$ of type (2) such that 0 is the fixed element in H and the following conditions are satisfied for all $\alpha, \beta, \gamma \in A$:*

$(SBG-1)$ $(((\gamma|(\alpha|\alpha))|(\gamma|(\alpha|\alpha)))|(((\beta|(\alpha|\alpha))|(\gamma|(\beta|\beta)))|((\beta|(\alpha|\alpha))|(\gamma|(\beta|\beta)))))|(((\gamma|(\alpha|\alpha))|(\gamma|(\alpha|\alpha)))|(((\beta|(\alpha|\alpha))|(\gamma|(\beta|\beta)))|((\beta|(\alpha|\alpha))|(\gamma|(\beta|\beta))))) = 0$,
$(SBG-2)$ $\alpha|\alpha = \alpha|(0|0)$,
$(SBG-3)$ $(\alpha|(\beta|\beta))|(\alpha|(\beta|\beta)) = 0$ *and* $(\beta|(\alpha|\alpha))|(\beta|(\alpha|\alpha)) = 0 \Rightarrow \alpha = \beta$.

Proposition 2.1. *[**7**] Let $\langle A, | \rangle$ be an SBG-algebra. Then the binary relation $\alpha \supseteq \beta$ if and only if $(\beta|(\alpha|\alpha))|(\beta|(\alpha|\alpha)) = 0$ is a partial order on H.*

Definition 2.3. *[**7**] A nonempty subset G of a Sheffer stroke BG-algebra H is called an SBG-subalgebra of H if $(\alpha|(\beta|\beta))|(\alpha|(\beta|\beta)) \in G$ for all $\alpha, \beta \in G$.*

Definition 2.4. *[**7**] A nonempty subset G of a Sheffer stroke BG-algebra H is called an SBG-ideal of H if:*

(1) $0 \in G$,
(2) $(\beta|(\alpha|\alpha))|(\beta|(\alpha|\alpha)) \in G$ *and* $\alpha \in G \Rightarrow \beta \in G$.

Lemma 2.1. *[**6**] In a Sheffer stroke BG-algebra H, the following property holds:*

$$((\alpha|\alpha^{\beta})|(\alpha|\alpha^{\beta}))|\beta^{\alpha} = \alpha|\alpha^{\beta}$$

for all $\alpha, \beta \in H$.

Definition 2.5. *[**7**] A nonempty subset G of a Sheffer stroke BG-algebra H is called an implicative SBG-ideal of H if:*

(1) $0 \in G$,
(2) $((((\alpha|(\beta|(\alpha|\alpha)))|(\alpha|(\beta|(\alpha|\alpha))))|(\gamma|\gamma))|(((\alpha|(\beta|(\alpha|\alpha)))|(\alpha|(\beta|(\alpha|\alpha))))|(\gamma|\gamma)) \in G$ *and* $\gamma \in G \Rightarrow \alpha \in G$.

Lemma 2.2. *[**7**] Let H be a Sheffer stroke BG-algebra. Then the following properties hold:*

(1) $(\alpha_1|(\alpha_2|\alpha_2))|(\alpha_1|(\alpha_2|\alpha_2)) = (\alpha_3|(\alpha_2|\alpha_2))|(\alpha_3|(\alpha_2|\alpha_2))$ *implies* $\alpha_1 = \alpha_3$,
(2) *If* $(\alpha_1|(\alpha_2|\alpha_2))|(\alpha_1|(\alpha_2|\alpha_2)) = 0$ *then* $\alpha_1 = \alpha_2$,
(3) $(\alpha_1|(\alpha_1|\alpha_1))|(\alpha_1|\alpha_1) = \alpha_1$.

for all $\alpha_1, \alpha_2, \alpha_3 \in A$.

Definition 2.6. *[**6**] A SBG-algebra H is termed implicative if it satisfies the condition:*

$$\alpha|\alpha^{\beta} = \beta|\beta^{\alpha}$$

for all $\alpha, \beta \in H$.

Theorem 2.1. *[**6**] Every medial SBG-algebra is also an implicative SBG-algebra.*

Definition 2.7. *[**6**] A SBG-algebra H is termed medial if it satisfies the condition:*

$$\alpha|\alpha^{\beta} = \beta|\beta,$$

for all $\alpha, \beta \in H$.

Definition 2.8. *[**16**] Let X be a reference set. A hesitant fuzzy set on X is defined in terms of a function h that, when applied to X, returns a subset of* $[0, 1]$, *that is,* $h : X \to P([0, 1])$.

If $Y \subset X$, the characteristic hesitant fuzzy set h_Y on X is a function of X into $P([0, 1])$ defined as follows:

$$h_Y(x) = \begin{cases} [0,1] & \text{if } x \in Y \\ \emptyset & \text{if } x \notin Y. \end{cases}$$

By the definition of characteristic hesitant fuzzy sets, h_Y is a function of X into $\emptyset, [0, 1]$. Hence h_Y is a hesitant fuzzy set on X.

Theorem 2.2. *[**7**] Let $\langle A, |_A\rangle$ and $\langle B, |_B\rangle$ be SBG-algebras. Then $\langle A\times B, |_{A\times B} \rangle$ is also an SBG-algebra, where $A \times B$ represents the Cartesian product of A and B, and the operation $|_{A\times B}$ is defined as $(a_1, b_1) \;|_{A\times B}\; (a_2, b_2) = (a_1 \;|_A\; a_2, b_1 \;|_B\; b_2)$. The fixed element in this algebra is given by $0_{A\times B} = (0_A, 0_B)$.*

3. Hesitant fuzzy sets in Sheffer stroke BG-algebras

In this section, we introduced the concept of hesitant fuzzy sets within the framework of Sheffer stroke BG-algebras (SBG-algebras). Our main focus was to formalize hesitant fuzzy structures such as hesitant fuzzy SBG-subalgebras and hesitant fuzzy SBG-ideals. To enhance readability, we introduced the abbreviation α^β, defined as $\alpha|(\beta|\beta)$, which simplifies the complex expressions involving the Sheffer stroke operation. We then provided the necessary conditions that a hesitant fuzzy set must satisfy to be considered a hesitant fuzzy SBG-subalgebra or a hesitant fuzzy SBG-ideal. These conditions are centered around the behavior of hesitant fuzzy sets in relation to the algebraic elements, and we demonstrated that certain properties, like being a hesitant fuzzy implicative SBG-ideal, imply weaker properties like being a hesitant fuzzy SBG-ideal.

Furthermore, we examined the interplay between different hesitant fuzzy structures within SBG-algebras. Through a series of propositions and theorems, we established that specific types of SBG-algebras, such as implicative and medial SBG-algebras, exhibit particular behaviors with respect to these fuzzy structures. For example, we proved that every hesitant fuzzy implicative SBG-ideal is also a hesitant fuzzy SBG-ideal. These theoretical results deepen the understanding of how hesitant fuzzy sets can be integrated into the algebraic framework of SBG-algebras and reveal important connections between different algebraic properties and their fuzzy counterparts.

Definition 3.1. *A hesitant fuzzy set $\boldsymbol{\psi}$ within the set H is termed a hesitant fuzzy SBG-subalgebra of $\mathcal{H} = (H, |)$ if it satisfies the following condition:*

$$(\forall \alpha, \beta \in H) \left(\ \boldsymbol{\psi}(\alpha^\beta|\alpha^\beta) \supseteq \boldsymbol{\psi}(\alpha) \cap \boldsymbol{\psi}(\beta) \ \right). \tag{3.1}$$

Definition 3.2. *A hesitant fuzzy set $\boldsymbol{\psi}$ on an SBG-algebra H is termed a hesitant fuzzy SBG-ideal of H if the following conditions hold:*

$$(\forall \alpha, \beta \in H) \left(\begin{array}{l} \boldsymbol{\psi}(0) \supseteq \boldsymbol{\psi}(\alpha) \\ \boldsymbol{\psi}(\alpha) \supseteq \boldsymbol{\psi}(\beta) \cap \boldsymbol{\psi}(\alpha^\beta|\alpha^\beta) \end{array} \right). \tag{3.2}$$

Definition 3.3. *A hesitant fuzzy set $\boldsymbol{\psi}$ on an SBG-algebra H is termed a hesitant fuzzy implicative SBG-ideal of H if the following condition holds:*

$$(\forall \alpha, \beta \in H) \left(\ \boldsymbol{\psi}(0) \supseteq \boldsymbol{\psi}(\alpha) \supseteq \boldsymbol{\psi}(((\alpha|\beta^\alpha)|(\alpha|\beta^\alpha))^\gamma|((\alpha|\beta^\alpha)|(\alpha|\beta^\alpha))^\gamma) \cap \boldsymbol{\psi}(\gamma) \ \right). \tag{3.3}$$

Proposition 3.1. *Every hesitant fuzzy implicative SBG-ideal in a SBG-algebra H is also a hesitant fuzzy SBG-ideal of H.*

Proof. Let $\boldsymbol{\psi}$ be a hesitant fuzzy implicative SBG-ideal of a Sheffer stroke BG-algebra H. By definition, we have:

$$\boldsymbol{\psi}(0) \supseteq \boldsymbol{\psi}(\alpha).$$

Additionally,

$$\begin{array}{rcl} \boldsymbol{\psi}(\alpha) & \supseteq & \boldsymbol{\psi}(\beta) \cap \boldsymbol{\psi}((\alpha^\alpha|\alpha^\alpha)^\beta|(\alpha^\alpha|\alpha^\alpha)^\beta) \\ & = & \boldsymbol{\psi}(\beta) \cap \boldsymbol{\psi}((\alpha^0|\alpha^0)^\beta|(\alpha^0|\alpha^0)^\beta) \\ & = & \boldsymbol{\psi}(\beta) \cap \boldsymbol{\psi}(\alpha^\beta|\alpha^\beta). \end{array}$$

Therefore, $\boldsymbol{\psi}$ satisfies the conditions of being a hesitant fuzzy SBG-ideal of H. □

DEFINITION 3.4. *A hesitant fuzzy subset $\boldsymbol{\psi}$ of a SBG-algebra H is termed a hesitant fuzzy sub-implicative SBG-ideal of H if the following conditions are satisfied:*

$$(\forall \alpha, \beta, \gamma \in H)\left(\begin{array}{l} \boldsymbol{\psi}(0) \supseteq \boldsymbol{\psi}(\alpha), \\ \boldsymbol{\psi}((\beta|\beta^{\alpha})|(\beta|\beta^{\alpha})) \supseteq \boldsymbol{\psi}(((\alpha|\alpha^{\beta})|(\alpha|\alpha^{\beta}))^{\gamma}|((\alpha|\alpha^{\beta})|(\alpha|\alpha^{\beta}))^{\gamma}) \cap \boldsymbol{\psi}(\gamma) \end{array}\right). \tag{3.4}$$

PROPOSITION 3.2. *Let H be a SBG-algebra. Then every hesitant fuzzy sub-implicative SBG-ideal in H is also a hesitant fuzzy SBG-ideal of H.*

PROOF. Let $\boldsymbol{\psi}$ be a hesitant fuzzy sub-implicative SBG-ideal of H. By the definition, we have:

$$\boldsymbol{\psi}(0) \supseteq \boldsymbol{\psi}(\alpha),$$

and

$$\begin{array}{rcl} \boldsymbol{\psi}(\alpha) & = & \boldsymbol{\psi}(\alpha^0|\alpha^0) \\ & = & \boldsymbol{\psi}((\alpha|\alpha^{\alpha})|(\alpha|\alpha^{\alpha}) \\ & \supseteq & \boldsymbol{\psi}(((\alpha|\alpha^{\alpha})|(\alpha|\alpha^{\alpha}))^{\gamma}|((\alpha|\alpha^{\alpha})|(\alpha|\alpha^{\alpha}))^{\gamma}) \cap \boldsymbol{\psi}(\gamma) \\ & = & \boldsymbol{\psi}((\alpha^0|\alpha^0)^{\gamma}|(\alpha^0|\alpha^0)^{\gamma}) \cap \boldsymbol{\psi}(\gamma) \\ & = & \boldsymbol{\psi}(\alpha^{\gamma}|\alpha^{\gamma}) \cap \boldsymbol{\psi}(\gamma). \end{array}$$

Therefore, $\boldsymbol{\psi}$ satisfies the conditions of being a hesitant fuzzy SBG-ideal of H. □

THEOREM 3.1. *Let H be a SBG-algebra, and let $\boldsymbol{\psi}$ be a hesitant fuzzy SBG-ideal of H. Then $\boldsymbol{\psi}$ is a hesitant fuzzy sub-implicative SBG-ideal of H if and only if the following relation*

$$\boldsymbol{\psi}((\beta|\beta^{\alpha})|(\beta|\beta^{\alpha})) \supseteq \boldsymbol{\psi}((\alpha|\alpha^{\beta})(\alpha|\alpha^{\beta})).$$

is verified for each $\alpha, \beta \in H$.

PROOF. Let $\boldsymbol{\psi}$ be a hesitant fuzzy sub-implicative SBG-ideal of H. We have:

$$\begin{array}{rcl} \boldsymbol{\psi}((\beta|\beta^{\alpha})|(\beta|\beta^{\alpha})) & \supseteq & \boldsymbol{\psi}(((\alpha|\alpha^{\beta})|(\alpha|\alpha^{\beta}))^0|((\alpha|\alpha^{\beta})|(\alpha|\alpha^{\beta}))^0) \cap \boldsymbol{\psi}(0) \\ & = & \boldsymbol{\psi}(((\alpha|\alpha^{\beta})|(\alpha|\alpha^{\beta}))) \cap \boldsymbol{\psi}(0) \\ & = & \boldsymbol{\psi}(((\alpha|\alpha^{\beta})|(\alpha|\alpha^{\beta}))). \end{array}$$

Conversely, since $\boldsymbol{\psi}$ is a hesitant fuzzy SBG-ideal, we know that:

$$\boldsymbol{\psi}(0) \supseteq \boldsymbol{\psi}(\alpha),$$

and

$$\begin{array}{rcl} \boldsymbol{\psi}((\beta|\beta^{\alpha})|(\beta|\beta^{\alpha})) & \supseteq & \boldsymbol{\psi}(((\alpha|\alpha^{\beta})|(\alpha|\alpha^{\beta}))) \\ & \supseteq & \boldsymbol{\psi}(((\alpha|\alpha^{\beta})|(\alpha|\alpha^{\beta}))^{\gamma}|((\alpha|\alpha^{\beta})|(\alpha|\alpha^{\beta}))^{\gamma}) \cap \boldsymbol{\psi}(\gamma). \end{array}$$

Therefore, $\boldsymbol{\psi}$ is a hesitant fuzzy sub-implicative SBG-ideal of H. □

THEOREM 3.2. *Let H be an implicative SBG-algebra. Then every fuzzy SBG-ideal of H is also a hesitant fuzzy sub-implicative SBG-ideal of H.*

Proof. Let $\boldsymbol{\psi}$ be a hesitant fuzzy SBG-ideal of H. Then, by definition, we have:

$$\boldsymbol{\psi}(0) \supseteq \boldsymbol{\psi}(\alpha),$$

and

$$\begin{aligned}\boldsymbol{\psi}((\beta|\beta^{\alpha})|(\beta|\beta^{\alpha})) &\supseteq \boldsymbol{\psi}((((\beta|\beta^{\alpha})|(\beta|\beta^{\alpha}))^{\gamma}|((\beta|\beta^{\alpha})|(\beta|\beta^{\alpha}))^{\gamma}) \cap \boldsymbol{\psi}(\gamma)\\ &= \boldsymbol{\psi}((((\alpha|\alpha^{\beta})|(\alpha|\alpha^{\beta}))^{\gamma}|((\alpha|\alpha^{\beta})|(\alpha|\alpha^{\beta}))^{\gamma}) \cap \boldsymbol{\psi}(\gamma).\end{aligned}$$

Therefore, $\boldsymbol{\psi}$ is a hesitant fuzzy sub-implicative SBG-ideal of H. □

Theorem 3.3. *Every hesitant fuzzy SBG-ideal in a medial Sheffer stroke BG-algebra H is also a hesitant fuzzy sub-implicative SBG-ideal of H.*

Proof. Let $\boldsymbol{\psi}$ be a hesitant fuzzy SBG-ideal of a medial Sheffer stroke BG-algebra H. Then, we have:

$$\boldsymbol{\psi}(0) \supseteq \boldsymbol{\psi}(\alpha),$$

and

$$\begin{aligned}\boldsymbol{\psi}((\beta|\beta^{\alpha})|(\beta|\beta^{\alpha})) &= \boldsymbol{\psi}(\alpha)\\ &\supseteq \boldsymbol{\psi}(\alpha^{\gamma}|\alpha^{\gamma}) \cap \boldsymbol{\psi}(\gamma)\\ &= \boldsymbol{\psi}((((\alpha|\alpha^{\beta})|(\alpha|\alpha^{\beta}))^{\gamma}|((\alpha|\alpha^{\beta})|(\alpha|\alpha^{\beta}))^{\gamma}) \cap \boldsymbol{\psi}(\gamma)\\ &= \boldsymbol{\psi}(((((\alpha|\alpha^{\beta})|(\alpha|\alpha^{\beta}))|\beta^{\alpha})^{\gamma}|(((\alpha|\alpha^{\beta})|(\alpha|\alpha^{\beta}))|\beta^{\alpha})^{\gamma}) \cap \boldsymbol{\psi}(\gamma)\\ &= \boldsymbol{\psi}((((\alpha|\alpha^{\beta})|(\alpha|\alpha^{\beta}))^{\gamma}|((\alpha|\alpha^{\beta})|(\alpha|\alpha^{\beta}))^{\gamma}) \cap \boldsymbol{\psi}(\gamma).\end{aligned}$$

Hence, $\boldsymbol{\psi}$ is a hesitant fuzzy sub-implicative SBG-ideal of H. □

Theorem 3.4. *Let H be a Sheffer stroke BG-algebra that satisfies the following condition:*

$$(\forall \alpha, \beta, \gamma \in H)\left(\boldsymbol{\psi}(\beta^{\gamma}|\beta^{\gamma}) \supseteq \boldsymbol{\psi}((((\alpha|\alpha^{\beta})|(\alpha|\alpha^{\beta}))^{\gamma}|((\alpha|\alpha^{\beta})|(\alpha|\alpha^{\beta}))^{\gamma}) \right). \tag{3.5}$$

Then every hesitant fuzzy SBG-ideal of H is also a hesitant fuzzy sub-implicative SBG-ideal of H.

Proof. We derive that

$$\begin{aligned}\boldsymbol{\psi}((\beta|\beta^{\alpha})|(\beta|\beta^{\alpha})) &\supseteq \boldsymbol{\psi}(((((\alpha|\alpha^{\beta})|(\alpha|\alpha^{\beta}))|\beta^{\alpha})|(((\alpha|\alpha^{\beta})|(\alpha|\alpha^{\beta}))|\beta^{\alpha}))\\ &= \boldsymbol{\psi}(((\alpha|\alpha^{\beta})|(\alpha|\alpha^{\beta})))\end{aligned}$$

for each $\alpha, \beta \in H$. Therefore, $\boldsymbol{\psi}$ is a hesitant fuzzy sub-implicative SBG-ideal of H. □

Theorem 3.5. *Let $\boldsymbol{\psi}$ be a hesitant fuzzy SBG-ideal of a SBG-algebra H. Then $\boldsymbol{\psi}$ is a hesitant fuzzy SBG-implicative ideal of H if and only if it satisfies the following condition:*

$$(\forall \alpha, \beta \in H)\left(\boldsymbol{\psi}(\alpha) \supseteq \boldsymbol{\psi}((\alpha|\beta^{\alpha})|(\alpha|\beta^{\alpha})) \right). \tag{3.6}$$

Proof. Let $\boldsymbol{\psi}$ be a hesitant fuzzy SBG-implicative ideal of H. Then, we have:

$$\begin{aligned}\boldsymbol{\psi}(\alpha) &\supseteq \boldsymbol{\psi}(0)\cap\boldsymbol{\psi}(((\alpha|\beta^{\alpha})|(\alpha|\beta^{\alpha}))^{0}|((\alpha|\beta^{\alpha})|(\alpha|\beta^{\alpha}))^{0}) \\ &= \boldsymbol{\psi}(0)\cap\boldsymbol{\psi}((\alpha|\beta^{\alpha})|(\alpha|\beta^{\alpha})) \\ &= \boldsymbol{\psi}((\alpha|\beta^{\alpha})|(\alpha|\beta^{\alpha})),\end{aligned}$$

for all $\alpha,\beta\in H$.

Conversely, let $\boldsymbol{\psi}$ be a hesitant fuzzy SBG-ideal of H satisfying the inequality (3.6). Then, we get $\boldsymbol{\psi}(0)\supseteq\boldsymbol{\psi}(\alpha)$ for all $\alpha\in H$. Since:

$$\begin{aligned}\boldsymbol{\psi}(\alpha) &\supseteq \boldsymbol{\psi}((\alpha|\beta^{\alpha})|(\alpha|\beta^{\alpha})) \\ &\supseteq \boldsymbol{\psi}(((\alpha|\beta^{\alpha})|(\alpha|\beta^{\alpha}))^{\gamma}|((\alpha|\beta^{\alpha})|(\alpha|\beta^{\alpha}))^{\gamma})\cap\boldsymbol{\psi}(\gamma)\end{aligned}$$

for all $\alpha,\beta,\gamma\in H$, it follows that $\boldsymbol{\psi}$ is a hesitant fuzzy SBG-implicative ideal of H. □

Theorem 3.6. *Let H be a medial SBG-algebra that satisfies the following condition:*

$$(\forall\alpha,\beta\in H)\left(\ \boldsymbol{\psi}((\alpha|\alpha^{\beta})|(\alpha|\alpha^{\beta}))\supseteq\boldsymbol{\psi}((\alpha|\beta^{\alpha})|(\alpha|\beta^{\alpha}))\ \right). \tag{3.7}$$

Then every hesitant fuzzy sub-implicative SBG-ideal of H is also a hesitant fuzzy SBG-implicative ideal of H.

Proof. Let $\boldsymbol{\psi}$ be a hesitant fuzzy sub-implicative SBG-ideal of a medial Sheffer stroke BG-algebra H that satisfies the inequality (3.7). Then, we get

$$\begin{aligned}\boldsymbol{\psi}(\alpha) &= \boldsymbol{\psi}((\beta|\beta^{\alpha})|(\beta|\beta^{\alpha})) \\ &\supseteq \boldsymbol{\psi}(((\alpha|\alpha^{\beta})|(\alpha|\alpha^{\beta}))^{0}|((\alpha|\alpha^{\beta})|(\alpha|\alpha^{\beta}))^{0})\cap\boldsymbol{\psi}(0) \\ &= \boldsymbol{\psi}((\alpha|\alpha^{\beta})|(\alpha|\alpha^{\beta}))\cap\boldsymbol{\psi}(0) \\ &= \boldsymbol{\psi}((\alpha|\alpha^{\beta})|(\alpha|\alpha^{\beta})) \\ &\supseteq \boldsymbol{\psi}((\alpha|\beta^{\alpha})|(\alpha|\beta^{\alpha})).\end{aligned}$$

for each $\alpha,\beta\in H$. Therefore, $\boldsymbol{\psi}$ is a hesitant fuzzy SBG-implicative ideal of H. □

Theorem 3.7. *Let H be an implicative SBG-algebra. Then every hesitant fuzzy SBG-implicative ideal in H is also a hesitant fuzzy sub-implicative SBG-ideal of H.*

Proof. Let $\boldsymbol{\psi}$ be a hesitant fuzzy SBG-implicative ideal of an implicative SBG-algebra H. Since $\boldsymbol{\psi}$ is also a hesitant fuzzy SBG-ideal of H, it is clear that $\boldsymbol{\psi}(0)\supseteq\boldsymbol{\psi}(\alpha)$ for all $\alpha\in H$. Thus, we have:

$$\begin{aligned}\boldsymbol{\psi}((\beta|\beta^{\alpha})|(\beta|\beta^{\alpha})) &= \boldsymbol{\psi}((\alpha|\alpha^{\beta})|(\alpha|\alpha^{\beta})) \\ &\supseteq \boldsymbol{\psi}(((\alpha|\alpha^{\beta})|(\alpha|\alpha^{\beta}))^{\gamma}|((\alpha|\alpha^{\beta})|(\alpha|\alpha^{\beta}))^{\gamma})\cap\boldsymbol{\psi}(\gamma)\end{aligned}$$

for all $\alpha,\beta,\gamma\in H$. Hence, $\boldsymbol{\psi}$ is a hesitant fuzzy sub-implicative SBG-ideal of H. □

Corollary 3.1. *Let H be a medial Sheffer stroke BG-algebra. Then every hesitant fuzzy SBG-implicative ideal in H is also a hesitant fuzzy sub-implicative SBG-ideal of H.*

Definition 3.5. *A hesitant fuzzy SBG-ideal* $\boldsymbol{\psi}$ *of a Sheffer stroke BG-algebra* H *is called hesitant fuzzy closed if it satisfies the following condition:*

$$(\forall \alpha \in H)\left(\ \boldsymbol{\psi}(0^{\alpha}|0^{\alpha}) \supseteq \boldsymbol{\psi}(\alpha) \ \right).$$

Definition 3.6. *Let* $\boldsymbol{\psi}$ *be a hesitant fuzzy SBG-ideal of a Sheffer stroke BG-algebra* H*. Then* $\boldsymbol{\psi}$ *is called a hesitant fuzzy completely closed ideal of* H *if it satisfies the following condition:*

$$(\forall \alpha, \beta \in H)\left(\ \boldsymbol{\psi}(\alpha^{\beta}|\alpha^{\beta}) \supseteq \boldsymbol{\psi}(\alpha) \cap \boldsymbol{\psi}(\beta) \ \right).$$

Theorem 3.8. *Let* H *be a Sheffer stroke BG-algebra that satisfies the following condition:*

$$(\forall \alpha, \beta, \gamma \in H)\left(\ (((\alpha^{\beta}|\alpha^{\beta})|\alpha^{\gamma})|((\alpha^{\beta}|\alpha^{\beta})|\alpha^{\gamma}))|\alpha^{\beta} = 0|0. \ \right) \tag{3.8}$$

In this case, H *is implicative if and only if each hesitant fuzzy closed ideal of* H *is a hesitant fuzzy SBG-implicative ideal.*

Proof. Let H be a Sheffer stroke BG-algebra that satisfies equation (3.8). Assume H is implicative and that $\boldsymbol{\psi}$ is a hesitant fuzzy closed ideal of H. Since $\boldsymbol{\psi}$ is a hesitant fuzzy SBG-ideal, it follows that $\boldsymbol{\psi}(0) \supseteq \boldsymbol{\psi}(\alpha)$. Additionally, we have:

$$\begin{aligned}
\boldsymbol{\psi}(\alpha) &\supseteq \boldsymbol{\psi}(\gamma) \cap \boldsymbol{\psi}(\alpha^{\gamma}|\alpha^{\gamma}) \\
&= \boldsymbol{\psi}(\gamma) \cap \boldsymbol{\psi}(((\alpha|\alpha)^{\alpha}|(\alpha|\beta))^{\gamma}|((\alpha|\alpha)^{\alpha}|(\alpha|\beta))^{\gamma}) \\
&= \boldsymbol{\psi}(\gamma) \cap \boldsymbol{\psi}(((\alpha|\beta^{\alpha})|(\alpha|\beta^{\alpha}))^{\gamma}|((\alpha|\beta^{\alpha})|(\alpha|\beta^{\alpha}))^{\gamma})
\end{aligned}$$

which implies that $\boldsymbol{\psi}$ is a hesitant fuzzy SBG-implicative ideal of H.

Conversely, assume every hesitant fuzzy closed ideal of H is a hesitant fuzzy SBG-implicative ideal. From equation (3.8), using Definition 2.1 (S1)-(S2) and Lemma 2.2 (2), we derive that $\gamma^{\beta} = \alpha^{\gamma}|(\alpha^{\beta}|\alpha^{\beta})$. Since we have

$$\gamma^{\beta} = \alpha^{\gamma}|(\alpha^{\beta}|\alpha^{\beta}) = ((\alpha|\alpha^{\gamma})|(\alpha|\alpha^{\gamma}))^{\beta},$$

it follows from Definition 2.1 (S1) and (S3) that $\gamma = (\alpha|\alpha^{\gamma})|(\alpha|\alpha^{\gamma})$.

Thus, from Definition 2.1 (S1)-(S3) and Lemma 2.2 (3), we get:

$$\begin{aligned}
\alpha|\alpha^{\beta} &= ((\beta|\beta^{\alpha})|(\beta|\beta^{\alpha}))|((\beta|\beta^{\alpha})|(\beta|\beta^{\alpha}))^{\beta} \\
&= ((\beta|\beta^{\alpha})|(\beta|\beta^{\alpha}))|(\beta|(((\beta|\beta)|\beta^{\alpha})|((\beta|\beta)|\beta^{\alpha}))) \\
&= ((\beta|\beta^{\alpha})|(\beta|\beta^{\alpha}))|\beta^{\beta} \\
&= ((\beta^{\beta}|(\beta|\beta)^{\beta})|(\beta^{\beta}|(\beta|\beta)^{\beta}))|\beta^{\alpha} \\
&= \beta|\beta^{\alpha},
\end{aligned}$$

for all $\alpha, \beta \in H$, which confirms that H is implicative. □

Proposition 3.3. *Let* H *be an implicative Sheffer stroke BG-algebra that satisfies equation* (3.8)*. Then, every hesitant fuzzy completely closed ideal of* H *is also a hesitant fuzzy SBG-implicative ideal of* H*.*

Proof. Let $\boldsymbol{\psi}$ be a hesitant fuzzy completely closed ideal in an implicative Sheffer stroke BG-algebra H. Then $\boldsymbol{\psi}$ is also a hesitant fuzzy SBG-ideal of H. Since $\boldsymbol{\psi}(0^{\beta}|0^{\beta}) \supseteq \boldsymbol{\psi}(0) \cap \boldsymbol{\psi}(\beta) = \boldsymbol{\psi}(\beta)$, it follows that $\boldsymbol{\psi}$ is a hesitant fuzzy closed ideal of H. Consequently, $\boldsymbol{\psi}$ is a hesitant fuzzy SBG-implicative ideal of H. □

Corollary 3.2. *Let H be a medial Sheffer stroke BG-algebra that satisfies equation (3.8). In this case, every hesitant fuzzy completely closed ideal of H is also a hesitant fuzzy SBG-implicative ideal of H.*

Definition 3.7. *A hesitant fuzzy set $\boldsymbol{\psi}$ on a SBG-algebra H is defined as a hesitant fuzzy p-ideal of H if, for all $\alpha, \beta, \gamma \in H$,*

$$\left(\begin{array}{l} \boldsymbol{\psi}(0) \supseteq \boldsymbol{\psi}(\alpha), \\ \boldsymbol{\psi}(((\alpha^{\gamma}|\alpha^{\gamma})|\beta^{\gamma})|((\alpha^{\gamma}|\alpha^{\gamma})|\beta^{\gamma})) \cap \boldsymbol{\psi}(\beta). \end{array} \right) \tag{3.9}$$

Definition 3.8. *[6] Let H be a Sheffer stroke BG-algebra. The set*

$$A_+ = \{\alpha \in A : 0^{\alpha}|0^{\alpha} = 0\}$$

is referred to as the BCA-part of H.

Theorem 3.9. *Let $A = A_+$ be an SBG-algebra. In this case, every hesitant fuzzy p-ideal of H is also a hesitant fuzzy SBG-implicative ideal of H.*

Proof. Let $\boldsymbol{\psi}$ be a hesitant fuzzy p-ideal of H. Then, we obtain the follows:

$$\begin{aligned} \boldsymbol{\psi}(\alpha) &\supseteq \boldsymbol{\psi}((((\alpha|\beta^{\alpha})|(\alpha|\beta^{\alpha}))|(0|\beta^{\alpha}))|(((\alpha|\beta^{\alpha})|(\alpha|\beta^{\alpha}))|(0|\beta^{\alpha}))) \cap \boldsymbol{\psi}(0) \\ &= \boldsymbol{\psi}(((\alpha|\beta^{\alpha})|(\alpha|\beta^{\alpha}))^{0}|((\alpha|\beta^{\alpha})|(\alpha|\beta^{\alpha}))^{0}) \cap \boldsymbol{\psi}(0) \\ &= \boldsymbol{\psi}((\alpha|\beta^{\alpha})|(\alpha|\beta^{\alpha})) \cap \boldsymbol{\psi}(0) \\ &= \boldsymbol{\psi}((\alpha|\beta^{\alpha})|(\alpha|\beta^{\alpha})). \end{aligned}$$

Therefore, $\boldsymbol{\psi}$ is a hesitant fuzzy SBG-implicative ideal of H. □

Lemma 3.1. *The constant 0 of H belongs to a nonempty subset B of H if and only if $\boldsymbol{\psi}_B(0) \supseteq \boldsymbol{\psi}_B(\alpha)$ for all $\alpha \in H$.*

Proof. Assume $0 \in B$. In this case, $\boldsymbol{\psi}_B(0) = [0,1]$. Therefore, $\boldsymbol{\psi}_B(0) = [0,1] \supseteq \boldsymbol{\psi}_B(\alpha)$ for all $\alpha \in H$.

Conversely, suppose that $\boldsymbol{\psi}_B(0) \supseteq \boldsymbol{\psi}_B(\alpha)$ for every $\alpha \in H$. Since B is nonempty, there exists some element $a \in B$. Thus, $\boldsymbol{\psi}_B(0) \supseteq \boldsymbol{\psi}_B(a) = [0,1]$, which implies that $\boldsymbol{\psi}_B(0) = [0,1]$. Hence, we attain that $0 \in B$. □

Theorem 3.10. *A nonempty subset S of H is an SBG-subalgebra of H if and only if the characteristic hesitant fuzzy set $\boldsymbol{\psi}_S$ is a hesitant fuzzy SBG-subalgebra of H.*

Proof. Assume that S is an SBG-subalgebra of H. Let $\alpha, \beta \in H$. We examine the following cases:

- *Case 1* : Suppose $\alpha, \beta \in S$. Then $\boldsymbol{\psi}_S(\alpha) = [0,1]$ and $\boldsymbol{\psi}_S(\beta) = [0,1]$, so $\boldsymbol{\psi}_S(\alpha) \cap \boldsymbol{\psi}_S(\beta) = [0,1]$. Since S is an SBG-subalgebra, we know that $\alpha^{\beta}|\alpha^{\beta} \in S$, which gives $\boldsymbol{\psi}_S(\alpha^{\beta}|\alpha^{\beta}) = [0,1]$. Thus, $\boldsymbol{\psi}_S(\alpha^{\beta}|\alpha^{\beta}) = [0,1] \supseteq [0,1] = \boldsymbol{\psi}_S(\alpha) \cap \boldsymbol{\psi}_S(\beta)$.
- *Case 2*: Suppose $\alpha \in S$ and $\beta \notin S$. Then $\boldsymbol{\psi}_S(\alpha) = [0,1]$ and $\boldsymbol{\psi}_S(\beta) = \emptyset$, so $\boldsymbol{\psi}_S(\alpha) \cap \boldsymbol{\psi}_S(\beta) = [0,1]$. Hence, $\boldsymbol{\psi}_S(\alpha^{\beta}|\alpha^{\beta}) \supseteq [0,1] = \boldsymbol{\psi}_S(\alpha) \cap \boldsymbol{\psi}_S(\beta)$.
- *Case 3*: Suppose $\alpha \notin S$ and $\beta \in S$. Then $\boldsymbol{\psi}_S(\alpha) = \emptyset$ and $\boldsymbol{\psi}_S(\beta) = [0,1]$, giving $\boldsymbol{\psi}_S(\alpha) \cap \boldsymbol{\psi}_S(\beta) = [0,1]$. Thus, $\boldsymbol{\psi}_S(\alpha^{\beta}|\alpha^{\beta}) \supseteq [0,1] = \boldsymbol{\psi}_S(\alpha) \cap \boldsymbol{\psi}_S(\beta)$.
- *Case 4*: Suppose $\alpha \notin S$ and $\beta \notin S$. Then $\boldsymbol{\psi}_S(\alpha) = \emptyset$ and $\boldsymbol{\psi}_S(\beta) = \emptyset$, so $\boldsymbol{\psi}_S(\alpha) \cap \boldsymbol{\psi}_S(\beta) = \emptyset$. Therefore, $\boldsymbol{\psi}_S(\alpha^{\beta}|\alpha^{\beta}) \supseteq \emptyset = \boldsymbol{\psi}_S(\alpha) \cap \boldsymbol{\psi}_S(\beta)$.

Thus, $\boldsymbol{\psi}_S$ is a hesitant fuzzy SBG-subalgebra of H.

Conversely, suppose that $\boldsymbol{\psi}_S$ is a hesitant fuzzy SBG-subalgebra of H. Let $\alpha, \beta \in S$. Then $\boldsymbol{\psi}_S(\alpha) = [0,1]$ and $\boldsymbol{\psi}_S(\beta) = [0,1]$, so $\boldsymbol{\psi}_S(\alpha^\beta|\alpha^\beta) \supseteq \boldsymbol{\psi}_S(\alpha) \cap \boldsymbol{\psi}_S(\beta) = [0,1]$, which implies that $\boldsymbol{\psi}_S(\alpha^\beta|\alpha^\beta) = [0,1]$. Hence, $\alpha^\beta|\alpha^\beta \in S$, and so S is an SBG-subalgebra of H. □

Theorem 3.11. *A nonempty subset D of H is an SBG-ideal of H if and only if its characteristic hesitant fuzzy set $\boldsymbol{\psi}_D$ is a hesitant fuzzy SBG-ideal of H.*

Proof. Assume that D is an SBG-ideal of H. Since $0 \in D$, it follows from Lemma 3.1 that $\boldsymbol{\psi}_D(0) \supseteq \boldsymbol{\psi}_D(\alpha)$ for all $\alpha \in H$. Now, let $\alpha, \beta \in H$.

- *Case 1*: Suppose $\alpha^\beta|\alpha^\beta, \beta \in D$. Then $\boldsymbol{\psi}_D(\alpha^\beta|\alpha^\beta) = [0,1]$ and $\boldsymbol{\psi}_D(\beta) = [0,1]$. Therefore, $\boldsymbol{\psi}_D(\alpha) \supseteq \boldsymbol{\psi}_D(\alpha^\beta|\alpha^\beta) \cap \boldsymbol{\psi}_D(\beta)$.
- *Case 2*: Suppose $\alpha^\beta|\alpha^\beta \notin D$ and $\beta \in D$. Then $\boldsymbol{\psi}_D(\alpha^\beta|\alpha^\beta) = \emptyset$ and $\boldsymbol{\psi}_D(\beta) = [0,1]$. Thus, $\boldsymbol{\psi}_D(\alpha) \supseteq [0,1] = \boldsymbol{\psi}_D(\alpha^\beta|\alpha^\beta) \cap \boldsymbol{\psi}_D(\beta)$.
- *Case 3*: Suppose $\alpha^\beta|\alpha^\beta \in D$ and $\beta \notin D$. Then $\boldsymbol{\psi}_D(\alpha^\beta|\alpha^\beta) = [0,1]$ and $\boldsymbol{\psi}_D(\beta) = \emptyset$. Thus, $\boldsymbol{\psi}_D(\alpha) \supseteq [0,1] = \boldsymbol{\psi}_D(\alpha^\beta|\alpha^\beta) \cap \boldsymbol{\psi}_D(\beta)$.
- *Case 4*: Suppose $\alpha^\beta|\alpha^\beta \notin D$ and $\beta \notin D$. Then $\boldsymbol{\psi}_D(\alpha^\beta|\alpha^\beta) = \emptyset$ and $\boldsymbol{\psi}_D(\beta) = \emptyset$. Thus, $\boldsymbol{\psi}_D(\alpha) \supseteq \emptyset = \boldsymbol{\psi}_D(\alpha^\beta|\alpha^\beta) \cap \boldsymbol{\psi}_D(\beta)$.

Hence, $\boldsymbol{\psi}_D$ is a hesitant fuzzy SBG-ideal of H.

Conversely, assume that $\boldsymbol{\psi}_D$ is a hesitant fuzzy SBG-ideal of H. Since $\boldsymbol{\psi}_D(0) \supseteq \boldsymbol{\psi}_D(\alpha)$ for all $\alpha \in A$, it follows from Lemma 3.1 that $0 \in D$. Now, let $\alpha, \beta \in A$ such that $\alpha^\beta|\alpha^\beta \in D$ and $\beta \in D$. Then $\boldsymbol{\psi}_D(\alpha^\beta|\alpha^\beta) = [0,1]$ and $\boldsymbol{\psi}_D(\beta) = [0,1]$. Thus, $\boldsymbol{\psi}_D(\alpha) \supseteq \boldsymbol{\psi}_D(\alpha^\beta|\alpha^\beta) \cap \boldsymbol{\psi}_D(\beta) = [0,1]$, so $\boldsymbol{\psi}_D(\alpha) = [0,1]$. Therefore, $\alpha \in D$, and hence D is an SBG-ideal of H. □

Theorem 3.12. *Let $\langle A, |_A\rangle$ and $\langle B, |_B\rangle$ be SBG-algebras. If $\boldsymbol{\psi}_A$ and $\boldsymbol{\psi}_B$ are hesitant fuzzy SBG-subalgebras of $\langle A, |_A\rangle$ and $\langle B, |_B\rangle$, respectively, then $\boldsymbol{\psi}_{A\times B}$ is a hesitant fuzzy SBG-subalgebra of $\langle A \times B, |_{A\times B}\rangle$.*

Proof. Let $\langle A, |_A\rangle$ and $\langle B, |_B\rangle$ be SBG-algebras, and let $\boldsymbol{\psi}_A$ and $\boldsymbol{\psi}_B$ be hesitant fuzzy SBG-subalgebras of $\langle A, |_A\rangle$ and $\langle B, |_B\rangle$, respectively. Consider any elements $(\alpha_1, \beta_1), (\alpha_2, \beta_2) \in A \times B$. We have:

$$\begin{aligned}
\boldsymbol{\psi}_{A\times B}((\alpha_1,\beta_1)^{(\alpha_2,\beta_2)}|_{A\times B}(\alpha_1,\beta_1)^{(\alpha_2,\beta_2)}) &= \boldsymbol{\psi}_{A\times B}(\alpha_1^{\alpha_2}|_A\alpha_1^{\alpha_2}, \beta_1^{\beta_2}|_B\beta_1^{\beta_2}) \\
&= \boldsymbol{\psi}_A(\alpha_1^{\alpha_2}|_A\alpha_1^{\alpha_2}) \cap \boldsymbol{\psi}_B(\beta_1^{\beta_2}|_B\beta_1^{\beta_2}) \\
&\supseteq \boldsymbol{\psi}_A(\alpha_1) \cap \boldsymbol{\psi}_A(\alpha_2) \cap \boldsymbol{\psi}_B(\beta_1) \cap \boldsymbol{\psi}_B(\beta_2) \\
&= \boldsymbol{\psi}_{A\times B}(\alpha_1,\beta_1) \cap \boldsymbol{\psi}_{A\times B}(\alpha_2,\beta_2).
\end{aligned}$$

Thus, $\boldsymbol{\psi}_{A\times B}$ is a hesitant fuzzy SBG-subalgebra of $\langle A \times B, |_{A\times B}\rangle$. □

Theorem 3.13. *Let $\langle A, |_A\rangle$ and $\langle B, |_B\rangle$ be SBG-algebras. If $\boldsymbol{\psi}_A$ and $\boldsymbol{\psi}_B$ are hesitant fuzzy SBG-ideals of $\langle A, |_A\rangle$ and $\langle B, |_B\rangle$, respectively, then $\boldsymbol{\psi}_{A\times B}$ is a hesitant fuzzy SBG-ideal of $\langle A \times B, |_{A\times B}\rangle$.*

Proof. Let $\langle A, |_A\rangle$ and $\langle B, |_B\rangle$ be SBG-algebras, and let $\boldsymbol{\psi}_A$ and $\boldsymbol{\psi}_B$ be hesitant fuzzy SBG-ideals of $\langle A, |_A\rangle$ and $\langle B, |_B\rangle$, respectively. Consider any $(a_1, b_1) \in$

$A \times B$. Then:

$$\begin{aligned}\boldsymbol{\psi}_{A\times B}(0_A, 0_B) &= \boldsymbol{\psi}_A(0_A) \cap \boldsymbol{\psi}_B(0_B) \\ &\supseteq \boldsymbol{\psi}_A(\alpha_1) \cap \boldsymbol{\psi}_B(\beta_1) \\ &= \boldsymbol{\psi}_{A\times B}(\alpha_1, \beta_1),\end{aligned}$$

Now consider $\boldsymbol{\psi}_{A\times B}(\alpha_1, \beta_1)$:

$$\begin{aligned}\boldsymbol{\psi}_{A\times B}(\alpha_1, \beta_1) &= \boldsymbol{\psi}_A(\alpha_1) \cap \boldsymbol{\psi}_B(\beta_1) \\ &\supseteq \boldsymbol{\psi}_A(\alpha_2) \cap \boldsymbol{\psi}_A(\alpha_1^{\alpha_2}|_A\alpha_1^{\alpha_2}) \cap \boldsymbol{\psi}_B(\beta_2) \cap \boldsymbol{\psi}_B(\beta_1^{\beta_2}|_B\beta_1^{\beta_2}) \\ &= \boldsymbol{\psi}_A(\alpha_2) \cap \boldsymbol{\psi}_B(\beta_2) \cap \boldsymbol{\psi}_A(\alpha_1^{\alpha_2}|_A\alpha_1^{\alpha_2}) \cap \boldsymbol{\psi}_B(\beta_1^{\beta_2}|_B\beta_1^{\beta_2}) \\ &= \boldsymbol{\psi}_{A\times B}(\alpha_2, \beta_2) \cap \boldsymbol{\psi}_{A\times B}((\alpha_1, \beta_1)^{(\alpha_2,\beta_2)}|_{A\times B}(\alpha_1, \beta_1)^{(\alpha_2,\beta_2)})\end{aligned}$$

Thus, $\boldsymbol{\psi}_{A\times B}$ is a hesitant fuzzy SBG-ideal of $\langle A \times B, |_{A\times B}\rangle$. □

THEOREM 3.14. *Let $\langle A, |_A\rangle$ and $\langle B, |_B\rangle$ be SBG-algebras. If $\boldsymbol{\psi}_A$ and $\boldsymbol{\psi}_B$ are hesitant fuzzy implicative SBG-ideals of $\langle A, |_A\rangle$ and $\langle B, |_B\rangle$, respectively, then $\boldsymbol{\psi}_{A\times B}$ is a hesitant implicative fuzzy SBG-ideal of $\langle A \times B, |_{A\times B}\rangle$.*

PROOF. The proof follows in a similar manner to that of Theorem 3.13. □

THEOREM 3.15. *Let $\boldsymbol{\psi}$ be a hesitant fuzzy SBG-ideal of an SBG-algebra H. Then the subset $I = \{\alpha \in H : \boldsymbol{\psi}(\alpha) = \boldsymbol{\psi}(0)\}$ is an SBG-ideal of H.*

PROOF. Let $\boldsymbol{\psi}$ be a hesitant fuzzy SBG-ideal of H, and let $I = \{\alpha \in H : \boldsymbol{\psi}(\alpha) = \boldsymbol{\psi}(0)\}$ be a subset of H. Clearly, $0 \in I$. Now, suppose $\alpha, \beta \in H$ are such that $\alpha^\beta|\alpha^\beta$ and $\beta \in I$. Since $\boldsymbol{\psi}(\alpha^\beta|\alpha^\beta) = \boldsymbol{\psi}(0)$ and $\boldsymbol{\psi}(\beta) = \boldsymbol{\psi}(0)$ for any $\alpha, \beta \in H$, we have $\boldsymbol{\psi}(0) \supseteq \boldsymbol{\psi}(\alpha) \supseteq \boldsymbol{\psi}(\alpha^\beta|\alpha^\beta) \cap \boldsymbol{\psi}(\beta) = \boldsymbol{\psi}(0) \cap \boldsymbol{\psi}(0) = \boldsymbol{\psi}(0)$.

Therefore, $\boldsymbol{\psi}(\alpha) = \boldsymbol{\psi}(0)$, implying $\alpha \in I$. Hence, I is an SBG-ideal of H. □

THEOREM 3.16. *Let $\boldsymbol{\psi}$ be a hesitant fuzzy implicative SBG-ideal of an SBG-algebra H. Then the subset $I = \{\alpha \in H : \boldsymbol{\psi}(\alpha) = \boldsymbol{\psi}(0)\}$ is an implicative SBG-ideal of H.*

PROOF. Let $\boldsymbol{\psi}$ be a hesitant fuzzy implicative SBG-ideal of H, and let $I = \{\alpha \in H : \boldsymbol{\psi}(\alpha) = \boldsymbol{\psi}(0)\}$ be a subset of H. It is clear that $0 \in I$. Now, suppose $\alpha, \beta, \gamma \in H$ such that $((\alpha|\beta^\alpha)|(\alpha|\beta^\alpha))^\gamma|((\alpha|\beta^\alpha)|(\alpha|\beta^\alpha))^\gamma$, with $\gamma \in I$. Since $\boldsymbol{\psi}(((((\alpha \mid (\beta \mid (\alpha \mid \alpha))) \mid (\alpha \mid (\beta \mid (\alpha \mid \alpha)))) \mid (\gamma \mid \gamma)) \mid (((\alpha \mid (\beta \mid (\alpha \mid \alpha))) \mid (\alpha \mid (\beta \mid (\alpha \mid \alpha)))) \mid (\gamma \mid \gamma))) = \boldsymbol{\psi}(0)$ and $\boldsymbol{\psi}(\gamma) = \boldsymbol{\psi}(0)$, for any $\alpha, \beta, \gamma \in H$, we have:

$$\boldsymbol{\psi}(0) \supseteq \boldsymbol{\psi}(\alpha) \supseteq \boldsymbol{\psi}(((\alpha|\beta^\alpha)|(\alpha|\beta^\alpha))^\gamma|((\alpha|\beta^\alpha)|(\alpha|\beta^\alpha))^\gamma) \cap \boldsymbol{\psi}(\gamma) = \boldsymbol{\psi}(0).$$

Thus, $\boldsymbol{\psi}(\alpha) = \boldsymbol{\psi}(0)$, which implies $\alpha \in I$. Therefore, I is an implicative SBG-ideal of H. □

DEFINITION 3.9. *Let $\boldsymbol{\psi}$ be a hesitant fuzzy set on H. For any $\pi \in P([0,1])$, the subset $U(\boldsymbol{\psi}, \pi) = \{\alpha \in H \mid \boldsymbol{\psi}(\alpha) \supseteq \pi\}$ is referred to as the upper π-level subset of H.*

THEOREM 3.17. *A hesitant fuzzy set $\boldsymbol{\psi}$ on H is a hesitant fuzzy SBG-subalgebra of H if and only if, for all $\pi \in P([0,1])$, the nonempty subset $U(\boldsymbol{\psi}, \pi)$ of H is an SBG-subalgebra of H.*

Proof. Assume that $\boldsymbol{\psi}$ is a hesitant fuzzy SBG-subalgebra of H. Let $\pi \in P([0,1])$ be such that $U(\boldsymbol{\psi}, \pi) \neq \emptyset$, and let $\alpha, \beta \in U(\boldsymbol{\psi}, \pi)$. This implies $\boldsymbol{\psi}(\alpha) \supseteq \pi$ and $\boldsymbol{\psi}(\beta) \supseteq \pi$. Since $\boldsymbol{\psi}$ is a hesitant fuzzy SBG-subalgebra, we have $\boldsymbol{\psi}(\alpha^{\beta} \mid \alpha^{\beta}) \supseteq \boldsymbol{\psi}(\alpha) \cap \boldsymbol{\psi}(\beta) \supseteq \pi$, which means that $\alpha^{\beta} \mid \alpha^{\beta} \in U(\boldsymbol{\psi}, \pi)$. Therefore, $U(\boldsymbol{\psi}, \pi)$ is an SBG-subalgebra of H.

Conversely, assume that for all $\pi \in P([0,1])$, the nonempty subset $U(\boldsymbol{\psi}, \pi)$ is an SBG-subalgebra of H. Let $\alpha, \beta \in H$, and choose $\pi = \boldsymbol{\psi}(\alpha) \cap \boldsymbol{\psi}(\beta) \in P([0,1])$. Then $\boldsymbol{\psi}(\alpha) \supseteq \pi$ and $\boldsymbol{\psi}(\beta) \supseteq \pi$, so $\alpha, \beta \in U(\boldsymbol{\psi}, \pi) \neq \emptyset$. By assumption, $U(\boldsymbol{\psi}, \pi)$ is an SBG-subalgebra of H, which implies $\alpha^{\beta} \mid \alpha^{\beta} \in U(\boldsymbol{\psi}, \pi)$. Thus, $\boldsymbol{\psi}(\alpha^{\beta} \mid \alpha^{\beta}) \supseteq \pi = \boldsymbol{\psi}(\alpha) \cap \boldsymbol{\psi}(\beta)$. Hence, $\boldsymbol{\psi}$ is a hesitant fuzzy SBG-subalgebra of H. □

Theorem 3.18. *A hesitant fuzzy set $\boldsymbol{\psi}$ on H is a hesitant fuzzy SBG-ideal of H if and only if, for all $\pi \in P([0,1])$, the nonempty subset $U(\boldsymbol{\psi}, \pi)$ of H is an SBG-ideal of H.*

Proof. Assume that $\boldsymbol{\psi}$ is a hesitant fuzzy SBG-ideal of H. Let $\pi \in P([0,1])$ such that $U(\boldsymbol{\psi}, \pi) \neq \emptyset$, and let $\alpha \in U(\boldsymbol{\psi}, \pi)$, meaning $\boldsymbol{\psi}(\alpha) \supseteq \pi$. Since $\boldsymbol{\psi}$ is a hesitant fuzzy SBG-ideal, we also have $\boldsymbol{\psi}(0) \supseteq \boldsymbol{\psi}(\alpha) \supseteq \pi$, so $0 \in U(\boldsymbol{\psi}, \pi)$. Now, let $\alpha, \beta \in H$ be such that $\alpha^{\beta} \mid \alpha^{\beta}, \beta \in U(\boldsymbol{\psi}, \pi)$. This implies $\boldsymbol{\psi}(\alpha^{\beta} \mid \alpha^{\beta}) \supseteq \pi$ and $\boldsymbol{\psi}(\beta) \supseteq \pi$. Since $\boldsymbol{\psi}$ is a hesitant fuzzy SBG-ideal, we have $\boldsymbol{\psi}(\alpha) \supseteq \boldsymbol{\psi}(\alpha^{\beta} \mid \alpha^{\beta}) \cap \boldsymbol{\psi}(\beta) \supseteq \pi$, implying $\alpha \in U(\boldsymbol{\psi}, \pi)$. Therefore, $U(\boldsymbol{\psi}, \pi)$ is an SBG-ideal of H.

Conversely, assume that for all $\pi \in P([0,1])$, the nonempty subset $U(\boldsymbol{\psi}, \pi)$ of H is an SBG-ideal. Let $\alpha \in H$. Since $\boldsymbol{\psi}(\alpha) \in P([0,1])$, choose $\pi = \boldsymbol{\psi}(\alpha) \in P([0,1])$. Then $\boldsymbol{\psi}(\alpha) \supseteq \pi$, so $\alpha \in U(\boldsymbol{\psi}, \pi)$. By assumption, $U(\boldsymbol{\psi}, \pi)$ is an SBG-ideal of H, and hence $0 \in U(\boldsymbol{\psi}, \pi)$, meaning $\boldsymbol{\psi}(0) \supseteq \pi = \boldsymbol{\psi}(\alpha)$.

Moreover, let $\alpha, \beta \in H$, and note that $\boldsymbol{\psi}(\alpha^{\beta} \mid \alpha^{\beta})$ and $\boldsymbol{\psi}(\beta) \in P([0,1])$. Choose $\pi = \boldsymbol{\psi}(\alpha^{\beta} \mid \alpha^{\beta}) \cap \boldsymbol{\psi}(\beta) \in P([0,1])$. Then $\boldsymbol{\psi}(\alpha^{\beta} \mid \alpha^{\beta}) \supseteq \pi$ and $\boldsymbol{\psi}(\beta) \supseteq \pi$. Since $\alpha^{\beta} \mid \alpha^{\beta}, \beta \in U(\boldsymbol{\psi}, \pi) \neq \emptyset$, and $U(\boldsymbol{\psi}, \pi)$ is an SBG-ideal of H, it follows that $\alpha \in U(\boldsymbol{\psi}, \pi)$. Therefore, $\boldsymbol{\psi}(\alpha) \supseteq \pi = \boldsymbol{\psi}(\alpha^{\beta} \mid \alpha^{\beta}) \cap \boldsymbol{\psi}(\beta)$. Thus, $\boldsymbol{\psi}$ is a hesitant fuzzy SBG-ideal of H. □

4. Conclusion

In this paper, we introduced and explored the concept of hesitant fuzzy structures within the framework of Sheffer stroke BG-algebras. We specifically focused on defining hesitant fuzzy SBG-subalgebras, SBG-ideals, and implicative SBG-ideals, which extend classical algebraic structures by incorporating the notion of hesitation. By leveraging the Sheffer stroke operation, we formalized various properties and conditions that hesitant fuzzy sets must satisfy to belong to these algebraic categories. Through detailed definitions and theorems, we provided a comprehensive understanding of how hesitant fuzzy structures behave in relation to the elements of Sheffer stroke BG-algebras. Our results revealed that certain algebraic properties, such as being a hesitant fuzzy implicative SBG-ideal, imply weaker properties like being a hesitant fuzzy SBG-ideal.

Throughout our investigation, we examined the relationships between different types of hesitant fuzzy structures. We proved that in specific types of Sheffer stroke BG-algebras, such as medial and implicative algebras, hesitant fuzzy implicative SBG-ideals also satisfy the conditions of hesitant fuzzy sub-implicative SBG-ideals. These findings not only deepen the understanding of fuzzy algebraic systems but also establish connections between different types of ideals, demonstrating the structural robustness of hesitant fuzzy sets in this context. Additionally, we developed several propositions and lemmas to support these results and to outline the conditions under which hesitant fuzzy sets operate within Sheffer stroke BG-algebras.

Moving forward, our future research will aim to extend the concepts introduced in this paper to other types of algebraic structures beyond Sheffer stroke BG-algebras. We plan to investigate how hesitant fuzzy structures can be applied to other logical systems and algebraic frameworks, such as BZ-algebras and BCI-algebras. Furthermore, we aim to explore potential applications of hesitant fuzzy sets in decision-making systems and artificial intelligence, where handling uncertainty is crucial. These extensions will allow for a broader understanding of how hesitant fuzzy sets interact with different algebraic systems, offering potential insights into both theoretical and practical domains.

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Received by editors 23.9.2024; Revised version 24.2.2025; Available online 31.5.2025.

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