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SYMMETRIC BI-DERIVATIONS ON d-ALGEBRAS

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Abstract. In this paper, we introduce the notions of symmetric bi-derivations of d-algebras, along with the associated sets $Ker_\delta(\Upsilon)$ and $Fix_\delta(\Upsilon)$. We investigate several of their properties and provide illustrative examples and counterexamples to support our findings.

1. Introduction

Two important classes of logic algebras, BCK algebras and BCI algebras, were introduced in 1960s [**5**], [**6**] and extensively investigated by many authors. It is known that BCK-algebras form a proper subclass of BCI-algebras. One of the generalizations of BCK-algebras, d-algebras, was presented by Neggers and Kim [**13**]. The notions of d-subalgebra, d-ideal and some related concepts defined and relations among them investigated by Neggers, Jun and Kim [**14**].

The notion of derivation was given by Posner who established two very striking results on derivations in prime rings [**17**]. Many types of derivations on BCK-BCI-algebras and d-algebras provide a field of study for many researchers (see [**1**], [**2**], [**3**], [**7**], [**8**], [**9**], [**19**]). The notion of symmetric bi-derivation was introduced by Maksa (see [**11**]). For symmetric bi-derivations on lattices, some types of algebras and types of rings can refer to [**4**], [**10**], [**12**], [**15**], [**16**], [**18**], [**20**].

Based on the previous results, it is natural to ask whether it is possible to define symmetric bi-derivations in d-algebras. This paper presents the notions of (l, r)-symmetric bi-derivations and (r, l)-symmetric bi-derivations on d-algebras. Some different properties provided by symmetric bi-derivations are given with examples and counterexamples.

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2. Preliminaries

In this section, we recall some definitions and properties that are usefull for developing the our main results. For information used in this section, see [**2**], [**8**], [**13**], [**14**].

Definition 2.1. *A non-empty set $\Upsilon = (\Upsilon; *, 0)$ of type $(2, 0)$ is called a d-algebra if for all $\upsilon, \omega \in \Upsilon$ the following conditions hold:*

(1) $\upsilon * \upsilon = 0$
(2) $0 * \upsilon = 0$
(3) $\upsilon * \omega = 0$ and $\omega * \upsilon = 0$ implies $\upsilon = \omega$.

Definition 2.2. *Suppose $\Upsilon = (\Upsilon; *, 0)$ be a d-algebra and $\emptyset \neq \Phi \subseteq \Upsilon$. If $\upsilon * \omega \in \Phi$ whenever $\upsilon, \omega \in \Phi$, then Φ is called a d-subalgebra of Υ. Moreover, Φ is called a d-ideal of Υ if it satisfies the following:*

(1) $\upsilon * \omega \in \Phi$ and $\omega \in \Phi$ imply $\upsilon \in \Phi$
(2) $\upsilon \in \Phi$ and $\omega \in \Upsilon$ imply $\upsilon * \omega \in \Phi$, that is, $\Phi * \Upsilon \subseteq \Phi$.

Lemma 2.1. *If Φ is a d-ideal of a d-algebra Υ, then $0 \in \Phi$.*

Definition 2.3. *Let Υ be a d-algebra and $\upsilon \in \Upsilon$. Define $\upsilon * \Upsilon = \{\upsilon * \omega | \omega \in \Upsilon\}$. If for any $\upsilon \in \Upsilon$, $\upsilon * \Upsilon = \{\upsilon, 0\}$, then Υ is called an edge d-algebra.*

Lemma 2.2. *If Υ is an edge d-algebra, then $\upsilon * 0 = \upsilon$ for any $\upsilon \in \Upsilon$.*

Lemma 2.3. *If Υ is an edge d-algebra, then $(\upsilon * (\upsilon * \omega)) * \omega = 0$ for all $\upsilon, \omega \in \Upsilon$.*

Remark 2.1. Let Υ be a d-algebra and $\upsilon, \omega \in \Upsilon$. We write $\upsilon \wedge \omega = \omega * (\omega * \upsilon)$. Moreover, Υ is called a commutative d-algebra if $\upsilon \wedge \omega = \omega \wedge \upsilon$ for all $\upsilon, \omega \in \Upsilon$.

Definition 2.4. *Let Υ be a d-algebra. A map $g : \Upsilon \to \Upsilon$ is a left-right derivation (briefly, (l,r)-derivation) of Υ, if it satisfies $g(\upsilon * \omega) = (g(\upsilon) * \omega) \wedge (\upsilon * g(\omega))$ for all $\upsilon, \omega \in \Upsilon$. If g satisfies $g(\upsilon * \omega) = (\upsilon * g(\omega)) \wedge (g(\upsilon) * \omega)$ for all $\upsilon, \omega \in \Upsilon$, then g is a right-left derivation (briefly, (r,l)-derivation) of Υ. If g is both a (l,r)-derivation and (r,l)-derivation, then g is a derivation of Υ.*

3. Symmetric bi-derivations on d-algebras

Definition 3.1. *Let Υ be a d-algebra. A mapping $\Delta : \Upsilon \times \Upsilon \to \Upsilon$ is called symmetric if $\Delta(\upsilon, \omega) = \Delta(\omega, \upsilon)$ holds for all $\upsilon, \omega \in \Upsilon$.*

Definition 3.2. *Let Υ be a d-algebra and $\delta : \Upsilon \to \Upsilon$ be a mapping defined by $\delta(\upsilon) = \Delta(\upsilon, \upsilon)$. Then δ is called the trace of Δ, where $\Delta : \Upsilon \times \Upsilon \to \Upsilon$ is a symmetric mapping.*

Definition 3.3. *Let Υ be a d-algebra and $\Delta : \Upsilon \times \Upsilon \to \Upsilon$ be a symmetric mapping. If Δ satisfies the identity*

$$\Delta(\upsilon * \omega, \eta) = (\Delta(\upsilon, \eta) * \omega) \wedge (\upsilon * \Delta(\omega, \eta))$$

for all $\upsilon, \omega, \eta \in \Upsilon$, *then* Δ *is called left-right symmetric bi-derivation (briefly (l,r)-symmetric bi-derivation). If* Δ *satisfies the identity*

$$\Delta(\upsilon * \omega, \eta) = (\upsilon * \Delta(\omega, \eta)) \wedge (\Delta(\upsilon, \eta) * \omega)$$

for all $\upsilon, \omega, \eta \in \Upsilon$, *then* Δ *is called right-left symmetric bi-derivation (briefly (r,l)-symmetric bi-derivation). Moreover,* Δ *is symmetric bi-derivation if* Δ *is both (l,r)-symmetric bi-derivation and (r,l)-symmetric bi-derivation. It is clear that the definitions of (l,r)-symmetric bi-derivation and (r,l)-symmetric bi-derivation coincide if* Υ *is commutative d-algebra.*

The existence of the symmetric bi-derivations of d-algebras is showed by the following examples.

EXAMPLE 3.1. *Consider a commutative d-algebra* $\Upsilon = \{0, \upsilon, \omega\}$ *with the following Cayley table:*

$*$	0	υ	ω
0	0	0	0
υ	υ	0	0
ω	ω	υ	0

Define a mapping $\Delta : \Upsilon \times \Upsilon \to \Upsilon$ by

$$\Delta(a, b) = \begin{cases} \upsilon, & (a, b) = (\omega, \omega) \\ 0, & otherwise \end{cases} .$$

Then it can be checked that Δ is a symmetric bi-derivation.

EXAMPLE 3.2. *Let* $\Upsilon = \{0, 1, 2, 3, 4, 5\}$ *be a d-algebra in which the operation is defined as follows:*

$*$	0	1	2	3	4	5
0	0	0	0	0	0	0
1	1	0	0	0	0	0
2	2	2	0	0	0	0
3	3	3	1	0	0	0
4	4	2	1	1	0	0
5	5	5	3	3	1	0

Define a map $\Delta : \Upsilon \times \Upsilon \to \Upsilon$ *by*

$$\Delta(\upsilon, \omega) = \begin{cases} 3, & (\upsilon, \omega) = (4, 4) \\ 5, & (\upsilon, \omega) = (5, 5) \\ 0, & otherwise \end{cases} .$$

Although Δ *is a (r,l)-symmetric bi-derivation, it is not a (l,r)-symmetric bi-derivation. The reason is explained below:*

$$\begin{aligned} \Delta(4 * 0, 4) &= (4 * \Delta(0, 4)) \wedge (\Delta(4, 4) * 0) \\ &= 4 \wedge 3 \\ &= 3 \end{aligned}$$

and

$$\begin{aligned}\Delta(4*0,4) &= (\Delta(4,4)*0)\wedge(4*\Delta(0,4))\\ &= 3\wedge 4\\ &= 2.\end{aligned}$$

PROPOSITION 3.1. *Let Υ be a d-algebra and Δ be a (l,r)-symmetric bi-derivation on Υ. Then $\Delta(0,\upsilon)=0$ for all $\upsilon\in\Upsilon$.*

PROOF. For all $\upsilon\in\Upsilon$, we have

$$\begin{aligned}\Delta(0,\upsilon) &= \Delta(0*\upsilon,\upsilon)\\ &= (\Delta(0,\upsilon)*\upsilon)\wedge(0*\delta(\upsilon))\\ &= (\Delta(0,\upsilon)*\upsilon)\wedge 0\\ &= 0*(0*(\Delta(0,\upsilon)*\upsilon))\\ &= 0.\end{aligned}$$

□

The same result is achieved for (r,l)-symmetric bi-derivations when Υ is an edge d-algebra.

PROPOSITION 3.2. *Let Υ be an edge d-algebra and Δ be a (r,l)-symmetric bi-derivation on Υ. Thus $\Delta(0,\upsilon)=0$ for all $\upsilon\in\Upsilon$.*

PROOF. Since Δ is a (r,l)-symmetric bi-derivation, we have

$$\begin{aligned}\Delta(0,\upsilon) &= \Delta(0*\upsilon,\upsilon)\\ &= (0*\delta(\upsilon))\wedge(\Delta(0,\upsilon)*\upsilon)\\ &= 0\wedge(\Delta(0,\upsilon)*\upsilon)\\ &= (\Delta(0,\upsilon)*\upsilon)*((\Delta(0,\upsilon)*\upsilon)*0)\\ &= (\Delta(0,\upsilon)*\upsilon)*(\Delta(0,\upsilon)*\upsilon)\\ &= 0.\end{aligned}$$

□

PROPOSITION 3.3. *Let Υ be an edge d-algebra and Δ be a (r,l)-symmetric bi-derivation on Υ. Thus for all $\upsilon,\omega,\eta\in\Upsilon$*

(1) $\Delta(\upsilon,\omega)=\upsilon\wedge\Delta(\upsilon,\omega)$
(2) $\delta(\upsilon)=\upsilon\wedge\delta(\upsilon)$
(3) $\Delta(\upsilon,\omega)\leqslant\upsilon$
(4) $\delta(\upsilon)\leqslant\upsilon$
(5) $\delta(\delta(\upsilon)*\upsilon)=0$
(6) $\Delta(\upsilon*\omega,\eta)\leqslant\upsilon*\Delta(\omega,\eta)$
(7) $\Delta(\delta(\upsilon),\omega)\leqslant\delta(\upsilon)$.

PROOF. (1) Let $\upsilon, \omega \in \Upsilon$. Using Proposition 3.2, we have

$$\begin{aligned} \Delta(\upsilon,\omega) &= \Delta(\upsilon * 0, \omega) \\ &= (\upsilon * \Delta(0,\omega)) \wedge (\Delta(\upsilon,\omega) * 0) \\ &= (\upsilon * 0) \wedge \Delta(\upsilon,\omega) \\ &= \upsilon \wedge \Delta(\upsilon,\omega). \end{aligned}$$

(2) Since δ is the trace of Δ, the proof is obtained from (1).

(3) Let $\upsilon, \omega \in \Upsilon$. By using (1) and Lemma 2.3, we get

$$\begin{aligned} \Delta(\upsilon,\omega) &= \upsilon \wedge \Delta(\upsilon,\omega) \\ &= \Delta(\upsilon,\omega) * (\Delta(\upsilon,\omega) * \upsilon) \end{aligned}$$

and

$$\Delta(\upsilon,\omega) * \upsilon = (\Delta(\upsilon,\omega) * (\Delta(\upsilon,\omega) * \upsilon)) * \upsilon = 0.$$

Therefore, $\Delta(\upsilon,\omega) \leqslant \upsilon$ for all $\upsilon, \omega \in \Upsilon$.

(4) It can be easily obtained from (3).

(5) By using (4) and (2), we have $\delta(0) = 0 \wedge \delta(0) = 0$. Hence, we get $\delta(\delta(\upsilon) * \upsilon) = \delta(0) = 0$ for all $\upsilon \in \Upsilon$.

(6) Let $\upsilon, \omega, \eta \in \Upsilon$. Then, we get

$$\begin{aligned} \Delta(\upsilon * \omega, \eta) &= (\upsilon * \Delta(\omega,\eta)) \wedge (\Delta(\upsilon,\eta) * \omega) \\ &= (\Delta(\upsilon,\eta) * \omega) * ((\Delta(\upsilon,\eta) * \omega) * (\upsilon * \Delta(\omega,\eta))) \end{aligned}$$

and

$$\begin{aligned} & \Delta(\upsilon * \omega, \eta) * (\upsilon * \Delta(\omega,\eta)) \\ = \; & [(\Delta(\upsilon,\eta) * \omega) * ((\Delta(\upsilon,\eta) * \omega) * (\upsilon * \Delta(\omega,\eta)))] * (\upsilon * \Delta(\omega,\eta)) \\ = \; & 0. \end{aligned}$$

Therefore, we get $\Delta(\upsilon * \omega, \eta) \leqslant \upsilon * \Delta(\omega,\eta)$ for all $\upsilon, \omega, \eta \in \Upsilon$, as required.

(7) Let $\upsilon, \omega \in \Upsilon$. By using (1), we have

$$\begin{aligned} \Delta(\delta(\upsilon),\omega) &= \delta(\upsilon) \wedge \Delta(\delta(\upsilon),\omega) \\ &= \Delta(\delta(\upsilon),\omega) * (\Delta(\delta(\upsilon),\omega) * \delta(\upsilon)) \end{aligned}$$

and

$$\begin{aligned} \Delta(\delta(\upsilon),\omega) * \delta(\upsilon) &= (\Delta(\delta(\upsilon),\omega) * (\Delta(\delta(\upsilon),\omega) * \delta(\upsilon))) * \delta(\upsilon) \\ &= 0. \end{aligned}$$

□

PROPOSITION 3.4. *Let Υ be an edge d-algebra and Δ be a (l,r)-symmetric bi-derivation on Υ. Then, for all $\upsilon, \omega, \eta \in \Upsilon$*

(1) $\Delta(\upsilon,\omega) = \Delta(\upsilon,\omega) \wedge \upsilon$

(2) $\delta(\upsilon) = \delta(\upsilon) \wedge \upsilon$

(3) $\Delta(\upsilon * \omega, \eta) \leqslant \Delta(\upsilon,\eta) * \omega$.

PROOF. (1) Let $\upsilon, \omega \in \Upsilon$. Using Proposition 3.1, we get

$$\begin{aligned}\Delta(\upsilon, \omega) &= \Delta(\upsilon * 0, \omega) \\ &= (\Delta(\upsilon, \omega) * 0) \wedge (\upsilon * \Delta(0, \omega)) \\ &= \Delta(\upsilon, \omega) \wedge \upsilon.\end{aligned}$$

(2) Since δ is the trace of Δ, the proof is obtained from (1).
(3) Let $\upsilon, \omega, \eta \in \Upsilon$. Hence, we get

$$\begin{aligned}\Delta(\upsilon * \omega, \eta) &= (\Delta(\upsilon, \eta) * \omega) \wedge (\upsilon * \Delta(\omega, \eta)) \\ &= (\upsilon * \Delta(\omega, \eta)) * ((\upsilon * \Delta(\omega, \eta)) * (\Delta(\upsilon, \eta) * \omega))\end{aligned}$$

and so

$$\begin{aligned}&\Delta(\upsilon * \omega, \eta) * (\Delta(\upsilon, \eta) * \omega) \\ =\ &[(\upsilon * \Delta(\omega, \eta)) * ((\upsilon * \Delta(\omega, \eta)) * (\Delta(\upsilon, \eta) * \omega))] * (\Delta(\upsilon, \eta) * \omega) \\ =\ &0.\end{aligned}$$

It implies that $\Delta(\upsilon * \omega, \eta) \leqslant \Delta(\upsilon, \eta) * \omega$ for all $\upsilon, \omega, \eta \in \Upsilon$. $\square$

EXAMPLE 3.3. *Let $\Upsilon = \{0, \upsilon, \omega, \eta\}$ be a d-algebra in which the operation is defined as follows:*

$*$	0	υ	ω	η
0	0	0	0	0
υ	υ	0	υ	0
ω	ω	ω	0	0
η	ω	ω	ω	0

Let $\Delta : \Upsilon \times \Upsilon \to \Upsilon$ by

$$\Delta(x, y) = \begin{cases} \upsilon, & (x, y) = (\eta, \eta) \\ 0, & otherwise \end{cases}.$$

It is easy to check that Δ is both a (l,r) symmetric bi-derivation and a (r,l) symmetric bi-derivation on Υ. We remark that Υ is not an edge d-algebra. On the other hand, $\delta(\eta) = \upsilon \neq \delta(\eta) \wedge \eta = \upsilon \wedge \eta = \omega$ and $\Delta(0, \eta) = 0 \neq \Delta(0, \eta) \wedge \eta = 0 \wedge \eta = \omega$. Therefore, in Proposition 3.3 and Proposition 3.4 the condition of edge cannot be omitted.

PROPOSITION 3.5. *Let Υ be an edge d-algebra and Δ be a (r,l)-symmetric bi-derivation on Υ. If there exists $\upsilon \in \Upsilon$ such that $\upsilon \leqslant \Delta(\omega, \eta)$ for all $\omega, \eta \in \Upsilon$, then $\upsilon \leqslant \omega$.*

PROOF. Assume that there exists $\upsilon \in \Upsilon$ such that $\upsilon \leqslant \Delta(\omega, \eta)$ for all $\omega, \eta \in \Upsilon$. By Proposition 3.3(1), we get $\upsilon \leqslant \Delta(\omega, \eta) * (\Delta(\omega, \eta) * \omega)$ and $\upsilon * \omega \leqslant (\Delta(\omega, \eta) * (\Delta(\omega, \eta) * \omega)) * \omega$ which implies that $\upsilon * \omega = 0$. Thus, we have $\upsilon \leqslant \omega$. $\square$

DEFINITION 3.4. *Let Υ be a d-algebra and B be a non-empty subset of Υ. Then, B is said to be Δ-invariant if*
$\Delta(B, B) \subseteq B$, where $\Delta(B, B) = \{\Delta(\omega_1, \omega_2) | \omega_1, \omega_2 \in B\}$.

PROPOSITION 3.6. *Let Δ be a (r,l)-symmetric bi-derivation of an edge d-algebra Υ. Then, every d-ideal B of Υ is Δ-invariant.*

PROOF. Let $\eta \in \Delta(B,B)$. Thus, we have $\eta = \Delta(\omega_1, \omega_2)$ for some $\omega_1, \omega_2 \in B$. By Proposition 3.3(3), we get $\Delta(\omega_1, \omega_2) \leqslant \omega_1$ and so $\Delta(\omega_1, \omega_2) * \omega_1 = 0$. Since B is a d-ideal and $\omega_1 \in B$, we have $\Delta(\omega_1, \omega_2) = \eta \in B$. Thus, we obtain $\Delta(B,B) \subseteq B$. $\square$

DEFINITION 3.5. *Let Υ be a d-algebra and Δ be a (l,r)-(or (r,l)-)symmetric bi-derivation of Υ. For a fixed element $\upsilon \in \Upsilon$, we define a map $\delta_\upsilon : \Upsilon \to \Upsilon$ by $\delta_\upsilon(\omega) = \Delta(\upsilon, \omega)$ for all $\omega \in \Upsilon$.*

THEOREM 3.1. *Let Υ be a d-algebra and Δ be a (l,r)-(resp. (r,l)-)symmetric bi-derivation of Υ. Then, the map δ_υ defined in Definition 3.5 is a (l,r)-(resp. (r,l)-) derivation of Υ for all $\upsilon \in \Upsilon$.*

PROOF. For all $\omega, \eta \in \Upsilon$, we have

$$\begin{aligned}
\delta_\upsilon(\omega * \eta) &= \Delta(\upsilon, \omega * \eta) \\
&= (\Delta(\upsilon, \omega) * \eta) \wedge (\omega * \Delta(\upsilon, \eta)) \\
&= (\delta_\upsilon(\omega) * \eta) \wedge (\omega * \delta_\upsilon(\eta)).
\end{aligned}$$

$\square$

DEFINITION 3.6. *Let Υ be a d-algebra and Δ be a (l,r)-(or (r,l)-)symmetric bi-derivation of Υ and δ be the trace of Δ. We define a subset $Ker_\delta(\Upsilon)$ of Υ by $Ker_\delta(\Upsilon) = \{\upsilon \in \Upsilon |\, \delta(\upsilon) = 0\}$.*

THEOREM 3.2. *Let Υ be an edge d-algebra and Δ be a (l,r)-(or (r,l)-)symmetric bi-derivation of Υ and δ be the trace of Δ. Then $Ker_\delta(\Upsilon)$ is a d-subalgebra of Υ.*

PROOF. Since $\delta(0) = 0$, we have $0 \in Ker_\delta(\Upsilon)$ and so $Ker_\delta(\Upsilon) \neq \emptyset$. Let $\upsilon, \omega \in Ker_\delta(\Upsilon)$. Thus, we have $\delta(\upsilon) = \delta(\omega) = 0$. By using the definition of (l,r)-symmetric bi-derivation, we have

$$\begin{aligned}
\delta(\upsilon * \omega) &= \Delta(\upsilon * \omega, \upsilon * \omega) \\
&= (\Delta(\upsilon, \upsilon * \omega) * \omega) \wedge (\upsilon * \Delta(\omega, \upsilon * \omega)) \\
&= [((\delta(\upsilon) * \omega) \wedge (\upsilon * \Delta(\upsilon, \omega))) * \omega] \\
&\quad \wedge [\upsilon * ((\Delta(\omega, \upsilon) * \omega) \wedge (\upsilon * \delta(\omega)))] \\
&= 0 \wedge [\upsilon * ((\Delta(\omega, \upsilon) * \omega) \wedge \upsilon] \\
&= 0.
\end{aligned}$$

Therefore, $Ker_\delta(\Upsilon)$ is a d-subalgebra of Υ. $\square$

EXAMPLE 3.4. $\Upsilon = \{0, \upsilon, \omega, \eta\}$ *be a d-algebra in which the operation is defined as follows:*

$*$	0	υ	ω	η
0	0	0	0	0
υ	υ	0	0	υ
ω	ω	ω	0	0
η	η	0	η	0

Define a map $\Delta : \Upsilon \times \Upsilon \to \Upsilon$ *by*

$$\Delta(x,y) = \begin{cases} \eta, & (x,y) = (\eta,\eta) \\ 0, & otherwise \end{cases}.$$

Then we can see that Δ *is a both (r,l)- and (l,r)-symmetric bi-derivation and* $Ker_\delta(\Upsilon) = \{0, \upsilon, \omega\}$. *Since* $\eta * \upsilon \in Ker_\delta(\Upsilon)$ *and* $\upsilon \in Ker_\delta(\Upsilon)$ *but* $\eta \notin Ker_\delta(\Upsilon)$, *we have* $Ker_\delta(\Upsilon)$ *is not a d-ideal.*

Theorem 3.3. *Let* Δ *be a (r,l)-(resp. (l,r)-)symmetric bi-derivation of d-algebra (resp. edge d-algebra)* Υ *and* δ *be the trace of* Δ. *If* $\upsilon \in \Upsilon$ *and* $\omega \in Ker_\delta(\Upsilon)$, *then* $\upsilon \wedge \omega \in Ker_\delta(\Upsilon)$ *(resp.* $\omega \wedge \upsilon \in Ker_\delta(\Upsilon)$*).*

Proof. Suppose that Δ is a (r,l)-symmetric bi-derivation of Υ. Let $\upsilon \in \Upsilon$ and $\omega \in Ker_\delta(\Upsilon)$. Then, we have $\delta(\omega) = 0$ and

$$\begin{aligned}
\delta(\upsilon \wedge \omega) &= \Delta(\upsilon \wedge \omega, \upsilon \wedge \omega) \\
&= \Delta(\upsilon \wedge \omega, \omega * (\omega * \upsilon)) \\
&= (\omega * \Delta(\upsilon \wedge \omega, \omega * \upsilon)) \wedge (\Delta(\omega, \upsilon \wedge \omega) * (\omega * \upsilon)) \\
&= (\omega * \Delta(\upsilon \wedge \omega, \omega * \upsilon)) \wedge (\Delta(\omega, \omega * (\omega * \upsilon)) * (\omega * \upsilon)) \\
&= (\omega * \Delta(\upsilon \wedge \omega, \omega * \upsilon)) \wedge [((\omega * \Delta(\omega, \omega * \upsilon)) \wedge (\delta(\omega) * (\omega * \upsilon))) * (\omega * \upsilon)] \\
&= (\omega * \Delta(\upsilon \wedge \omega, \omega * \upsilon)) \wedge 0 \\
&= 0.
\end{aligned}$$

Therefore, we get $\upsilon \wedge \omega \in Ker_\delta(\Upsilon)$ for all $\upsilon \in \Upsilon$ and $\omega \in Ker_\delta(\Upsilon)$. □

example 3.5. $\Upsilon = \{0, \upsilon, \omega, \eta\}$ *be a d-algebra (not an edge d-algebra) in which the operation is defined as follows:*

$*$	0	υ	ω	η
0	0	0	0	0
υ	υ	0	0	ω
ω	ω	ω	0	0
η	η	η	υ	0

Define a map $\Delta : \Upsilon \times \Upsilon \to \Upsilon$ *by*

$$\Delta(x,y) = \begin{cases} \eta, & (x,y) = (\eta,\eta) \\ 0, & otherwise \end{cases}.$$

Then we can see that Δ *is a both (r,l)- and (l,r)-symmetric bi-derivation. In the conditions of Theorem 3.3 for (l,r)-symmetric bi-derivations, the edge condition cannot be removed. Because* $\eta \in \Upsilon$ *and* $\omega \in Ker_\delta(\Upsilon)$ *but* $\omega \wedge \eta = \eta \notin Ker_\delta(\Upsilon)$.

Definition 3.7. *Let* Δ *be a (l,r)-(or (r,l)-) symmetric bi-derivation of a d-algebra* Υ. *For a fixed element* $\eta \in \Upsilon$, *we define a set* $Fix_\Delta(\Upsilon) = \{\upsilon \in \Upsilon | \, \Delta(\upsilon, \eta) = \upsilon\}$.

Proposition 3.7. *Let* Δ *be a (l,r)-(or (r,l)-)symmetric bi-derivation on an edge d-algebra* Υ. *Then* $Fix_\Delta(\Upsilon)$ *is a d-subalgebra of* Υ.

PROOF. Since $\Delta(0,\eta)=0$, we get $0 \in Fix_\Delta(\Upsilon)$ and so $Fix_\Delta(\Upsilon) \neq \emptyset$. Let $\upsilon,\omega \in Fix_\Delta(\Upsilon)$. Thus,

$$\begin{aligned} \Delta(\upsilon*\omega,\eta) &= (\Delta(\upsilon,\eta)*\omega) \wedge (\upsilon*\Delta(\omega,\eta)) \\ &= (\upsilon*\omega)\wedge(\upsilon*\omega) \\ &= \upsilon*\omega. \end{aligned}$$

Therefore, $\upsilon*\omega \in Fix_\Delta(\Upsilon)$ and so $Fix_\Delta(\Upsilon)$ is a d-subalgebra of Υ. □

EXAMPLE 3.6. *$\Upsilon=\{0,\upsilon,\omega,\eta\}$ be a d-algebra in which the operation is defined as follows:*

$*$	0	υ	ω	η
0	0	0	0	0
υ	υ	0	υ	0
ω	ω	ω	0	0
η	η	η	0	0

Define a map $\Delta:\Upsilon\times\Upsilon\to\Upsilon$ by

$$\Delta(x,y)=\begin{cases} \omega, & (x,y)\in\{(\omega,\eta),(\eta,\omega)\} \\ 0, & otherwise \end{cases}.$$

Then we can see that Δ is a (r,l)-symmetric bi-derivation but it is not a (l,r)-symmetric bi-derivation. Because

$$\Delta(\eta*\upsilon,\omega)=(\eta*\Delta(\upsilon,\omega))\wedge(\Delta(\eta,\omega)*\upsilon)=\eta*\omega=\omega$$

and

$$\Delta(\eta*\upsilon,\omega)=(\Delta(\eta,\omega)*\upsilon)\wedge(\eta*\Delta(\upsilon,\omega))=\omega*\eta=\eta.$$

Moreover, and $Fix_\Delta(\Upsilon)=\{0,\omega\}$. Since $\eta*\omega\in Fix_\Delta(\Upsilon)$ and $\omega\in Fix_\Delta(\Upsilon)$ but $\eta\notin Fix_\Delta(\Upsilon)$, we have $Fix_\Delta(\Upsilon)$ is not a d-ideal.

PROPOSITION 3.8. *Let Δ be a (l,r)-(or (r,l)-)symmetric bi-derivation on an edge d-algebra Υ. If $\upsilon,\omega\in Fix_\Delta(\Upsilon)$, then $\upsilon\wedge\omega\in Fix_\Delta(\Upsilon)$.*

PROOF. Let $\upsilon,\omega\in Fix_\Delta(\Upsilon)$. Hence, we get

$$\begin{aligned} \Delta(\upsilon\wedge\omega,\eta) &= \Delta(\omega*(\omega*\upsilon),\eta) \\ &= (\Delta(\omega,\eta)*(\omega*\upsilon))\wedge(\omega*\Delta(\omega*\upsilon,\eta)) \\ &= (\omega*(\omega*\upsilon))\wedge(\omega*((\Delta(\omega,\eta)*\upsilon)\wedge(\omega*\Delta(\upsilon,\eta)) \\ &= (\upsilon\wedge\omega)\wedge(\omega*((\omega*\upsilon)\wedge(\omega*\upsilon))) \\ &= (\upsilon\wedge\omega)\wedge(\upsilon\wedge\omega) \\ &= \upsilon\wedge\omega. \end{aligned}$$

□

References

1. K. Alnefaie, On Reverse Derivations in d-algebras. *European Journal of Pure and Applied Mathematics*, **17**(1) (2024), 362–371.
2. R. M. Al-Omary, M. S. Khan, and N. Rehman, On generalized derivations in d-algebras. *J. Adv. Res. Pure Math*, **7**(3) (2015), 23–34.
3. R. M. Al-Omary, On $(alpha, beta)$-derivations in d-algebras. *Bollettino dell'Unione Matematica Italiana*, **12** (2019), 549–556.
4. Y. Çeven, Symmetric bi-derivations of lattices. *Quaestiones mathematicae*, **32**(2) (2009), 241–245.
5. Y. Imai and K. Iséki, On axiom systems of propositional calculi. *I. Proceedings of the Japan Academy*, **41**(6) (1965), 436–439.
6. K. Iséki, An algebra related with a propositional calculus. *Proceedings of the Japan Academy*, **42**(1) (1966), 26–29.
7. Y. B. Jun and X. L. Xin, On derivations of BCI-algebras. *Information Sciences*, **159**(3-4) (2004), 167–176.
8. N. Kandaraj and M. Chandramouleeswaran, On Left derivations of d-algebras. *Int. J Math. Arch*, **3**(11) (2012), 3961–3966.
9. Y. H. Kim, Some derivations on d-algebras. *International Journal of Fuzzy Logic and Intelligent Systems*, **18**(4) (2018), 298–302.
10. S. Koç and H. Yazarlı, Generalized symmetric bi-derivations in MV-algebras. *New Trends in Mathematical Sciences*, **10**(4) (2022), 107–114.
11. G. Maksa, On the trace of symmetric bi-derivations. *R Math. Rep. Acad. Sci. Canada*, **9**(6) (1987), 303–307.
12. G. Muhiuddin et al., On Symmetric Left Bi-Derivations in BCI-Algebras. *International Journal of Mathematics and Mathematical Sciences*, **2013** (2013), 1–6.
13. J. Neggers and H. S. Kim, On d-algebras. *Mathematica Slovaca*, **49**(1) (1999), 19–26.
14. J. Neggers, Y. B. Jun, and H. S. Kim, On d-ideals in d-algebras. *Mathematica Slovaca*, **49**(3) (1999), 243–251.
15. M. A. Öztürk, Symmetric bi-derivations on prime gamma rings. *Scientiae Mathematicae*, **3**(2) (2000), 273–281.
16. M. A. Öztürk and Y. B. Jun, On trace of symmetric bi-derivations in near-ring. *International Journal of Pure and Applied Mathematics*, **17**(1) (2004), 93–100.
17. E. C. Posner, Derivation in prime rings. *Proc. Am. Math. Soc.*, **8** (1957), 1093–1100.
18. I. Sabahattin, A. Firat, and Y. B. Jun, On symmetric bi-derivations of BCI-algebras. *Applied mathematical sciences*, **5**(57-60) (2011), 2957–2966.
19. A. Şenbakar and D. Yılmaz, On Generalized $(alpha, beta)$-derivations in d-algebras. *Bull. Int. Math. Virtual Inst.*, **13**(2) (2023), 239–247.
20. D. Yılmaz, Symmetric bi-derivations of UP (BCC)-algebras. *Journal of Applied Non-Classical Logics*, **34**(1) (2024), 155–169.

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