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NEW BOUNDS ON SOMBOR AND ELLIPTIC SOMBOR INDEX

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Abstract. The Sombor index (SO) is a vertex-degree-based graph invariant, equal to the sum of the terms $\sqrt{d(u)^2 + d(v)^2}$ over all pairs of adjacent vertices u, v of the underlying graph, where $d(u)$ is the degree of the vertex u. The elliptic Sombor index (ESO) is a recently introduced variant of SO, equal to the sum of terms $[d(u)+d(v)]\,\sqrt{d(u)^2 + d(v)^2}$. In this paper, we establish new lower and upper bounds for SO and ESO, as well as bounds involving their coindices.

1. Introduction

In this paper we are concerned with simple graphs, i.e., graphs without directed, multiple or weighted edges, and without self-loops [**2**, **14**]. Let G be such a graph, with vertex set $\mathbf{V}(G)$ and edge set $\mathbf{E}(G)$. The number of vertices and edges of G are $n = |\mathbf{V}(G)|$ and $m = |\mathbf{E}(G)|$, respectively. An edge connecting the vertices $u, v \in \mathbf{V}(G)$ will be denoted by uv. The degree (= number of first neighbors) of a vertex $u \in \mathbf{V}(G)$ is denoted by $d(u)$. For other graph theoretic notations and terminology see [**2**, **14**].

In contemporary mathematics and mathematical chemistry, a large number of graph invariants are studied, aimed at modeling structural properties of chemical compounds [**17**, **28**]. A large group of such invariants is of the form

$$TI = TI(G) = \sum_{uv \in \mathbf{E}(G)} \varphi\big(d(u), d(v)\big)$$

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where $\varphi(x,y)$ is a conveniently chosen function with property $\varphi(x,y)=\varphi(y,x)$. These are usually referred to as vertex-degree-based (VDB) topological indices.

The coindex of the invariant TI is defined as

$$\overline{TI}=\overline{TI}(G)=\sum_{uv\notin\mathbf{E}(G)}\varphi\big(d(u),d(v)\big)$$

where it is assumed that the vertices $u,v\in\mathbf{V}(G)$ are distinct, i.e., $u\neq v$.

Two recently introduced VDB graph invariants are the Sombor index (SO) [**7**] and the elliptic Sombor index (ESO) [**11**], both conceived by using geometric considerations. These are defined as

$$SO=SO(G)=\sum_{uv\in\mathbf{E}(G)}\sqrt{d(u)^2+d(v)^2}$$

and

$$ESO=ESO(G)=\sum_{uv\in\mathbf{E}(G)}[d(u)+d(v)]\,\sqrt{d(u)^2+d(v)^2}\,.$$

Of their several noteworthy applications we mention here just a few [**1**, **15**, **25**, **26**]. Their mathematical properties are found in the review [**18**] and in the recent papers [**6**, **23**, **24**, **27**]. For researches on Sombor coindex see in [**3**, **4**, **19**, **22**].

Several lower and upper bounds for Sombor and elliptic Sombor indices were earlier communicated [**8**, **9**, **12**, **16**, **20**]. In the present paper we offer a few more such results.

In order to present our findings, we need some preparations.

2. Preparations

In he below considerations, the following well-known VDB topological indices will be applied [**5**, **13**, **21**]: the fist Zagreb index (M_1), the second Zagreb index (M_2), and the forgotten index (F). These are are defined as

$$\begin{aligned}
M_1(G) &= \sum_{uv\in\mathbf{E}(G)}[d(u)+d(v)]=\sum_{u\in\mathbf{V}(G)}d(u)^2\\
M_2(G) &= \sum_{uv\in\mathbf{E}(G)}d(u)\,d(v)\\
F(G) &= \sum_{uv\in\mathbf{E}(G)}[d(u)^2+d(v)^2]=\sum_{u\in\mathbf{V}(G)}d(u)^3\,.
\end{aligned}$$

The respective coindices are

$$\begin{aligned}\overline{M_1}(G) &= \sum_{uv\notin \mathbf{E}(G)} [d(u)+d(v)]\\ \overline{M_2}(G) &= \sum_{uv\notin \mathbf{E}(G)} d(u)\,d(v)\\ \overline{F}(G) &= \sum_{uv\notin \mathbf{E}(G)} [d(u)^2+d(v)^2]\,.\end{aligned}$$

Bearing in mind that

$$TI(G)+\overline{TI}(G)=\frac{1}{2}\left[\sum_{u\in\mathbf{V}(G)}\sum_{v\in\mathbf{V}(G)}\varphi\big(d(u),d(v)\big)-\sum_{u\in\mathbf{V}(G)}\varphi\big(d(u),d(u)\big)\right]$$

and therefore

$$M_1(G)+\overline{M_1}(G)=\frac{1}{2}\left[\sum_{u\in\mathbf{V}(G)}\sum_{v\in\mathbf{V}(G)}[d(u)+d(v)]-\sum_{u\in\mathbf{V}(G)}[d(u)+d(u)]\right]$$

and recalling that

$$\sum_{u\in\mathbf{V}(G)} d(u)=2m$$

we get

$$M_1(G)+\overline{M_1}(G)=\frac{1}{2}\,[2mn+2mn-4m]$$

and thus arrive at:

Lemma 2.1. [**10**] *Let G be a graph with n vertices and m edges. Then*

$$M_1(G)+\overline{M_1}(G)=2m(n-1)\,.$$

In an analogous manner, we have:

Lemma 2.2. [**10**] *Let G be a graph with m edges. Then*

$$M_2(G)+\overline{M_2}(G)=2\,m^2-\frac{1}{2}\,M_1(G)\,.$$

Lemma 2.3. *Let G be a graph with n vertices. Then*

$$F(G)+\overline{F}(G)=(n-1)M_1(G)\,.$$

3. Main results

The starting point in several earlier studies of estimates of Sombor-type indices [**8**, **9**, **20**] were the inequalities

$$\frac{1}{\sqrt{2}}\,(a+b)\leqslant\sqrt{a^2+b^2}<a+b\,. \tag{3.1}$$

When applying (3.1) to vertex degrees, one must take into account that a and b are positive integers not greater than $n-1$. Therefore, the left-hand side equality holds whenever $a = b$, whereas the right-hand inequality is strict (i.e., equality would hold if either $a = 0$ or $b = 0$).

Applying (3.1) to the definitions of Sombor and elliptic Sombor indices, one immediately obtains the many-times reported relations:

$$\frac{1}{\sqrt{2}}\,M_1(G) \leqslant SO(G) < M_1(G) \tag{3.2}$$

and

$$\frac{1}{\sqrt{2}}\,\big[F(G) + 2\,M_2(G)\big] \leqslant ESO(G) < F(G) + 2\,M_2(G)\,. \tag{3.3}$$

In (3.3) we have used the fact that

$$\sum_{uv\in\mathbf{E}(G)} \big[d(u)+d(v)\big]^2 = F(G) + 2\,M_2(G)\,.$$

Equality on the left-hand side of (3.2) and (3.3) holds if and only if the graph G is regular.

We are now going to improve the bounds (3.2) and (3.3). From now on, without losing generality, it will be assumed that $a \geqslant b$.

Denote by Δ and δ the maximal and minimal vertex degree in the considered graph G. Then Δ is the greatest possible value of the parameter a, whereas δ is the smallest possible value of the parameter b.

Introduce the auxiliary functions

$$Q_1 = Q_1(a,b) = a + b - \sqrt{a^2+b^2}$$

and

$$Q_2 = Q_2(a,b) = \sqrt{a^2+b^2} - \frac{1}{\sqrt{2}}\,(a+b)$$

and note that that (3.1) is equivalent to the conditions $Q_1 > 0$ and $Q_2 \geqslant 0$.

Since

$$\frac{\partial Q_1}{\partial a} = 1 - \frac{a}{\sqrt{a^2+b^2}}$$

is positive-valued for any $a > 0$, the function Q_1 is monotonically increasing in the variable a. By symmetry, the same holds also for the variable b. Therefore, the maximal value of Q_1 is $Q_1(\Delta,\Delta)$. This implies

$$a + b - \sqrt{a^2+b^2} \leqslant \Delta + \Delta - \sqrt{\Delta^2+\Delta^2} = (2-\sqrt{2})\,\Delta$$

and by summation over all pairs of adjacent vertices of the graph G, we obtain:

$$M_1(G) - SO(G) \leqslant (2-\sqrt{2})\,\Delta\,m\,. \tag{3.4}$$

The case of the function Q_2 is somewhat different. The derivative

$$\frac{\partial Q_2}{\partial a} = \frac{a}{\sqrt{a^2+b^2}} - \frac{1}{\sqrt{2}}$$

is positive-valued if and only if $a > b$. On the other hand, in view of $b \leqslant a$, $\delta Q_2/\delta b$ is negative valued, i.e., Q_2 is monotonically decreasing in the variable b. Therefore, the maximal value of Q_2 is $Q_2(\Delta,\delta)$. This implies

$$\sqrt{a^2+b^2} - \frac{1}{\sqrt{2}}\,(a+b) \leqslant \sqrt{\Delta^2+\delta^2} - \frac{1}{\sqrt{2}}\,(\Delta+\delta)$$

and by summation over all pairs of adjacent vertices of the graph G, we obtain:

$$SO(G) - \frac{1}{\sqrt{2}}\,M_1(G) \leqslant \left[\sqrt{\Delta^2+\delta^2} - \frac{1}{\sqrt{2}}\,(\Delta+\delta)\right] m\,. \tag{3.5}$$

Combining relations (3.4) and (3.5) we arrive at our first main result.

THEOREM 3.1. *Let G be a graph with m edges, maximum vertex degree Δ, and minimum vertex degree δ. Then its Sombor index is bounded as:*

$$M_1(G) - (2-\sqrt{2})\,\Delta\, m \leqslant SO(G) \leqslant \frac{1}{\sqrt{2}}\,M_1(G) + \left[\sqrt{\Delta^2+\delta^2} - \frac{1}{\sqrt{2}}\,(\Delta+\delta)\right] m\,.$$

Equality on the left-hand side holds if and only G is regular. Equality on the right-hand side holds if and only if all edges of G connect a vertex of degree Δ with a vertex of degree δ (which includes regular graphs when $\Delta = \delta$).

In order to obtain analogous estimates for the elliptic Sombor index, we consider the auxiliary functions

$$\begin{aligned} R_1 &= R_1(a,b) = (a+b)^2 - (a+b)\,\sqrt{a^2+b^2} \\ R_2 &= R_2(a,b) = (a+b)\,\sqrt{a^2+b^2} - \frac{1}{\sqrt{2}}\,(a+b)^2\,. \end{aligned}$$

Since $R_1 = (a+b)\,Q_1$, and since Q_1 is monotonically increasing in both variables a and b, this must be the case also with R_1. Thus

$$(a+b)^2 - (a+b)\,\sqrt{a^2+b^2} \leqslant (\Delta+\Delta)^2 - (\Delta+\Delta)\,\sqrt{\Delta^2+\Delta^2} = 2(2-\sqrt{2})\,\Delta^2$$

and by summation over all pairs of adjacent vertices of the graph G, we get

$$F(G) + 2\,M_2(G) - ESO(G) \leqslant 2(2-\sqrt{2})\,\Delta^2\, m\,. \tag{3.6}$$

From $R_2 = (a+b)\,Q_2$ we conclude that provided $a > b$, R_2 monotonically increases in the variable a. In order to resolve the case of variable b, rewrite R_2 as

$$R_2 = a^2\left[(1+\gamma)\,\sqrt{1+\gamma^2} - \frac{1}{\sqrt{2}}\,(1+\gamma)^2\right]$$

where $\gamma = b/a$.

By numerical testing it can be shown that $(1+\gamma)\,\sqrt{1+\gamma^2} - \frac{1}{\sqrt{2}}\,(1+\gamma)^2$ monotonically decreases in the interval $(0,1)$. This implies that R_2 attains its greatest value for minimal γ, which must be $\gamma = \delta/\Delta$, that is $a = \Delta$ and $b = \delta$, that is at $R_2(\Delta,\delta)$. We thus have

$$(a+b)\,\sqrt{a^2+b^2} - \frac{1}{\sqrt{2}}\,(a+b)^2 \leqslant (\Delta+\delta)\,\sqrt{\Delta^2+\delta^2} - \frac{1}{\sqrt{2}}\,(\Delta+\delta)^2$$

and

$$\text{(3.7)}\quad ESO(G) - \frac{1}{\sqrt{2}}\,[F(G) + 2\,M_2(G)] \leqslant \left[(\Delta+\delta)\,\sqrt{\Delta^2+\delta^2} - \frac{1}{\sqrt{2}}\,(\Delta+\delta)^2\right] m\,.$$

By combining the relations (3.6) and (3.7), we obtain the second main result.

Theorem 3.2. *Let G be a graph with m edges, maximum vertex degree Δ, and minimum vertex degree δ. Then its elliptic Sombor index is bounded as:*

$$F(G) + 2\,M_2(G) - 2(2-\sqrt{2})\,\Delta^2\,m \leqslant ESO(G)$$

$$\leqslant \quad \frac{1}{\sqrt{2}}\,[F(G) + 2\,M_2(G)] + \left[(\Delta+\delta)\,\sqrt{\Delta^2+\delta^2} - \frac{1}{\sqrt{2}}\,(\Delta+\delta)^2\right] m\,.$$

Conditions for equality are same as in Theorem 3.1.

The coindex-version of the estimates (3.2) is

$$\frac{1}{\sqrt{2}}\,\overline{M_1}(G) \leqslant \overline{SO}(G) < \overline{M_1}(G)$$

which together with (3.2) yields

$$\frac{1}{\sqrt{2}}\left[M_1(G) + \overline{M_1}(G)\right] \leqslant SO(G) + \overline{SO}(G) < M_1(G) + \overline{M_1}(G)\,.$$

Taking into account Lemma 2.1, we obtain bounds involving the Sombor index and its coindex.

Theorem 3.3. *Let G be a graph with n vertices and m edges. Then the sum of its Sombor index and its coindex is bounded as:*

$$\sqrt{2}m(n-1) \leqslant SO(G) + \overline{SO}(G) < 2m(n-1)\,.$$

Equality on the left-hand side holds if and only if the graph G is regular.

In an analogous manner, using Eq. (3.3), and Lemmas 2.2 and 2.3 we have:

Theorem 3.4. *Let G be a graph with n vertices and m edges. Then the sum of its elliptic Sombor index and its coindex is bounded as:*

$$\frac{1}{\sqrt{2}}\left[(n-2)\,M_1(G) + 4m^2\right] \leqslant ESO(G) + \overline{ESO}(G) < (n-2)\,M_1(G) + 4m^2\,.$$

Equality on the left-hand side holds if and only if the graph G is regular.

If instead of Eqs. (3.2) and (3.3), we employ the results of Theorems 3.1 and 3.2, the following improved versions of Theorems 3.3 and 3.4 are obtained.

Theorem 3.5. *Let G be a graph with n vertices, m edges, maximal vertex degree Δ, and minimal vertex degree δ. Then the sum of its Sombor index and its coindex is bounded as:*

$$2m(n-1) - (2-\sqrt{2})\,\Delta\,m \leqslant SO(G) + \overline{SO}(G)$$

$$\leqslant \quad \sqrt{2}\,m(n-1) + \left[\sqrt{\Delta^2+\delta^2} - \frac{1}{\sqrt{2}}\,(\Delta+\delta)\right] m\,.$$

Conditions for equality are same as in Theorem 3.1.

Theorem 3.6. *Let G be a graph with n vertices, m edges, maximal vertex degree Δ, and minimal vertex degree δ. Then the sum of its elliptic Sombor index and its coindex is bounded as:*

$$\begin{aligned} &(n-2)\,M_1(G) + 4m^2 - 2(2-\sqrt{2})\,\Delta^2\,m \leqslant ESO(G) + \overline{ESO}(G) \\ \leqslant\ & \frac{1}{\sqrt{2}}\left[(n-2)\,M_1(G) + 4m^2\right] + \left[(\Delta+\delta)\sqrt{\Delta^2+\delta^2} - \frac{1}{\sqrt{2}}\,(\Delta+\delta)^2\right] m\,. \end{aligned}$$

Conditions for equality are same as in Theorem 3.1.

References

[1] S. Anwar, M. Azeem, M. K. Jamil, B. Almohsen, and Y. Shang, *Single-valued neutrosophic fuzzy Sombor numbers and their applications in trade flows between different countries via sea route*, J. Supercomput. **80** (2024) 19976–20019.

[2] J. A. Bondy and U. S. R. Murty, *Graph Theory with Applications*, Macmillan Press, New York, 1976.

[3] Z. Du, L. You, H. Liu, and Y. Huang, *The Sombor index and coindex of two-trees*, AIMS Math. **8**(8) (2023) 18982–18994.

[4] Z. Du, L. You, H. Liu, and Y. Huang, *The Sombor index and coindex of chemical graphs*, Polyc. Arom. Comp. **44** (2024) 2942–2964.

[5] B. Furtula and I. Gutman, *A forgotten topological index*, J. Math. Chem. **53** (2015) 1184–1190.

[6] N. Ghanbari and S. Alikhani, *Elliptic Sombor index of graphs from primary subgraphs*, Global Anal. Discr. Math. **8** (2023) 127–140.

[7] I. Gutman, *Geometric approach to degree-based topological indices: Sombor indices*, MATCH Commun. Math. Comput. Chem. **86** (2021) 11–16.

[8] I. Gutman, *Some basic properties of Sombor indices*, Open J. Discr. Appl. Math. **4**(1) (2021) 1–3.

[9] I. Gutman, *Improved estimates of Sombor index*, Iran. Math. Chem. **15** (2024) 1–5.

[10] I. Gutman, B. Furtula, Ž. Kovijanić Vukićević, and G. Popivoda, *On Zagreb indices and coindices*, MATCH Commun. Math. Comput. Chem. **74** (2015) 5–16.

[11] I. Gutman, B. Furtula, and M. S. Oz, *Geometric approach to vertex-degree-based topological indices -Elliptic Sombor index – theory and application*, Int. J. Quantum Chem. **124**(2) (2024) e27346.

[12] I. Gutman, N. K. Gürsoy, A. Gürsoy, and A. Ülker, *New bounds on Sombor index*, Commun. Comb. Optim. **8** (2023) 305–311.

[13] I. Gutman and N. Trinajstić, *Graph theory and molecular orbitals. Total π-electron energy of alternant hydrocarbons*, Chem. Phys. Lett. **17** (1972) 535–538.

[14] F. Harary, *Graph Theory*, Addison-Wesley, Reading, 1969.

[15] M. Imran, M. Azeem, M. K. Jamil, and M. Deveci, *Some operations on intuitionistic fuzzy graphs via novel versions of the Sombor index for internet routing*, Granular Comput. **9** (2024) #53.

[16] G. Kaya Gök and K. Çelik, *New bounds for Sombor index*, Bull. Int. Math. Virt. Inst. **14** (2024) 267–274.

[17] V. R. Kulli, Graph indices, in: M. Pal, S. Samanta, A. Pal (Eds.), *Handbook of Research of Advanced Applications of Graph Theory in Modern Society*, Global, Hershey, 2020, pp. 66–91.

[18] H. Liu, I. Gutman, L. You, and Y. Huang, *Sombor index: Review of extremal results and bounds*, J. Math. Chem. **66** (2022) 771–798.

[19] E. Milovanović, S. Stankov, M. Matejić, and I. Milovanović, *Some observations on Sombor coindex of graphs*, Commun. Comb. Optim. **9** (2024) 813–825.

[20] I. Milovanović, E. Milovanović, and M. Matejić, *On some mathematical properties of Sombor indices*, Bull. Int. Math. Virt. Inst. **11** (2021) 341–353.

[21] S. Nikolić, G. Kovačević, A. Miličević, and N. Trinajstić, *The Zagreb indices 30 years after*, Croat. Chem. Acta **76** (2003) 113–124.

[22] C. Phanjoubam, S. M. Mawiong, and A. M. Buhphang, *On Sombor coindex of graphs*, Commun. Comb. Optim. **8** (2023) 513–529.

[23] F. Qi and Z. Lin, *Maximal elliptic Sombor index of bicyclic graphs*, Contrib. Math. **10** (2024) 25–29.

[24] J. Rada, J. M. Rodríguez, and J. M. Sigarreta, *Optimization problems for general elliptic Sombor index*, MATCH Commun. Math. Comput. Chem. **93** (2025) 819–838.

[25] A. Rauf and S. Ahmad, *On Sombor indices of tetraphenylethylene, terpyridine rosettes and QSPR analysis on fluorescence properties of several aromatic hetero-cyclic species*, Int. J. Quantum Chem. **124**(1) (2024) e27261.

[26] I. Redžepović, *Chemical applicability of Sombor indices*, J. Serb. Chem. Soc. **86** (2021) 445–457.

[27] Z. Tang, Y. Li, and H. Deng, *Elliptic Sombor index of trees and unicyclic graphs*, El. J. Math. **7** (2024) 19–34.

[28] R. Todeschini and V. Consonni, *Molecular Descriptors for Chemoinformatics*, Wiley–VCH, Weinheim, 2009.

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