BULLETIN OF THE INTERNATIONAL MATHEMATICAL VIRTUAL INSTITUTE
ISSN (p) 2303-4874, ISSN (o) 2303-4955
www.imvibl.org /JOURNALS/BULLETIN
Bull. Int. Math. Virtual Inst., **14**(1)(2024), 31–40
DOI: 10.7251/BIMVI2401031R

Former
BULLETIN OF THE SOCIETY OF MATHEMATICIANS BANJA LUKA
ISSN 0354-5792 (o), ISSN 1986-521X (p)

ALMOST PRIME IDEALS IN $\Gamma-$SEMIRINGS

Hitesh Kumar Ranote

Abstract. In this paper, we generalize the results of almost prime ideals in a commutative $\Gamma-$semiring which are analogues to the results of almost prime ideals in commutative semirings. Further, we investigate the results regarding k-ideals, k-closure and local $\Gamma-$semirings. Finally, we prove a characterization theorem for n-almost prime ideals in $\Gamma-$semirings.

1. Introduction

The algebraic structure of a semiring, which is a common generalization of a ring was first introduced by Vandiver [**15**] in 1934. But non-trivial examples of a semiring appeared in the theory of commutative ideals of a ring studied by Dedekind in the 19th century. Despite the great efforts of some mathematicians on semiring theory in 1940, 1950 and early 1960, they were apparently not successful at drawing the attention of mathematical society to consider semiring theory as a serious line of mathematical research. Actually, it was in the late 1960 that semiring theory was considered a more important topic for research when real applications were found for semirings. Semirings are useful in the areas of theoretical computer science as well as in the solutions of graph theory in particular for studying automata theory, coding theory and formal languages. Semiring theory has many applications in other branches. A natural example of a semiring that is not a ring is the set of all non-negative integers under the usual addition and multiplication. In 1995, Rao [**8**] first introduced the concept of a $\Gamma-$ semiring. The important reason for the development of $\Gamma-$ semirings is a generalization of the results of rings, $\Gamma-$ rings, semirings, semi-groups and ternary semirings. The set of all negative integers $\mathbf{Z}^-$ is not a semiring with respect to usual addition and multiplication, but $\mathbf{Z}^-$ forms

2010 *Mathematics Subject Classification.* Primary 16Y60; Secondary 46J20.
Key words and phrases. k-ideals, Q-ideal, k-closure, local $\Gamma-$ semiring, almost prime ideals.
Communicated by Dusko Bogdanic.

a $\Gamma-$ semiring where $\Gamma = Z$. However, the properties of ideals in semirings and $\Gamma-$ semirings are somewhat different from the properties of the usual ring ideals. In order to address these differences, the concept of k-ideals in a semiring was introduced and considered by Torre [**7**] in 1965.

Bhatwadekar and Sharma [**5**] introduced the concept of almost prime ideals in commutative rings with non-zero identity, which arises from the study of factorization in Noetherian domains. Anderson and Smith [**2**] introduced the notion of weakly prime ideals that arise from the study of factorization in commutative rings with zero divisors. The concept of almost prime ideals over a commutative semiring with identity is studied by Atani [**3**]. In this paper, we generalize these results in almost prime ideals over a commutative $\Gamma-$ semiring with identity. Further, we study the relationship between almost prime ideals and weakly prime ideals in $\Gamma-$ semirings and provide several equivalent conditions for an ideal of a $\Gamma-$ semiring to be almost prime, which are analogous to conditions for an ideal to be prime. This study actually aims to generalize several findings about virtually prime ideals and weakly prime ideals from commutative semirings to commutative $\Gamma-$ semirings.

2. Preliminaries

In this section, we examine some of the basic definitions and fundamental concepts that are important to this paper. R represents a $\Gamma-$ semiring throughout this paper.

Definition 2.1. *[**8**] Let R and Γ be two additive commutative semigroups. Then R is called a $\Gamma-$ semiring if there exists a mapping $R \times \Gamma \times R \to R$ denoted by $x\alpha y$ for all $x, y \in R$ and $\alpha \in \Gamma$ satisfying the following conditions:*

(1) $(x + y)\alpha z = x\alpha z + y\alpha z$.
(2) $x(\alpha + \beta)z = x\alpha z + x\beta z$.
(3) $x\alpha(y + z) = x\alpha y + x\alpha z$.
(4) $(x\alpha y)\beta z = x\alpha(y\beta z)$ *for all* $x, y, z \in R$ *and* $\alpha, \beta \in \Gamma$.

Definition 2.2. *[**10**] A $\Gamma-$ semiring R is said to have a zero element if $0\gamma x = 0 = x\gamma 0$ and $x + 0 = x = 0 + x$ for all $x \in R$ and $\gamma \in \Gamma$.*

Definition 2.3. *[**10**] A $\Gamma-$ semiring R is said to have an identity element if for all $x \in R$, there exists $\alpha \in \Gamma$ such that $1\alpha x = x = x\alpha 1$.*

Definition 2.4. *[**8**] A $\Gamma-$ semiring R is said to be commutative if $x\gamma y = y\gamma x$ for all $x, y \in R$ and for all $\gamma \in \Gamma$.*

Definition 2.5. *[**6**] A $\Gamma-$ semiring R with a zero element is said to be right (left) multiplicative cancellable if for all $a, b, c \in R$ and $\alpha \in \Gamma$ we have that $a \neq 0$, $a\alpha b = a\alpha c$ and $b\alpha a = c\alpha a$ implies $b = c$.*

Definition 2.6. *[**10**] An element x of a $\Gamma-$ semiring R is said to be multiplicative $\Gamma-$ idempotent if there exists $\gamma \in \Gamma$ such that $x = x\gamma x$. If every element of R is multiplicative $\Gamma-$ idempotent then R is called multiplicative $\Gamma-$ idempotent $\Gamma-$ semiring.*

DEFINITION 2.7. *[**6**] A non empty subset I of R is said to be a left (right) ideal of R if I is sub semigroup of $(R,+)$ and $x\alpha y \in I(y\alpha x \in I)$ for all $y \in I, x \in R$ and $\alpha \in \Gamma$. If I is both a left and right ideal of R, then I is known to be an ideal of R.*

DEFINITION 2.8. *[**6**] An ideal I of a $\Gamma-$ semiring R is called k-ideal if for $x, y \in R, x+y \in I$ and $y \in I$ implies that $x \in I$.*

DEFINITION 2.9. *[**11**] A proper ideal M of a $\Gamma-$ semiring R is said to be maximal (resp. k-maximal) ideal if there does not exist any other proper ideal (resp. k-ideal) of R containing M properly.*

DEFINITION 2.10. *[**6**] Let R be a $\Gamma-$ semiring and J be an ideal of R. Then the k-closure of J is denoted by $cl(J)$ and defined by $cl(J) = \{x \in R : x + j \in J$ for some $j \in J\}$.*

DEFINITION 2.11. *[**6**] Let R be a $\Gamma-$ semiring. An ideal P of R is a prime ideal if for any two ideals A and B of R such that $A\Gamma B \subseteq P$ implies that either $A \subseteq P$ or $B \subseteq P$.*

DEFINITION 2.12. *A $\Gamma-$ semiring R is said to be $\Gamma-$ semidomain if $x, y \in R$ and $\alpha \in \Gamma$ such that $x\alpha y = 0$, then either $x = 0$ or $y = 0$.*

REMARK 2.1. All through here, R will signify with "0" and "1" as zero and identity element except if in any case expressed.

3. Almost prime ideals in a $\Gamma-$semiring

In this section, we investigate the properties of almost prime ideals and prove some results.

DEFINITION 3.1. *Let R be a commutative $\Gamma-$ semiring. A proper ideal M of R is said to be an n-almost prime $(n \geqslant 2)$ ideal of R in which $m \in M$ or $n \in M$ and $\alpha \in \Gamma$ such that $m\alpha n \in M - (M\Gamma)^{n-1}M$. In particular, the almost prime ideals are just the 2-almost prime ideals.*

EXAMPLE 3.1. *Let $R = (Z_{10}, +_{10})$ be an additive commutative semigroup of addition modulo 10 and $\Gamma = \{0, 2, 4\}$. Then R is a $\Gamma-$ semiring, since the function $R \times \Gamma \times R \to R$ with $(x, \alpha, y) \to (x\alpha y)$ usual under scalar multiplication of integers is well defined and satisfied all the properties, where $x, y \in R, \alpha \in \Gamma$. Let $M = 2Z_{10} = \{0, 2, 4, 8\}$ and let $M\Gamma M = \{0, 6\}$, then $M - M\Gamma M = \{2, 4, 8\}$. As $1, 2 \in R$ and $4 \in \Gamma$ such that $1.4.2 = 8 \in M - M\Gamma M$, where $2 \in M$. Therefore, M is an almost prime ideal in R.*

DEFINITION 3.2. *[**9**] An ideal P of $\Gamma-$ semiring R is said to be a weakly prime ideal if $0 \neq x\alpha y \in P$, $\alpha \in \Gamma$ implies $x \in P$ or $y \in P$.*

Every prime ideal is weakly prime, but converse need not be true.

EXAMPLE 3.2. *Let $R = (Z_6, +_6)$ be an additive commutative semigroup of addition modulo 6 and $\Gamma = \{0, 2, 4\}$. Then R is a $\Gamma-$ semiring. If $P = \{0\}$, then it is a weakly prime ideal by definition but it is not prime, since $1, 3 \in R$ and $2 \in \Gamma$ such that $1.2.3 = 0 \in P$, where neither $1 \in P$ nor $3 \in P$*

Definition 3.3. *[**6**] Let R be a $\Gamma-$ semiring. An ideal J of R is said to be a partitioning ideal ($Q-ideal$) if there exists a subset Q of R such that:*

1. $R = \cup\{q + J : q \in Q\}$.
2. *If $q_1, q_2 \in Q$, then $(q_1 + J) \cap (q_2 + J) \neq \phi$ if and only if $q_1 = q_2$.*

Let J be a Q-ideal of $\Gamma-$ semiring R and let $R/J = \{q + J : q \in Q\}$, then R/J form a $\Gamma-$ semiring under the binary operations $\oplus, \odot$ defined as follows: $((q_1 + J) \oplus (q_2 + J)) = q_3 + J$, where $q_3 \in Q$ is the unique element such that $q_1 + q_2 + J \subseteq q_3 + J$ and $((q_1 + J) \odot \alpha \odot (q_2 + J)) = q_4 + J$, where $q_4 \in Q$ is the unique element such that $q_1\alpha q_2 + J \subseteq q_4 + J$ for all $\alpha \in \Gamma$. This $\Gamma-$ semiring R/J is called the quotient $\Gamma-$ semiring of R by J [**6**].

Lemma 3.1. *[**9**] Let R be a $\Gamma-$ semiring.*

1. *If an ideal of R is the union of two k-ideals, then it is equal to one of them.*
2. *Let J and K be ideals of R with J being a k-ideal and $x \in R$. Then $(I : J) = \{r \in R : r\Gamma K \subseteq J\}$, $(0 : x)$ and $(J : x)$ are k-ideals of R.*

Theorem 3.1. *Let R be a $\Gamma-$ semiring. Then every weakly prime k-ideal of R is an almost prime ideal.*

Proof. Let M be a weakly prime k-ideal of R. Assume that $x, y \in R$ and $\alpha \in \Gamma$ such that $x\alpha y \in M - M\Gamma M$ with $x \notin M$. Obviously, $M \cup (0 : x) \subseteq (M : x)$. Suppose $z \in (M : x)$. If $z\alpha x \neq 0$, then M is weakly prime, giving $z \in M$. If $z\alpha x = 0$, then we have $z \in (0 : x)$. Thus, $M \cup (0 : x) = (M : x)$. Therefore by Lemma 3.1 (i), either $(M : x) = M$ or $(M : x) = (0 : x)$. As $y \notin (0 : x)$ and $y \in (M : x)$. Hence, $y \in M$. □

Theorem 3.2. *[**6**] Let R be a commutative $\Gamma-$ semiring with zero element and J be a Q-ideal in R. If $a \in Q$ and $a + J$ is the zero in R/J, then $a + J = J$.*

Theorem 3.3. *Let R be a $\Gamma-$ semiring and J be a proper Q-ideal of R. Then J is prime if and only if R/J is a $\Gamma-$ semidomain.*

Proof. Let p be the unique element in Q such that $p + J$ is the zero in R/J. Let J be a prime ideal of R and $p_1 + J$ and $p_2 + J$ be elements of R/J in such a way that $(p_1 + J) \odot \alpha \odot (p_2 + J) = p + J$ where $p_1, p_2 \in Q, \alpha \in \Gamma$. If $p_1 \in J$, then $p_1 \in (p + J) \cap (p_1 + J)$. Hence $p_1 + J = p + J$. Let $p_1 \notin J$. Assume that $p_1\alpha p_2 + J \subseteq p + J$, $\alpha \in \Gamma$, so $p + a = p_1\alpha p_2 + b \in p + J = J$ and gives $p_1\alpha p_2 \in J$ for some $a, b \in J$. Since J is prime, $p_2 \in J = p + J$. Therefore, $p_2 \in (p_2 + J) \cap (p + J)$ and it follows that $p_2 + J = p + J$. Hence, R/J is a $\Gamma-$ semidomain. Conversely, let R/J be a $\Gamma-$ semidomain. Assume that $x, y \in R$ and $\alpha \in \Gamma$ such that $x\alpha y \in J$ with $x \notin J$. Then by Theorem 3.2, we have $x\alpha y \in p + J$ and $x \notin p + J = J$. As J is a Q-ideal of R, then $p_1, p_2 \in Q$ such that $x + J \subseteq p_1 + J$ and $y + J \subseteq p_2 + J$. Therefore, $x = p_1 + i$ and $y = p_2 + j$ for some $i, j \in J$. Since $x \in p_1 + J$ and $x \notin p + J$, we have $p_1 + J \neq p + J$. It is easy to prove that $x\alpha y \in p_1\alpha p_2 + J$. Let p_3 be the unique element in Q such that $p_1\alpha p_2 + J \subseteq p_3 + J$. Since $x\alpha y \in (p + J) \cap (p_3 + J)$, we must have $p = p_3$ and $(p_1 + J) \odot \alpha \odot (p_2 + J) = p + J$. Hence $y \in p_2 + J = p + j = J$, as R/J is a $\Gamma-$ semidomain. Therefore, J is prime. □

Corollary 3.1. *Let R be a* $\Gamma-$ *semiring. Then* $\{0\}$ *is prime if and only if R is* $\Gamma-$ *semidomain.*

Proof. It follows from Theorem 3.3, since $\{0\}$ is a Q-ideal of $R \cong R/\{0\}$ with $Q = R$.

□

Definition 3.4. *[**6**] Let R be a* $\Gamma-$ *semiring with non-zero identity. A non-zero element x of R is said to be a semi-unit in R. If there exist* $r, s \in R$ *and* $\alpha, \beta \in \Gamma$ *such that* $1 + r\alpha x = s\beta x$.

Definition 3.5. *[**6**] A* $\Gamma-$ *semiring R is said to be local* $\Gamma-$ *semiring if and only if R has a unique maximal k-ideal.*

Moreover, a is a semi-unit of R if and only if a lies outside each maximal k-ideal of R ([**6**], Lemma 4.9). Let I be a proper Q-ideal of a $\Gamma-$ semiring R, then there exists a maximal k-ideal M of R with $I \subseteq M$ ([**6**], Theorem 4.4). The ideal $\{0\}$ is always weakly prime and hence almost prime by Theorem 3.1, but it is prime if and only if R is a $\Gamma-$ semidomain by Corollary 3.1. Thus, weakly prime ideals and almost prime ideals need not be prime ideals. Moreover, an idempotent ideal $I(I = I\Gamma I)$ is almost prime. We next give a non-trivial example of an almost prime ideal which is not a prime.

example 3.3. *Let* (R, M) *be a local* $\Gamma-$ *semiring with* $M\Gamma M = 0$. *Let J be a proper k-ideal of R such that* $0 \neq x\alpha y \in J, \alpha \in \Gamma$. *Since* $M\Gamma M = 0$ *and* $x\alpha y \neq 0$, *either x or y does not lie in M. If* $x \notin M$, *then x is a semi-unit. Thus, for some* $r, s \in R$ *and* $\alpha, \beta \in \Gamma$ *such that* $1 + r\alpha x = s\beta x$. *Therefore,* $(1 + r\alpha x)\gamma y = (s\beta x)\gamma y$ *implies that* $1\gamma y + (r\alpha x)\gamma y = (s\beta x)\gamma y$. *As J is a k-ideal of R, so* $y \in J$. *Hence, every proper k-ideal of J is weakly prime ideal and hence almost prime. While, if* $J \subset M$, *J is not prime, since M is the unique prime k-ideal of R.*

Theorem 3.4. *Let R be a* $\Gamma-$ *semiring. If J and K are k-ideals of R with* $J\Gamma K = J \cap K$, *then* $J\Gamma K$ *is a k-ideal of R.*

Proof. It is sufficient to show that $J\Gamma K = cl(J\Gamma K)$. As, it is clear that $J\Gamma K \subseteq cl(J\Gamma K)$. We will now show the reverse inclusion. Assume that $y \in cl(J\Gamma K)$, then $y + j = k$ for some $j, k \in J\Gamma K$. Therefore, $y \in J \cap K = J\Gamma K$. □

Theorem 3.5. *Let J be a Q-ideal of R and M be a k-ideal of R such that* $J \subseteq M$. *Then* $M/J = \{q + J : q \in M \cap Q\}$ *is a k-ideal of* R/J.

Proof. Let p be the unique element of Q such that $p + J$ is the zero in R/J. We first prove that $p + J \in M/J$. Let $x \in M \cap Q$ such that $x + J \in M/J \subseteq R/J$. Then $(x+J) \oplus (p+J) = x+J$ where $x+p+J \subseteq x+J$ implies that $x+p+j = x+k$ for some $j, k \in J$. As M is a k-ideal of R, $p \in M \cap Q$. Therefore, $p + J \in M/J$. Let $p_1, p_2 \in M \cap Q$ such that $p_1 = p'_1 + J, p_2 = p'_2 + J \in M/J$, then there exists a unique element $p_3 \in Q$ with $p_1 + p_2 = p_3 + J$ and $p_1 + p_2 + J \subseteq p_3 + J$. Thus, $p_1 + p_2 + l = p_3 + m \in M$ for some $l, m \in J$. Hence, $p_3 \in Q \cap M$, as M is a k-ideal of R. Therefore, $p_1 + p_2 \in M/J$. It is sufficient to prove that if $r + J \in R/J$ and $x + J \in M/J$ where $r \in Q, x \in M \cap Q$, then $(r + J) \odot \alpha \odot (x + J) \in M/J, \alpha \in \Gamma$.

There exists a unique element $p_4 \in Q$ such that $(r+J)\odot\alpha\odot(x+J) = p_4+J, \alpha \in \Gamma$ and $r\alpha x + J \subseteq p_4 + J$ so $r\alpha x + a = p_4 + b \in M$ for some $a, b \in J$. Consequently, $p_4 \in M \cap Q$. Therefore, $p_4 + J \in M/J$. Thus, M/J is an ideal of R/J. Finally, let $y + J \in M/J$ and $(y + J) \oplus (z + J) = w + J \in M/J$ where $y, w \in M \cap Q, z \in Q$ and $y + z + J \subseteq w + J$. Then $y + z + c = w + d \in M$ for some $c, d \in J$. Since M is a k-ideal of R, thus $w \in M \cap Q$. Therefore, $w + J \in M/J$. □

THEOREM 3.6. *Let R be a $\Gamma-$ semiring. Then the following holds:.*

1. *If I and J are k-ideals of R, then $I + J$ is a k-ideal of R.*
2. *An intersection of a family of k-ideals of R is a k-ideal.*

PROOF. The proof is straightforward. □

THEOREM 3.7. *Let J be a Q-ideal of R and M be an almost prime ideal of R with $J \subseteq M$. If $M\Gamma M$ is a k-ideal of R, then M/J is an almost prime ideal of R/J.*

PROOF. By Theorem 3.5, $M/J = \{q + J : q \in M \cap Q\}$ is a k-ideal of R/J. By Theorem 3.6, $M\Gamma M + J$ is a k-ideal of R. We first prove that $(M\Gamma M + J)/J = (M/J)\Gamma(M/J)$. Let $x_j, y_j \in M/J$ and $\alpha_j \in \Gamma$ such that $Z = \sum_{j=1}^{n} x_j\alpha_j y_j \in (M/J)\Gamma(M/J)$. It is sufficient to prove that $x_j\alpha_j y_j \in (M\Gamma M + J)/J$, for all j $(1 \leqslant j \leqslant n)$. As J is a Q-ideal of R, $x_j = q_j + J$ and $y_j = q'_j + J$ for some $q_j, q'_j \in Q \cap M$. Then there is a unique element $p \in Q$ such that $x_j\alpha_j y_j = p + J$ where $q_j\alpha_j q'_j + J \subseteq p+J$. Consequently, $q_j\alpha_j q'_j + l = p+k$ for some $k, l \in J$. Therefore, $p \in (M\Gamma M+J)\cap Q$ and so $x_j\alpha_j y_j \in (M\Gamma M+J)/J$. Hence, $(M/J)\Gamma(M/J) \subseteq (M\Gamma M+J)/J$. For the reverse inclusion, suppose that $q+J \in (M\Gamma M+J)/J$, where $q \in (M\Gamma M + J) \cap Q$. Then $q = \sum_{j=1}^{m} z_j\alpha_j w_j + a$ for some $z_j, w_j \in M(1 \leqslant j \leqslant m)$ and $a \in J$. Since J is a Q-ideal of R and M is a k-ideal of R, then there exist $u_j, v_j \in M \cap Q$ such that $q = \sum_{j=1}^{m} u_j\alpha_j v_j + b$, for some $b \in J, \alpha_j \in \Gamma$. An inspection will show that $q + J = \sum_{j=1}^{m}(u_j + J) \odot \alpha_j \odot ((v_j + J) \in (M/J)\Gamma(M/J)$. Let $m_1 + J, m_2 + J \in R/J$ such that $(m_1+J)\odot\alpha\odot(m_2+J) \in (M/J)-(M/J)\Gamma(M/J)$, where $m_1, m_2 \in Q$. Then there is a unique element $m_3 \in Q$ such that $m_1\alpha m_2+J \subseteq m_3 + J \in M/J - (M\Gamma M + J)/J$, so $m_3 \in M \cap Q$. Hence, $m_1\alpha m_2 \in M$ and $m_1\alpha m_2 \notin M\Gamma M + J$ and then $m_1\alpha m_2 \notin M\Gamma M$. Hence, $m_1 \in M$ or $m_2 \in M$, as M is an almost prime. Therefore, $m_1 + J \in M/J$ or $m_2 + J \in M/J$. □

DEFINITION 3.6. *Let R be a $\Gamma-$ semiring and J be a Q-ideal of R. An element $x \in R$ is called a zero divisor in R/J if there exists $y \in R - J$ and $\alpha \in \Gamma$ such that $x\alpha y \in J$.*

DEFINITION 3.7. *An ideal J of a $\Gamma-$ semiring R is said to be an invertible ideal if there is an ideal K of R such that $J\Gamma K = R$.*

Note that, if an ideal J of a $\Gamma-$ semiring R is invertible and $J\Gamma K = R$, for some ideal K of R, then K is unique and we denote that by J^{-1}.

THEOREM 3.8. *Let R be a* $\Gamma-$ *semiring and M be an n-almost prime Q-ideal of R such that* $(M\Gamma)^{n-1}M$ *is a k-ideal. Then the following holds:*

(1) *If* $r \in R$ *is a zero divisor in* R/M *then either* $r \in M$ *or* $r\Gamma M \subseteq (M\Gamma)^{n-1}M$.
(2) *If for any ideal J of R,* $J \subseteq M$ *and J consists of zero divisors on* R/M, *then* $J\Gamma(M\Gamma)^{n-2}M = (M\Gamma)^{n-1}M$.
(3) *If M is invertible, then M is a prime k-ideal of R.*

PROOF. **(i)** Let $r\alpha s \in M$ for some $s \in R - M$ and $\alpha \in \Gamma$. Assume that $r \notin M$. As M is an n-almost prime, then $r\alpha s \in (M\Gamma)^{n-1}M$. It is sufficient to prove that $r\alpha m \in (M\Gamma)^{n-1}M$, for all $m \in M$. Since every Q-ideal is a k-ideal, so assume that $m \in M$, then $m + s \notin M$ and $r\alpha(m + s) \in M$. Therefore, $r\alpha(m + s) = r\alpha m + r\alpha s \in (M\Gamma)^{n-1}M$, as M is n-almost prime. It follows that, $r\alpha s \in (M\Gamma)^{n-1}M$, $r\alpha m \in (M\Gamma)^{n-1}M$, since $(M\Gamma)^{n-1}M$ is a k-ideal. Hence, $r\Gamma M \subseteq (M\Gamma)^{n-1}M$.

(ii) Let $j \in J$ and $m \in (M\Gamma)^{n-2}M$. It is sufficient to show that $j\alpha m \in (M\Gamma)^{n-1}M$. Since j is a zero divisor in R/M, then by (i) either $j \in M$ or $j\Gamma M \subseteq (M\Gamma)^{n-1}M$. If $j \in M$, then the result is obvious. Assume that $j\Gamma M \subseteq (M\Gamma)^{n-1}M$. Hence, $j\alpha m \in j\Gamma((M\Gamma)^{n-2}M)) \subseteq j\Gamma M \subseteq (M\Gamma)^{n-1}M$, $\alpha \in \Gamma$.

(iii) Assume that $n \notin M$ and $\alpha \in \Gamma$ such that $m\alpha n \in M$. If $n \in M$, we are done. So assume that $n \notin M$, then $m \notin M$ and $n \notin M$ but $m\alpha n \in M$. Thus, n is a zero divisor of R/M, then by (i), $n\Gamma M \subseteq (M\Gamma)^{n-1}M$. Since M is invertible, $n\Gamma M\Gamma N \subseteq ((M\Gamma)^{n-1}M)\Gamma N$. Hence, $R\Gamma n \subseteq (M\Gamma)^{n-2}M$ implies that $n \in (M\Gamma)^{n-2}M \subseteq M$, which is a contradiction. Therefore, M is a prime k-ideal of R. □

THEOREM 3.9. *Let R be a local* $\Gamma-$ *semiring with a unique maximal k-ideal M and J be a Q-ideal of R such that* $M\Gamma M \subseteq J \subseteq M$ *and* $J\Gamma J$ *is a k-ideal. Then J is an almost prime if and only if* $M\Gamma M = J\Gamma J$.

PROOF. Let J be an almost prime ideal. Since $M\Gamma M \subseteq J$, then $m\alpha n \in M\Gamma M \subseteq J$, for all $m, n \in M$ and $\alpha \in \Gamma$. Now, we will prove that $m\alpha n \in J\Gamma J$. If $m\alpha n \notin J\Gamma J$, then J is an almost prime gives $m \in J$ or $n \in J$. Let $m \in J$, then $n \notin J$ otherwise $m\alpha n \in J\Gamma J$. As $n\alpha n \in M\Gamma M \subseteq J$, $\alpha \in \Gamma$, n is a zero divisor in R/J, Then by Theorem 3.8, $m\alpha n \in n\Gamma J \subseteq J\Gamma J$, which is a contradiction. Therefore, $M\Gamma M - J\Gamma J$. Conversely, suppose that $M\Gamma M = J\Gamma J$. Let $m, n \in R$ and $\alpha \in \Gamma$ such that $m\alpha n \in J - J\Gamma J$. If $m \notin M$, then it is a semi-unit in R, thus for some $r, s \in R$ and $\beta, \gamma \in \Gamma$ such that $1 + r\gamma m = s\beta m$. Hence, $(1 + r\gamma m)\alpha n = (s\beta m)\alpha n$ implies that $1\alpha n + r\gamma m\alpha n = s\beta m\alpha n$. Since J is a k-ideal, so $n \in J$. Assume that $m, n \in M$ and $\alpha \in \Gamma$. In this case $m\alpha n \in M\Gamma M = J\Gamma J$, which is not true. Therefore, J is almost prime. □

THEOREM 3.10. *Let R be a commutative* $\Gamma-$ *semiring and* $x \in R$. *Then* $cl(R\Gamma x\Gamma x) = cl(R\Gamma x)\Gamma cl(R\Gamma x)$. *In particular,* $cl(R\Gamma x)\Gamma cl(R\Gamma x)$ *is a k-ideal of R.*

Proof. Let $y \in cl(R\Gamma x\Gamma x)$, then for some $m, n \in R$ and $\alpha, \beta, \gamma \in \Gamma$ such that $y + m\beta x\alpha x = n\gamma x\alpha x$. Since $m\beta x\alpha x, n\gamma x\alpha x \in cl(R\Gamma x)\Gamma(R\Gamma x) \subseteq cl(R\Gamma x)\Gamma cl(R\Gamma x)$. We have $cl(R\Gamma x\Gamma x) \subseteq cl(R\Gamma x)\Gamma cl(R\Gamma x)$. For the reverse inclusion, suppose that $k \in cl(R\Gamma x)$ and $n \in R$ such that $l = n\beta k\alpha k \in cl(R\Gamma x)\Gamma cl(R\Gamma x)$. Then there are elements $u, v \in R$ and $\alpha \in \Gamma$ such that $k + u\alpha x = v\alpha x$ so $n\delta k\gamma k + n\delta k\gamma u\alpha x + n\delta k\gamma u\alpha x + n\delta u\alpha u\gamma x\alpha x + n\delta u\alpha u\gamma x\alpha x = n\delta v\alpha v\gamma x\alpha x + n\delta u\alpha u\gamma x\alpha x$ and $n\delta k\gamma u\alpha x + n\delta k\gamma u\alpha x + n\delta u\alpha u\gamma x\alpha x + n\delta u\alpha u\gamma x\alpha x = n\delta v\alpha u\gamma x\alpha x + n\delta v\alpha u\gamma x\alpha x$ for some $\alpha, \gamma, \delta \in \Gamma$. Hence, $n\delta k\gamma k + n\delta v\alpha u\gamma x\alpha x + n\delta v\alpha u\gamma x\alpha x = n\delta v\alpha v\gamma x\alpha x + n\delta u\alpha u\gamma x\alpha x$ for some $\alpha, \gamma, \delta \in \Gamma$. Therefore, $x \in cl(R\Gamma x\Gamma x)$. □

Theorem 3.11. *Let R be a cancellative $\Gamma-$ semiring and $x \in R$. Then $cl(R\Gamma x)$ is an almost prime if and only if $cl(R\Gamma x)$ is a prime ideal of R.*

Proof. Let $cl(R\Gamma x)$ be almost prime and $m\alpha n \in cl(R\Gamma x)$ for some $m, n \in R$ and $\alpha \in \Gamma$. Assume that $m \notin cl(R\Gamma x)$ and $n \notin cl(R\Gamma x)$. Then $cl(R\Gamma x)$ is an almost prime gives $m\alpha n \in cl(R\Gamma x)\Gamma cl(R\Gamma x)$, Thus, $m\alpha(n + x) \in cl(R\Gamma x)$ and $m, n + x \notin cl(R\Gamma x)$, as it is a k-ideal of R. By Theorem 3.10, we have $m\alpha(n + x) \in cl(R\Gamma x)\Gamma cl(R\Gamma x) = cl(R\Gamma x\Gamma x)$. Hence, $m\alpha x \in cl(R\Gamma x\Gamma x)$, then for some $k, l \in R$ and $\alpha, \beta \in \Gamma$ such that $m\alpha x + l\alpha x\alpha x = k\beta x\alpha x$ implies that $m + l\alpha x = k\beta x$, which is a contradiction. Thus, $cl(R\Gamma x)$ is a prime ideal of R. The converse is trivial. □

Theorem 3.12. *[**9**] Let R_1 and R_2 be Γ_1 and Γ_2 semirings respectively. If we define:*

(1) $(x, y) + (z, w) = (x + z, y + w)$
(2) $(x, y)(\alpha, \beta)(z, w) = (x\alpha z, y\beta w)$, *for all $(x, y), (z, w) \in R_1 \times R_2$ and $(\alpha, \beta) \in \Gamma_1 \times \Gamma_2$. Then $R_1 \times R_2$ is a $\Gamma_1 \times \Gamma_2-$ semiring.*

Theorem 3.13. *Let R_1 and R_2 be $\Gamma = \Gamma_1 \times \Gamma_2-$ semirings. An ideal J of $R = R_1 \times R_2$ is an almost prime if and only if J satisfies one of the following conditions:*

(1) *$J = M_1 \times R_2$ for some almost prime ideal M_1 of R_1.*
(2) *$J = R_1 \times M_2$ for some almost prime ideal M_2 of R_2.*
(3) *$J = M_1 \times M_2$ for some idempotent ideals M_1 and M_2 of R_1 and R_2.*

Proof. Let M_1 be an ideal of R_1 and M_2 be an ideal of R_2 such that $J = M_1 \times M_2$ is an almost prime ideal of R. So $J \neq R$. We prove this theorem in two cases.
Case-I: Assume that $M_2 = R_2$. It is sufficient to show that M_1 is an almost prime ideal of R_1. Let $x\alpha y \in M_1 - M_1\Gamma_1 M_1$ for some $x, y \in R_1$, $\alpha \in \Gamma$. Then $(x, 1)(\alpha, \beta)(y, 1) \in (M_1 - M_1\Gamma_1 M_1) \times R_2 = J - J\Gamma J$, $\alpha, \beta \in \Gamma$. Hence, J is an almost prime gives either $x \in M_1$ or $y \in M_1$. Therefore, M_1 is an almost prime ideal of R_1. Similarly, if $M_1 = R_1$, then M_2 is an almost prime ideal of R_2.
Case-II: Assume that $M_1 \neq R_1$ and $M_2 \neq R_2$. If $M_1 \neq M_1\Gamma_1 M_1$, then $x \notin M_1\Gamma_1 M_1$ for some $x \in M_1$. Thus $(x, 1)(\alpha, \beta)(1, 0) = (x\alpha 1, 0) \in J - J\Gamma J = ((M_1 - M_1\Gamma_1 M_1) \times M_2) \cup ((M_1 \times (M_2 - M_2\Gamma_2 M_2))$, either $1 \in M_1$ or $1 \in M_2$, which is a contradiction. By a similar argument, we have $M_2 = M_2\Gamma M_2$. Conversely, let M_1 be an almost prime ideal of R_1 such that $J = M_1 \times R_2$. Now, we show that

J is an almost prime ideal of R. Let $(u_1, u_2), (v_1, v_2) \in R$ and $\alpha, \beta \in \Gamma$ such that $(u_1\alpha v_1, u_2\beta v_2) \in J - J\Gamma J = (M_1 - M_1\Gamma_1 M_1) \times R_2$, so $u_1\alpha v_1 \in M_1 - M_1\Gamma_1 M_1$. Hence, M_1 is an almost prime gives either $u_1 \in M_1$ or $v_1 \in M_1$. Therefore, either $(u_1, u_2) \in J$ or $(v_1, v_2) \in J$. Thus J is an almost prime. The similar reason is true for $J = R_1 \times M_2$, where M_2 is an almost prime ideal of R_2. Finally, suppose that $J = M_1 \times M_2$, where $M_1 = M_1\Gamma_1 M_1$ and $M_2 = M_2\Gamma_2 M_2$. Then $J = J\Gamma J$. Hence J is an almost prime. □

The characterizations of n-almost prime ideals are provided by the following theorem.

THEOREM 3.14. *Let R be a $\Gamma-$ semiring and M be a proper k-ideal of R, then the following statements are equivalent:*

(1) *M is n-almost prime.*
(2) *For $x \in R - M$, $(M : x) = M \cup ((M\Gamma)^{n-1}M : x)$.*
(3) *For $x \in R - M$, $(M : x) = M$ or $(M : x) = ((M\Gamma)^{n-1}M : x)$.*
(4) *For an ideals J and K of R with $J\Gamma K \subseteq M$ and $J\Gamma K \nsubseteq (M\Gamma)^{n-1}M$, then $J \subseteq M$ or $K \subseteq M$.*

PROOF. **(i) implies (ii).** Let $m \in (M : x)$ where $x \in R - M$. Then $m\alpha x \in M, \alpha \in \Gamma$. Assume that $m\alpha x \in (M\Gamma)^{n-1}M$, then $m \in ((M\Gamma)^{n-1}M : x)$. If $m\alpha x \notin (M\Gamma)^{n-1}M$, then M is n-almost primes gives $m \in M$. Thus, $(M : x) \subseteq M \cup ((M\Gamma)^{n-1}M : x)$. For any ideal M, the reverse inclusion is true.
(ii) implies (iii). Follows from Lemma 3.1 (i).
(iii) implies (iv). Let J and K be two ideals of R such that $J\Gamma K \subseteq J$. Suppose $J \nsubseteq M$ and $K \nsubseteq M$. Then there exists $n \in J - M$ such that $n\Gamma K \subseteq M$, so $K \subseteq (M : n)$, but $K \nsubseteq M$. Then by (iii), we have $(M : n) = ((M\Gamma)^{n-1}M : n)$. Therefore, $K \subseteq ((M\Gamma)^{n-1}M : n)$ implies that $n\Gamma K \subseteq ((M\Gamma)^{n-1}M : n)$. By a similar way, $z\Gamma K \subseteq (M\Gamma)^{n-1}M$, for some $z \in K - M$. Furthermore, $n\alpha z \in (M\Gamma)^{n-1}M$, for all $n \in J \cap M$, $z \in K \cap M$ and $\alpha \in \Gamma$. Therefore, $J\Gamma K \subseteq (M\Gamma)^{n-1}M$ which is a contradiction. Hence, $J \subseteq M$ or $K \subseteq M$.
(iv) implies (i). Let $m, n \in R$ and $\alpha \in \Gamma$ with $m\alpha n \in M - (M\Gamma)^{n-1}M$. Assume that $J = R\Gamma m$ and $K = R\Gamma n$. Then $J\Gamma K \subseteq M$ but $J\Gamma K \nsubseteq (M\Gamma)^{n-1}M$. By (iv), either $J \subseteq M$ or $K \subseteq M$. Thus, M is an n-almost prime ideal of R. □

References

1. P. J. Allen and J. Neggers, Ideal theory in commutative A-semirings, *KYUNGPOOK Math. J.*, **46** (2006), 261-271.
2. D. D. Anderson and E. Smith, Weakly prime ideals, *Houston J. of Math.*, **29**(4) (2003), 831-840.
3. R. E. Atani, Generalizations of prime ideals of semirings, *Azerbaijan Journal of Mathematics*, **3**(1) (2013), 76-83.
4. S. E. Atani, The ideal theory in quotients of commutative semirings, *Glansik Matematicki*, **42**(62) (2007), 301 – 308.
5. S. M. Bhatwadekar and P.K. Sharma, Unique factorization and birth of almost primes, *Comm. Algebra*, **33** (2005), 43-49.
6. W. M. Fakieh and F. A. Alhawiti, Ideal theory in commutative $\Gamma-$ semirings, *International Journal of Algebra*, **15**(2) (2021), 77-101.

7. D. R. La Torre, On h-ideals and k-ideals in hemirings, *Publ. Math. Debrecen*, **12**(1965), 219-226.
8. M. M. K. Rao, $\Gamma-$ semirings-I, *South East Asian Bull. of Math.,* **19** (1995), 49-54.
9. M. M. K. Rao and B. Venkateswarlu, On k-weakly primary ideals of $\Gamma-$ semirings, *Palestine Journal of Mathematics*, **6**(2) (2017), 279-288.
10. T. R. Sharma and S. Gupta, Some conditions on $\Gamma-$ semirings, *Journal of Combinatorics and System Sciences*, **41** (2016), 79-87.
11. T. R. Sharma and S. Gupta, Ideals in $\Gamma-$ semirings, *Journal of Combinatorics and System Sciences*, **42** (2017), 165-177.
12. T. R. Sharma and H. K. Ranote, On some properties of a $\Gamma-$ semiring, *JP Journal of Algebra, Number Theory and Applications*, **52**(2) (2021), 163-177.
13. T. R. Sharma and H. K. Ranote, More conditions on a $\Gamma-$ semiring and ideals of Izuka and Bourne factor $\Gamma-$ semiring, *South East Asian J. of Mathematics and Mathematical Sciences*, **18**(1) (2022), 71-84.
14. T. R. Sharma and H. K. Ranote, *Gel'fand* $\Gamma-$ *semirings*, *Mathematica Pannonica New Series*, **29**(3) (2023), 1-7.
15. H. S. Vandiver, Note on a simple type of algebra in which the cancellation law of addition does not hold *Bull. Amer. Math. Soc.*, **40**(12) (1934), 914-920.

Received by editors 2.6.2023; Revised version 22.2.2024; Available online 15.4.2024.

Hitesh Kumar Ranote, Department of Mathematics, School of Basic and Applied Sciences,, Maharaja Agarsen University, Baddi (solan), Himachal Pradesh, (India)-174103

Email address: `hk05905@gmail.com`