BULLETIN OF THE INTERNATIONAL MATHEMATICAL VIRTUAL INSTITUTE
ISSN (p) 2303-4874, ISSN (o) 2303-4955
www.imvibl.org /JOURNALS/BULLETIN
Bull. Int. Math. Virtual Inst., **14**(1)(2024), 23–30
DOI: 10.7251/BIMVI2401023N

Former
BULLETIN OF THE SOCIETY OF MATHEMATICIANS BANJA LUKA
ISSN 0354-5792 (o), ISSN 1986-521X (p)

QUASI $B-$ALGEBRAS

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Abstract. In this paper, we introduce the concept of a Quasi $B-$algebra, building upon the foundational work on Post algebras by G. Epstein and the investigation of semi $B-$algebras. We systematically explore the properties of Quasi $B-$algebras, establishing equivalent conditions for the transition from a semi $B-$algebra to a Quasi $B-$algebra.

1. Introduction

The remarkable work of Emil L. Post in 1921 paved the way for a comprehensive theory of propositions with the introduction of the Post lattice. This lattice effectively organizes closed classes of Boolean functions and has proven invaluable in practical applications of many-valued logical systems. Post's investigation confirmed the finite basis of each class within the Post lattice, making it a powerful tool for navigating the complexities of Boolean circuits and propositional formulas.

George Epstein further advanced the exploration of Post algebraic characteristics by introducing B-algebras, BL-algebras, and P-algebras. These algebraic systems, each with their unique properties, found applications in various fields, including biological systems and neural science. Specifically, B-algebras and BL-algebras play a critical role in computer science by providing abstractions for specific aspects of Post algebras.

In contrast, A Heyting algebra, named after Arend Heyting, is a bounded lattice equipped with an implication binary operation denoted as $x \to y$. While Heyting algebras position the implication operation $x \to y$ within the distributive lattice A, $B-$algebras place $x \Rightarrow y$ in the Boolean center B of the distributive lattice A.

2010 *Mathematics Subject Classification.* Primary 03G10; Secondary 06D99.
Key words and phrases. $B-$algebra, quasi $B-$algebra, semi $B-$algebra.
Communicated by Dusko Bogdanic.

This distinction highlights the specific location of the implication operation within the broader mathematical structure, emphasizing the nuanced behavior of logical operations in Heyting algebras and B-algebras within the context of distributive lattices. The implication operator $\Rightarrow$ plays a significant role in programming and logic, serving as a crucial tool for expressing conditional statements.

Building upon the research of G. Epstein on Post algebras [**4**]and our previous work on semi B-algebras [**8**], this paper introduces the concept of a Quasi B-algebra. We explore its properties and establish equivalent conditions for a semi B-algebra to become a Quasi B-algebra. Additionally, we offer additional characterizations of B-algebras, contributing to an enhanced understanding of these algebraic structures in various applications.

2. Preliminaries

In this section, we revisit fundamental notations and key foundational results, ensuring the self-contained nature of this paper.

DEFINITION 2.1. *[**1**] An algebra $(A, \vee, \wedge)$ of type $(2, 2)$ is called a lattice if it satisfies the following identities.*

(i) *Idempotency:* $x \wedge x = x$ *and* $x \vee x = x$.
(ii) *Commutativity:* $x \wedge y = y \wedge x$ *and* $x \vee y = y \vee x$.
(iii) *Associativity:* $(x \wedge y) \wedge z = x \wedge (y \wedge z)$ *and* $(x \vee y) \vee z = x \vee (y \vee z)$.
(iv) *Absorption laws:* $x \wedge (x \vee y) = x$ *and* $x \vee (x \wedge y) = x$.

DEFINITION 2.2. *[**1**] A uniquely bounded complemented distributive lattice $(A, \vee, \wedge, ', 0, 1)$ is called a Boolean algebra. In other words a Boolean algebra is a Boolean lattice in which $0, 1$ and $'$(complementation) are also considered to be fundamental operations.*

For a comprehensive exploration of lattice theory, readers are referred to the works of [**1**] and [**6**]. The subsequent definition is extracted from [**4**].

DEFINITION 2.3. *Let A be a distributive lattice with $0, 1$ and B as the set of all complemented elements of A. An algebra $(A, \vee, \wedge, \Rightarrow, 0, 1)$ is termed a $B-$algebra if, for all $x, y \in A$, there exists a greatest element $b \in B$ such that $x \wedge b \leqslant y$. Here, the complemented element b is denoted by $x \Rightarrow y$.*

For further properties of $B-$algebras, readers are directed to [**4**].

In our publication [**8**], we introduced the concept of a semi $B-$algebra and delineated its properties. The ensuing definition was extracted from [**8**].

DEFINITION 2.4. *Let A be a distributive lattice with $0, 1$ and B as the set of all complemented elements of A. An algebra $(A, \vee, \wedge, \Rightarrow, 0, 1)$ is deemed a semi $B-$algebra if it satisfies the following conditions for all $x, y \in A$ and $b \in B$*

$B_1 :\ (x \Rightarrow x) = 1$
$B_2 :\ (x \wedge (x \Rightarrow b)) = x \wedge b$
$B_3 :\ b \wedge (x \Rightarrow y) = [b \wedge ((b \wedge x) \Rightarrow (b \wedge y))]$

For further properties of semi $B-$algebras, we direct the reader to [**8**].

3. Quasi $B-$algebras

In this section, we present the concept of a Quasi $B-$algebra and explore its properties. Additionally, we establish equivalent conditions for a semi $B-$algebra to transform into a Quasi $B-$algebra. We commence with the following:

DEFINITION 3.1. *Let A be a distributive lattice with elements $0, 1,$ and B as the set comprising all complemented elements of A. An element $x \in A$ is termed $\wedge-$irreducible if, for $y, z \in A$ such that $x = y \wedge z$, it follows that $x = y$ or $x = z$.*

Now, we establish the following theorem:

THEOREM 3.1. *Let A be a semi $B-$algebra with $0, 1$ and B as the set comprising all complemented elements of A. If $x < b \leqslant y$ and x is $\wedge-$irreducible, then for all $x, y \in A$ and $b \in B$,*

(i) $b \wedge (x \Rightarrow b) = b \wedge (x \Rightarrow y)$
(ii) $(x \Rightarrow y) = x$ *if and only if* $(x \Rightarrow b) = x$
(iii) $(0 \Rightarrow 1) = x$ *if and only if* $(0 \Rightarrow b) = x$
(iv) $(b \Rightarrow x) = x$ *if* $x \in B$

PROOF. Suppose $x < b \leqslant y$ and x is $\wedge-$irreducible.
(i). $(b \wedge (x \Rightarrow b)) = \{b \wedge [(x \wedge b) \Rightarrow (b \wedge y)]\}$ $\quad (\because x < b \leqslant y)$
$= \{b \wedge b \wedge (x \Rightarrow y)\}$ (as per B_3 in Definition 2.2)
$= \{b \wedge (x \Rightarrow y)\}$
(ii). Suppose $(x \Rightarrow y) = x$. Then $b \wedge (x \Rightarrow y) = b \wedge x$
implies $b \wedge (x \Rightarrow b) = x$ $\quad (\because x < b$ and by (i))
implies $x = b$ or $x = (x \Rightarrow b)$ ($\because x$ is $\wedge-$irreducible)
implies $x = (x \Rightarrow b)$
Similarly, we establish the converse.
(iii) Suppose $(0 \Rightarrow 1) = x$.
Then $b \wedge (0 \Rightarrow 1) = b \wedge x$ implies $\{b \wedge [(b \wedge 0) \Rightarrow (b \wedge 1)]\} = x$
implies $(b \wedge (0 \Rightarrow b)) = x$
implies $x = b$ or $x = (0 \Rightarrow b)$ ($\because x$ is $\wedge-$irreducible)
implies $x = (0 \Rightarrow b)$
Similarly, we establish the converse.
(iv). Suppose $(b \Rightarrow x) = x$. Now $x = (b \wedge x)$
$x = (b \wedge (b \Rightarrow x))$
implies $x = b$ or $x = (b \Rightarrow x)$($\because x$ is $\wedge-$irreducible)
implies $x = (b \Rightarrow x)$ □

THEOREM 3.2. *Let A be a semi $B-$algebra with $0, 1$ and B, the set of all complemented elements of A. Then, for $b \in B$, $(0 \Rightarrow 1) = 1$ if and only if $b \wedge (0 \Rightarrow b) = b$*

PROOF. Suppose $(0 \Rightarrow 1) = 1$. Then $b \wedge (0 \Rightarrow 1) = b \wedge 1$
implies $b \wedge [(b \wedge 0) \Rightarrow (b \wedge 1)] = b$
implies $b \wedge (0 \Rightarrow b) = b$.
We can readily confirm the converse. □

DEFINITION 3.2. *Let A be a semi $B-$algebra with $0, 1$ and B, the set of all complemented elements of A. Then A is said to be a Quasi $B-$algebra if $((x \Rightarrow b) \wedge b) = b$ for $x \in A, b \in B$.*

EXAMPLE 3.1. *Let $A = \{0, x, 1\}$ be three element chain and $0, 1 \in B$. Define a binary operation $\Rightarrow$ on A as follows*

$\Rightarrow$	0	x	1
0	1	x	1
x	0	1	1
1	0	x	1

A satisfies all the conditions of both a semi $B-$algebra and a Quasi $B-$algebra.

EXAMPLE 3.2. *Let $A = \{0, x, 1\}$ be three element chain and $0, 1 \in B$. Define a binary operation $\Rightarrow$ on A as follows*

$\Rightarrow$	0	x	1
0	1	1	1
x	0	1	x
1	0	x	1

Clearly, A is a semi $B-$algebra but not a Quasi $B-$algebra ($\because (x \Rightarrow 1) \wedge 1 \neq 1$).

THEOREM 3.3. *Let A be a Quasi $B-$algebra with $0, 1$ and B, representing the set of all complemented elements of A. Then, for $x, y \in A$, $b \in B$, the following conditions hold:*

(1) $(x \Rightarrow 1) = 1$
(2) $b \leqslant (b \Rightarrow b)$
(3) $b \leqslant (x \Rightarrow b)$ *and* $((x \Rightarrow b) \wedge b) = b$
(4) $[b \wedge ((x \Rightarrow b) \Rightarrow b)] = b$
(5) $[b \wedge (((x \Rightarrow b) \Rightarrow b) \Rightarrow b)] = b$
(6) $(x \Rightarrow y) \wedge [x \Rightarrow (x \Rightarrow y)] = (x \Rightarrow y)$
(7) $[b \wedge ((x \wedge y) \Rightarrow b)] = [b \wedge ((y \wedge x) \Rightarrow b)]$

PROOF. The proof is straightforward and is therefore omitted. □

THEOREM 3.4. *Let A be a semi $B-$algebra with $0, 1$ and B, the set of all complemented elements of A. Then A is a Quasi $B-$algebra if and only if $(x \Rightarrow 1) = 1$*

PROOF. Suppose $(x \Rightarrow 1) = 1$. Now

$$\begin{aligned} b &= b \wedge 1 \\ &= b \wedge (x \Rightarrow 1) \\ &= b \wedge [(b \wedge x) \Rightarrow (b \wedge 1)] \\ &= b \wedge [(b \wedge x) \Rightarrow (b \wedge b)] \\ &= b \wedge (x \Rightarrow b). \end{aligned}$$

Hence, A qualifies as a Quasi $B-$algebra. The converse can be readily confirmed. □

4. $B-$algebras

As previously mentioned, during the examination of Post algebra properties [**4**], G. Epstein introduced the concept of $B-$algebra, which holds significance in both logic and computer science. In our work [**8**], we presented various characterizations of a $B-$algebra. This section delves deeper into additional significant characterizations for a $B-$algebra. The ensuing theorems are extracted from [**8**].

Theorem 4.1. *[8] Let A be a distributive lattice with $0, 1$ and B, the set of all complemented elements of A. The algebra A is classified as a $B-$algebra if and only if it adheres to the following conditions: For all $x, y, z \in A$ and $b \in B$,"*

(i) $(x \Rightarrow x) = 1$
(ii) $((x \Rightarrow b) \wedge b) = b$
(iii) $(x \wedge (x \Rightarrow b)) = x \wedge b$
(iv) $(z \Rightarrow (x \wedge y)) = (z \Rightarrow x) \wedge (z \Rightarrow y)$
(v) $((x \vee y) \Rightarrow z)) = (x \Rightarrow z) \wedge (y \Rightarrow z)$

Throughout this section, we use the notation A to signify a distributive lattice with $0, 1$ and B, representing the set of all complemented elements of A, unless explicitly stated otherwise

Theorem 4.2. *[8] Let $x, y \in A$ and $b \in B$. Then A is a $B-$algebra if and only if*

(i) $(x \wedge (x \Rightarrow b)) = x \wedge b$
(ii) $((x \wedge y) \Rightarrow x) = 1$
(iii) $(b \wedge (x \Rightarrow y)) = (b \wedge [(b \wedge x) \Rightarrow (b \wedge y)])$

In the subsequent three theorems, we establish diverse axiomatizations of $B-$algebras.

Theorem 4.3. *Let $x, y \in A$ and $b \in B$. Then A is a $B-$algebra if and only if*

i $(x \wedge (x \Rightarrow b)) \leqslant b$
ii $b \wedge (x \Rightarrow y) = (b \wedge [(b \wedge x) \Rightarrow (b \wedge y)])$
iii $((x \wedge y) \Rightarrow y) = 1$

Proof. Assuming that A satisfies the provided conditions. Now
$b \wedge (x \Rightarrow b) = b \wedge [(b \wedge x) \Rightarrow (b \wedge b)]$ ($\because$ by condition (ii))
$= b \wedge [(b \wedge x) \Rightarrow b]$
$= b.$ ($\because$ by condition (iii))

Thus $b \leqslant (x \Rightarrow b)$ and hence $x \wedge b \leqslant (x \wedge (x \Rightarrow b))$ ...(1)
From (i), we have $x \wedge (x \Rightarrow b) \leqslant b$ and hence $(x \wedge (x \Rightarrow b)) \leqslant x \wedge b$ ....(2)
Combining equations (1) and (2), we obtain $(x \wedge (x \Rightarrow b)) = x \wedge b$. According to Theorem 4.2, this implies that A is a $B-$algebra. The converse follows straightforwardly and is therefore omitted. □

THEOREM 4.4. *Let $x, y, z \in A$ and $b \in B$. Then A is a $B-$algebra if and only if*

(i) $(x \Rightarrow x) = 1$
(ii) $(x \wedge (x \Rightarrow b)) \leqslant b$
(iii) $b \wedge ((b \wedge x) \Rightarrow y) \leqslant (b \wedge (x \Rightarrow y))$
(iv) $x \Rightarrow (y \wedge z) = (x \Rightarrow y) \wedge (x \Rightarrow z)$
(v) $((x \vee y) \Rightarrow z)) = (x \Rightarrow z) \wedge (y \Rightarrow z)$

PROOF. Assuming that A satisfies the provided conditions.

Now $1 = (z \Rightarrow z) = (((x \wedge z) \vee z) \Rightarrow z)$
$= ((x \wedge z) \Rightarrow z) \wedge (z \Rightarrow z)$ ($\because$ conditon (v))
$= ((x \wedge z) \Rightarrow z) \wedge 1$ ($\because$ conditon (i))

Therefore $1 = ((x \wedge z) \Rightarrow z)$ ...(1)

Now $(b \wedge (x \Rightarrow y)) = b \wedge [((b \wedge x) \vee x) \Rightarrow y]$
$= b \wedge [((b \wedge x) \Rightarrow y) \wedge (x \Rightarrow y)]$ ($\because$ conditon (v))
$= (b \wedge ((b \wedge x) \Rightarrow y)) \wedge (b \wedge (x \Rightarrow y))$

hence $(b \wedge (x \Rightarrow y)) \leqslant [b \wedge ((b \wedge x) \Rightarrow y)]$ ... (2)

Combining equations (iii) and (1), we obtain $b \wedge ((b \wedge x) \Rightarrow y) = (b \wedge (x \Rightarrow y))$ ...(3)

Now $b \wedge [(b \wedge x) \Rightarrow (b \wedge y)] = b \wedge [((b \wedge x) \Rightarrow b) \wedge ((b \wedge x) \Rightarrow y)]$ ($\because$ by (iv))

$= \{b \wedge [1 \wedge ((b \wedge x) \Rightarrow y)]\}$ ($\because ((x \wedge y) \Rightarrow y) = 1$)
$= \{b \wedge ((b \wedge x) \Rightarrow y)\}$
$= \{b \wedge (x \Rightarrow y)\}$ ($\because$ by eq (2))

According to Theorem 4.3, A qualifies as a $B-$algebra. The converse is straightforward and is therefore omitted. □

THEOREM 4.5. *Let $x, y, z \in A$ and $b \in B$. Then A is a $B-$algebra if and only if*

(i) $b \wedge (x \Rightarrow b) = b$
(ii) $(x \wedge (x \Rightarrow b)) = x \wedge b$
(iii) $(x \Rightarrow (y \wedge z)) = \{(x \Rightarrow y) \wedge (x \Rightarrow z)\}$
(iv) $(x \Rightarrow x) = 1$

PROOF. The proof is direct and is thus omitted. □

In conclusion, we finalize this paper with the following theorem

THEOREM 4.6. *If A is a semi B-algebra, the following conditions are equivalent for all $x, y, z \in A$:*

i. *A is a $B-$algebra*
ii. *$x \leqslant y$ implies $(x \Rightarrow y) = 1$*
iii. *$x \leqslant y$ implies $(x \Rightarrow z) \geqslant (y \Rightarrow z)$*
iv. *$x \leqslant y$ implies $z \Rightarrow x \leqslant z \Rightarrow y$*
v. $(x \vee y) \Rightarrow z \leqslant (x \Rightarrow z) \wedge (y \Rightarrow z)$
vi. $x \Rightarrow (y \wedge z) \leqslant (x \Rightarrow y) \wedge (x \Rightarrow z)$
vii. $(x \Rightarrow (y \Rightarrow z)) = ((x \wedge y) \Rightarrow z)$
viii. $((x \vee y) \Rightarrow y) \leqslant (x \Rightarrow y)$

ix. $(x \Rightarrow y) = x \Rightarrow (x \wedge y)$.

PROOF. According to Theorem 3.17 from [**8**], A is a B-algebra if and only if it fulfills all the conditions, namely conditions (ii) to (vi).

(vii)$\Longrightarrow$(i):

Suppose A satisfies the condition $(x \Rightarrow (y \Rightarrow z)) = ((x \wedge y) \Rightarrow z)$. Now

$(x \Rightarrow 1) = (x \Rightarrow (0 \Rightarrow 0)) = ((x \wedge 0) \Rightarrow 1) = (0 \Rightarrow 0) = 1$

Now $((x \wedge y) \Rightarrow y) = (x \Rightarrow (y \Rightarrow y))$

$= (x \Rightarrow 1)$

$= 1$

Hence A is a $B-$algebra.

(viii)$\Longrightarrow$(i):

Suppose A hold the condition $((x \vee y) \Rightarrow y) \leqslant (x \Rightarrow y)$.

Now $1 = (y \Rightarrow y)$

$= (((x \wedge y) \vee y) \Rightarrow y)$

$\leqslant ((x \wedge y) \Rightarrow y)$

Therefore $((x \wedge y) \Rightarrow y) = 1$.

Hence A is a $B-$algebra.

(ix)$\Longrightarrow$(i):

Suppose $(x \Rightarrow y) = (x \Rightarrow (x \wedge y))$.

Replace x by $x \wedge y$ in $(x \Rightarrow y) = x \Rightarrow (x \wedge y)$,

implies $((x \wedge y) \Rightarrow y) = ((x \wedge y) \Rightarrow (x \wedge y \wedge y))$

implies $((x \wedge y) \Rightarrow y) = 1$

Hence A is a $B-$algebra.

□

Acknowledgments

The first author expresses gratitude to Government College Autonomous, Rajamahendravaram, for their financial support.

References

1. G. Birkhoff, Lattice Theory. *Amer. Math. Soc. Colloq. Publ. XXV, Providence*(1967), U.S.A.
2. G. Epstein, *The lattice theory of Post algebras,* Trans. Amer. Math. Soc, **95** (1960), 300-317
3. G. Epstein, *An equational axiomatization for the disjoint system of Post algebras,* IEEE Transactions on Computers, C-**22** (1973), 422-423
4. G. Epstein and A. Horn, *P-algebras, an abstraction from Post algebras*, Vol 4, Number 1,195-206,1974, Algebra Universalis.
5. G. Epstein and A. Horn, *Chain based Lattices*, Pacific Journal Of Mathematics, Vol. **55**, No. 1, 1974, 65-84.
6. G. Gratzer, *General Lattice Theory,* Academic Press, New York, Sanfransisco (1978).
7. A. Heyting, *Die formalen Regeln der intuitionistischen Logik,* Sitzungsberichte der Preussischen Akad. der Wessenschaften Physikalisch-mathematische Klasse, 1930, pp. 42-56.
8. N. K. Kakumanu, B. Tharuni, Daniel A. Romano, and G.C. Rao, *Semi $B-$algebra,* Bull. of Int. Math. Virt. Inst., 14 (1) 2024, 13-20.
9. T. Katrinak and A. Mitscke, *Stonesche verbdnde der ordnung n und Postalgebren*, Mathematische Annalen, **199** (1972), 13-30.

10. E.L. Post, *The Two-Valued Iterative Systems of Mathematical Logic,* Annals of Mathematics Studies, no. 5. Princeton University Press, Princeton, N. J., 1941. viii+122 pp.
11. T. Traczyk, *Axioms and some properties of Post algebras,* Colloquium Mathematicum, **10** (1963), 193-209.

Received by editors 27.9.2023; Revised version 5.1.2023; Available online 10.4.2024.

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