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A STUDY ON A HARMONIC UNIVALENT FUNCTIONS IN TERMS OF Q DIFFERENCE OPERATOR

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Abstract. We introduce and investigate q-analogue of a new subclass of harmonic univalent functions defined by subordination. We first obtained a coefficient characterization of these functions. We give necessary and sufficient convolution conditions, distortion bounds, compactness and extreme points for this subclass of harmonic univalent functions with negative coefficients.

1. Introduction

Let H denote the class of continuous complex-valued harmonic functions which are harmonic in the open unit disk $U = \{z : z \in \mathbb{C} \quad and \quad |z| < 1\}$ and let $\mathcal{A}$ be the subclass of H consisting of functions which are analytic in U. A function harmonic in U may be written as $f = h + \overline{g}$, where h and g are analytic in U. We call the h analytic part and g co-analytic part of f. A necessary and sufficient condition for f to be locally univalent and sense-preserving in U is that $|g'(z)| < |h'(z)|$ (see [**5**]). To this end, without loss of generality, we may write

$$h(z) = z + \sum_{n=2}^{\infty} a_n z^n \qquad and \qquad g(z) = \sum_{n=1}^{\infty} b_n z^n. \tag{1.1}$$

Let HS denote the class of functions $f = h + \overline{g}$ which are harmonic, univalent and sense-preserving in U for which $h(0) = h^{'}(0) - 1 = 0 = g(0)$. One shows easily that the sense-preserving property implies that $|b_1| < 1$. The subclass of consist

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of all functions in which have the additional property $b_1 = 0$. Clunie and Sheil-Small [**5**] investigated the class HS as well as its geometric subclasses and obtained some coefficient bounds. Since then, there have been several related papers on HS and its subclasses [**1**], [**8**], [**13**], [**14**], [**16**], [**19**].

We recollect here the q-difference operator that was used in geometric function theory and in several areas of science. We give basic definitions and properties about the q-difference operator that are used in this study (for details see [**4**] and [**10**]). For $0 < q < 1$, we defined the q-integer $[n]_q$ by

$$[n]_q = \frac{1-q^n}{1-q} \qquad (n = 1, 2, 3, ...).$$

Notice that if $q \to 1^-$, than $[n]_q \to n$.

In 1990, Ismail et. al. [**9**] used q-calculus, in the theory of analytic univalent functions by defining a class of complex valued functions that are analytic on the open unit disk U with the normalizations $f(0) = 0, f^{'}(0) = 1$ and $|f(qz)| \leqslant |f(z)|$ for every $q \in (0, 1)$. Motivated by these authors, several researches used the theory of analytic and harmonic univalent functions and q-calculus; for example see [**1**] and [**2**]. The q-difference operator of analytic functions h and g given by (1.1) are by definiton, given as follows [**10**]

$$\partial_q h(z) = \begin{cases} \frac{h(z)-h(qz)}{(1-q)z} & ; z \neq 0 \\ h^{'}(0) & ; z = 0 \end{cases}$$

and

$$\partial_q g(z) = \begin{cases} \frac{g(z)-g(qz)}{(1-q)z} & ; z \neq 0 \\ g^{'}(0) & ; z = 0. \end{cases}$$

Thus, for the function h and g of the form (1.1), we have

$$\partial_q h(z) = 1 + \sum_{n=2}^{\infty} [n]_q a_n z^{n-1}$$

and

$$\partial_q g(z) = \sum_{n=1}^{\infty} [n]_q b_n z^{n-1}. \tag{1.2}$$

Let $f \in HS$, for $v \in N_0 = N \cup 0$, $N = 1, 2, 3, ...$, $n \geqslant 1$ and $0 < q < 1$,

$$D_q^0 f(z) = f(z) = h(z) + \overline{g(z)}$$

$$D_q^1 f(z) = D_q^1 h(z) - \overline{D_q^1 g(z)} = z\partial_q h(z) - \overline{z\partial_q g(z)}, \tag{1.3}$$

$$\vdots$$

$$D_q^\nu f(z) = D_q^\nu h(z) + (-1)^\nu \overline{D_q^\nu g(z)}$$
$$= z\partial_q(D_q^{\nu-1}h(z)) + (-1)^\nu \overline{z\partial_q(D_q^{\nu-1}g(z))},$$

where

$$D_q^\nu h(z) = z + \sum_{n=2}^{\infty} [n]_q^\nu a_n z^n, \qquad D_q^\nu g(z) = \sum_{n=1}^{\infty} [n]_q^\nu b_n z^n.$$

We note that

(i) $\lim_{q\to 1^-} D_q^\nu f(z) = D^\nu f(z) = z + \sum_{n=2}^{\infty} n^\nu a_n z^n$ for $f \in S$ (see [**15**]).

(ii) $\lim_{q\to 1^-} D_q^\nu f(z) = D^\nu h(z) + (-1)^\nu \overline{D^\nu g(z)}$ for $f \in HS$,
where

$D^\nu h(z) = z + \sum_{n=2}^{\infty} n^\nu a_n z^n$ and $D^\nu g(z) = \sum_{n=1}^{\infty} n^\nu b_n z^n$ (see [**13**]).

A harmonic function $f = h + \overline{g}$ defined by (1.1) is said to be q-harmonic, locally univalent and sense-preserving in U denoted by HS_q, if and only if the second dilatation w_q hards requirement

$$|w_q(z)| = \left|\frac{\partial_q g(z)}{\partial_q h(z)}\right| < 1$$

where $0 < q < 1$ and $z \in U$. Note that as $q \to 1^-$, HS_q reduces to family HS (see [**1**] and [**12**]).

We also let the subclass THS_q consist of harmonic functions $f = h + \overline{g}$ in HS_q so that h and g are the form

$$h(z) = z - \sum_{n=2}^{\infty} |a_n| z^n \quad \textit{and} \quad g(z) = (-1)^\nu \sum_{n=1}^{\infty} |b_n| z^n. \tag{1.4}$$

We say that an analytic function f is subordinate to an analytic function F and write $f \prec F$ if there are complex valued function w which maps U into oneself with $w(0) = 0$ such that $f(z) = F(w(z))$.

Furthermore, if the function F is univalent in U, then we have the following equivalence:

$$f(z) \prec F(z) \Leftrightarrow f(0) = F(0) \quad \textit{and} \quad f(U) \subset F(U).$$

Denote by $HS_q^\nu(\lambda, \delta, A, B)$ the subclass of HS_q consisting of functions f of the form (1) that satisfy the condition

$$\frac{D_q^{\nu+1} f(z)}{\lambda D_q^{\nu+1} f(z) + (1-\lambda) D_q^\nu f(z)} \prec (1-\delta)\frac{1+Az}{1+Bz} + \delta = \frac{1 + [A + (B-A)\delta]z}{1+Bz}, \tag{1.5}$$

where $0 \leqslant \delta < 1$, $0 \leqslant \lambda < 1$, $0 < q < 1$ and $-B \leqslant A < B \leqslant 1$.

Finally, we let $THS_q^{\nu}(\lambda,\delta,A,B) \equiv HS_q^{\nu}(\lambda,\delta,A,B) \bigcap THS_q$. By suitably specializing the parameters, the classes $HS_q^{\nu}(\lambda,\delta,A,B)$ reduces to the various subclasses of harmonic univalent functions given below:

- ⋆ $HS_q^{\nu}(0,\delta,A,B) = HS_q^{\nu}(\delta,A,B)$ (see [**19**]),
- ⋆ $HS_q^{\nu}(0,0,A,B) = S_H(\nu,q,A,B)$ (see [**16**]),
- ⋆ $HS_q^{0}(0,0,A,B) = S_H^{*}(q,A,B)$ (see [**16**] and [**18**]),
- ⋆ $HS_q^{1}(0,0,A,B) = CH_q(A,B)$ (see [**16**]),
- ⋆ $HS_q^{\nu}(0,0,(1+q)\alpha-1,q) = H_q^{\nu}(\alpha)$ for $0 \leqslant \alpha < 1$ (see [**12**]),
- ⋆ $HS_q^{0}(0,0,(1+q)\alpha-1,q) = S_{H_q}^{*}(\alpha)$ for $0 \leqslant \alpha < 1$ (see [**2**]),
- ⋆ $HS_q^{1}(0,0,(1+q)\alpha-1,q) = S_{H_q}^{C}(\alpha)$ for $0 \leqslant \alpha < 1$ (see [**1**]),
- ⋆ $HS_q^{\nu}(0,\delta,A,B) = SH(\nu,\delta,A,B)$ for $q \to 1^-$ (see [**3**]),
- ⋆ $HS_q^{\nu}(0,0,A,B) = SH(\nu,A,B)$ for $q \to 1^-$ (see [**8**]),
- ⋆ $HS_q^{0}(0,0,A,B) = S_H^{*}(A,B)$ for $q \to 1^-$ (see [**11**] and [**12**]),
- ⋆ $HS_q^{1}(0,0,A,B) = CH(A,B)$ for $q \to 1^-$ (see [**8**]),
- ⋆ $HS_q^{\nu}(0,0,(1+q)\alpha-1,q) = H^{\nu}(\alpha)$ for $0 \leqslant \alpha < 1$ and $q \to 1^-$ (see [**13**]),
- ⋆ $HS_q^{0}(0,0,(1+q)\alpha-1,q) = S_H^{*}(\alpha)$ for $0 \leqslant \alpha < 1$ and $q \to 1^-$ (see [**11**] and [**14**]),
- ⋆ $HS_q^{1}(0,0,(1+q)\alpha-1,q) = S_H^{C}(\alpha)$ for $0 \leqslant \alpha < 1$ and $q \to 1^-$ (see [**11**] and [**14**]),
- ⋆ $HS_q^{\nu}(0,0,-1,q) = H^{\nu}(0)$ for $q \to 1^-$ (see [**13**]),
- ⋆ $HS_q^{0}(0,0,-1,q) = S_H^{*}$ for $q \to 1^-$ (see [**17**]),
- ⋆ $HS_q^{1}(0,0,-1,q) = C_H$ for $q \to 1^-$ (see [**17**]).

By techniques and methodology used by Dziok (see [**6**] and [**7**]), Dziok et al. (see [**8**]), in this paper we find necessary and sufficient conditions, distortion bounds, compactness and extreme points for the above defined class $THS_q^{\nu}(\lambda,\delta,A,B)$.

2. Main results

For functions f_1 and $f_2 \in HS_q$ of the form

$$f_j(z) = z + \sum_{n=2}^{\infty} a_{n,j} z^n + \sum_{n=1}^{\infty} \overline{b_{n,j} z^n}, \qquad (z \in U, j = 1,2), \tag{2.1}$$

we define the Hadamard product of f_1 and f_2 by

$$(f_1 * f_2)(z) = z + \sum_{n=2}^{\infty} a_{n,1} a_{n,2} z^n + \sum_{n=1}^{\infty} \overline{b_{n,1} b_{n,2} z^n}, \qquad (z \in U).$$

Firstly we will give necessary and sufficient conditions for harmonic function in $HS_q^{\nu}(\lambda,\delta,A,B)$.

THEOREM 2.1. *Let $f \in HS_q$. Then $f \in HS_q^{\nu}(\lambda,\delta,A,B)$ if and only if*

$$D_q^{\nu} f(z) * \Theta(z;\zeta) \neq 0, \quad (\zeta \in C, |\zeta| = 1, z \in U \backslash \{0\}),$$

where

$$\Theta(z;\zeta)=\frac{(B-A)(1-\delta)\zeta z+(1-\lambda)\{1+(M-1)\zeta\}qz^2}{(1-z)(1-qz)}$$

$$-\frac{\left\{2+[A+B+(B-A)\delta]\zeta-2\lambda\{1+(M-1)\zeta\}\overline{z}-(1-\lambda)\left\{1+(M-1)\zeta\right\}q\overline{z}^2\right\}}{(1-\overline{z})(1-q\overline{z})},$$

where $M=1+A+(B-A)\delta$.

Proof. Let $f\in HS_q$. Then $f\in HS_q^{\nu}(\lambda,\delta,A,B)$ if and only if (1.5) holds or equivalently

(2.2)
$$\frac{D_q^{\nu+1}f(z)}{\lambda D_q^{\nu+1}f(z)+(1-\lambda)D_q^{\nu}f(z)}\neq\frac{1+(M-1)\zeta}{1+B\zeta}\qquad(\zeta\in C,|\zeta|=1,z\in U\backslash\{0\}).$$

Now for

$$D_q^{\nu}f(z)=D_q^{\nu}f(z)*(\frac{z}{1-z}+\frac{\overline{z}}{1-\overline{z}})$$

and

$$D_q^{\nu+1}f(z)=D_q^{\nu}f(z)*\left(\frac{z}{(1-z)(1-qz)}-\frac{\overline{z}}{(1-\overline{z})(1-q\overline{z})}\right)$$

the inequality (2.2) yields

$$(1+B\zeta)D_q^{\nu+1}f(z)-\{1+(M-1)\zeta\}\{\lambda D_q^{\nu+1}f(z)+(1-\lambda)D_q^{\nu}f(z)\}$$
$$=D_q^{v}h(z)*\left\{(1+B\zeta)-\lambda\{1+(M-1)\zeta\}\tfrac{z}{(1-z)(1-qz)}-(1-\lambda)\{1+(M-1)\zeta\}\tfrac{z}{1-z}\right\}$$
$$-(-1)^{v}\overline{D_q^{v}g(z)}*\left\{(1+B\zeta)-\lambda\{1+(M-1)\zeta\}\tfrac{\overline{z}}{(1-\overline{z})(1-q\overline{z})}+(1-\lambda)\{1+(M-1)\zeta\}\tfrac{\overline{z}}{1-\overline{z}}\right\}$$

$$=D_q^{v}f(z)*\Theta(z;\zeta)\neq 0,$$

where $M=1+A+(B-A)\delta$. □

Now we derive a sufficient coefficient bound for the class $HS_q^{\nu}(\lambda,\delta,A,B)$.

Theorem 2.2. *Let f be of the form (1.1). If $-B\leqslant A<B\leqslant 1$, $0\leqslant\delta<1$ and* $0\leqslant\lambda\leqslant 1$

$$\sum_{n=1}^{\infty}(\Phi_n|a_n|+\Psi_n|b_n|)\leqslant 2(B-A)(1-\delta),\tag{2.3}$$

where

$$\Phi_n=[n]_q^{\nu}\left\{[n]_q(1+B-\lambda M)-(1-\lambda)M\right\}\tag{2.4}$$

and

$$\Psi_n = [n]_q^\nu \{[n]_q(1+B-\lambda M)+(1-\lambda)M\}, \tag{2.5}$$

where $M = 1+A+(B-A)\delta$, then f is harmonic, sense-preserving, locally univalent in U and $f \in HS_q^\nu(\lambda,\delta,A,B)$.

PROOF. Since

$$\begin{aligned}
|D_q h(z)| &\geqslant 1-\sum_{n=2}^{\infty}[n]_q|a_n||z|^{n-1} \\
&> 1-\sum_{n=2}^{\infty}\frac{[n]_q^\nu\{[n]_q(1+B-\lambda M)-(1-\lambda)M\}}{(B-A)(1-\delta)}|a_n| \\
&\geqslant \sum_{n=1}^{\infty}\frac{[n]_q^\nu\{[n]_q(1+B-\lambda M)+(1-\lambda)M\}}{(B-A)(1-\delta)}|b_n| \\
&> \sum_{n=1}^{\infty}\frac{[n]_q^\nu\{[n]_q(1+B-\lambda M)+(1-\lambda)M\}}{(B-A)(1-\delta)}|b_n||z|^{n-1} \\
&\geqslant \sum_{n=2}^{\infty}[n]_q|b_n||z|^{n-1} \geqslant |D_q g(z)|,
\end{aligned}$$

it follows that $f \in HS_q$. On the other hand, $f \in HS_q^\nu(\lambda,\delta,A,B)$ if and only if there exists a complex valued function w; $w(0)=0$, $|w(z)|<1 \quad z\in U$ so that

$$\frac{D_q^{\nu+1}f(z)}{\lambda D_q^{\nu+1}f(z)+(1-\lambda)D_q^\nu f(z)} = \frac{Mw(z)}{1+Bw(z)}$$

or equivalently

$$\left|\frac{(1-\lambda)(D_q^{\nu+1}f(z)-D_q^\nu f(z))}{\{B-\lambda M\}D_q^{\nu+1}f(z)-(1-\lambda)MD_q^\nu f(z)}\right| < 1. \tag{2.6}$$

Substituting for $D_q h(z)$ and $D_q g(z)$ in (2.6), we obtain

$$\begin{aligned}
&\left|(1-\lambda)(D_q^{\nu+1}f(z)-D_q^\nu f(z))\right| - \left|\{B-\lambda M\}D_q^{\nu+1}f(z)-(1-\lambda)MD_q^\nu f(z)\right| \\
&= \left|(1-\lambda)\textstyle\sum_{n=2}^{\infty}[n]_q^\nu([n]_q-1)a_n z^n-(1-\lambda)(-1)^\nu\sum_{n=1}^{\infty}[n]_q^\nu([n]_q+1)\overline{b_n z^n}\right| \\
&\qquad - \left|(B-A)(1-\delta)z+\textstyle\sum_{n=2}^{\infty}[n]_q^\nu\{[n]_qB-\lambda M-(1-\lambda)M\}a_n z^n\right. \\
&\qquad \left. -(-1)^\nu\textstyle\sum_{n=1}^{\infty}[n]_q^\nu\{[n]_qB-\lambda M+(1-\lambda)M\}\overline{b_n z^n}\right|
\end{aligned}$$

$$\begin{aligned}
&\leqslant \sum_{n=2}^{\infty}[n]_q^\nu(1-\lambda)([n]_q-1)|a_n||z|^n+\sum_{n=1}^{\infty}[n]_q^\nu(1-\lambda)([n]_q+1)|b_n||z|^n \\
&\qquad -(B-A)(1-\delta)+\textstyle\sum_{n=2}^{\infty}[n]_q^\nu\{(B-\lambda M)[n]_q-(B-\lambda M)\}|a_n||z|^n
\end{aligned}$$

$$+\sum_{n=1}^{\infty}[n]_q^{\nu}\left\{(B-\lambda M)[n]_q+(B-\lambda M)\right\}|b_n||z|^n$$

$$\leqslant |z|\left\{\sum_{n=2}^{\infty}\Phi_n|a_n||z|^{n-1}+\sum_{n=1}^{\infty}\Psi_n|b_n||z|^{n-1}-(B-A)(1-\delta)\right\}<0,$$

where $M=1+A+(B-A)\delta$.

The harmonic function

$$f(z)=z+\sum_{n=2}^{\infty}\frac{(B-A)(1-\delta)x_n}{\Phi_n}z^n+\sum_{n=1}^{\infty}\frac{(B-A)(1-\delta)y_n}{\Psi_n}\overline{z^n}, \tag{2.7}$$

where

$$\sum_{n=1}^{\infty}|x_n|+\sum_{n=1}^{\infty}|y_n|=1$$

shows that the coefficient bound given by (2.3) is sharp. The functions of the form (2.7) are in $HS_q^{\nu}(\lambda,\delta,A,B)$ because

$$\sum_{n=1}^{\infty}\left(\frac{\Phi_n}{2(B-A)(1-\delta)}|a_n|+\frac{\Psi_n}{2(B-A)(1-\delta)}|b_n|\right)=\sum_{n=1}^{\infty}(|x_n|+|y_n|)=1.$$

by (2.3). □

Next we show that the bound (2.3) is also necessary for $THS_q^{\nu}(\lambda,\delta,A,B)$.

THEOREM 2.3. *Let $f=h+\overline{g}$ with h and g of the form (1.4). Then $f\in THS_q^{\nu}(\lambda,\delta,A,B)$ if and only if the condition (2.3) holds.*

PROOF. From Theorem 2.2, we only need to show that $f\notin THS_q^{\nu}(\lambda,\delta,A,B)$ if condition (2.3)does not hold. We note that a necessary and sufficient condition for $f=h+\overline{g}$ given by (1.4) to be in $THS_q^{\nu}(\lambda,\delta,A,B)$ is that the coefficient condition (2.3) to be fulfilled. Equivalently, we must have

$$\left|\frac{(1-\lambda)(D_q^{\nu+1}f(z)-D_q^{\nu}f(z))}{B-\lambda MD_q^{\nu+1}f(z)-(1-\lambda)MD_q^{\nu}f(z)}\right|<1,$$

where $M=1+A+(B-A)\delta$. For $z=r<1$, we get

(2.8)
$$\frac{(1-\lambda)\left\{\sum_{n=2}^{\infty}[n]_q^{\nu}([n]_q-1)|a_n|r^{n-1}+\sum_{n=1}^{\infty}[n]_q^{\nu}([n]_q+1)|b_n|r^{n-1}\right\}}{(B-A)(1-\delta)+\sum_{n=2}^{\infty}[n]_q^{\nu}\left\{(B-\lambda M)[n]_q-(1-\lambda)M\right\}|a_n|r^{n-1}-\sum_{n=1}^{\infty}[n]_q^{\nu}\left\{(B-\lambda M)[n]_q+(1-\lambda)M\right\}|b_n|r^{n-1}}<1.$$

If condition (2.3) does not hold, then condition (2.8) does not hold for r sufficiently close to 1. Thus there exists $z_0=r_0$ in $(0,1)$ for which the quotient (2.8) is greater than 1. This contradicts the required condition for $f \in THS_q^{\nu}(\lambda,\delta,A,B)$ and so the proof is complete.

□

THEOREM 2.4. *Let $f \in THS_q^{\nu}(\lambda,\delta,A,B)$. Then for $|z| = r < 1$ we have*

$$f(z) \leqslant (1+|b_1|)r + \frac{(B-A)(1-\delta) - [2+A+B+(B-A)\delta]|b_1|}{[2]_q^{\nu}\{\{1+B-\lambda[1+A+(B-A)]\}q + (B-A)(1-\delta)\}} r^2$$

and

$$f(z) \geqslant (1-|b_1|)r - \frac{(B-A)(1-\delta) - [2+A+B+(B-A)\delta]|b_1|}{[2]_q^{\nu}\{\{1+B-\lambda[1+A+(B-A)]\}q + (B-A)(1-\delta)\}} r^2.$$

PROOF. We only proof the right hand inequality. The proof for the left hand inequality is similiar and will be omitted. Let $f \in THS_q^{\nu}(\lambda,\delta,A,B)$. Taking absolute value of f we get

$$|f(z)| \leqslant (1+|b_1|)r + \sum_{n=2}^{\infty}(|a_n|+|b_n|)r^n$$

$$\leqslant (1+|b_1|)r + \tfrac{r^2}{[2]_q^{\nu}\{\{1+B-\lambda[1+A+(B-A)]\}q+(B-A)(1-\delta)\}} \textstyle\sum_{n=2}^{\infty}(|a_n|+|b_n|)r^n$$

$$\leqslant (1+|b_1|)r + \frac{(B-A)(1-\delta) - [2+A+B+(B-A)\delta]|b_1}{[2]_q^{\nu}\{\{1+B-\lambda[1+A+(B-A)]\}q + (B-A)(1-\delta)\}} r^2.$$

□

THEOREM 2.5. *Set*

$$h_1(z) = z, \qquad h_n(z) = z - \frac{(B-A)(1-\delta)}{\Phi_n} z^n, \qquad (n = 2,3,...)$$

and

$$g_n(z) = z + (-1)^{\nu}\frac{(B-A)(1-\delta)}{\Psi n}\overline{z}^n, \qquad (n = 1,2,...).$$

Then $f \in THS_q^{\nu}(\lambda,\delta,A,B)$ if and only if it can be expressed as

$$f(z) = \sum_{n=1}^{\infty}(x_n h_n(z) + y_n g_n(z)),$$

where $x_n \geqslant 0$, $y_n \geqslant 0$ and $\sum_{n=1}^{\infty}(x_n+y_n) = 1$. In particular, the extreme points of $THS_q^{\nu}(\lambda,\delta,A,B)$ are $\{h_n\}$ and $\{g_n\}$.

PROOF. Suppose

$$f(z) = \sum_{n=1}^{\infty}(x_n h_n(z) + y_n g_n(z))$$

$$= \sum_{n=1}^{\infty}(x_n+y_n)z - \sum_{n=2}^{\infty}\frac{(B-A)(1-\delta)}{\Phi_n}x_n z^n + (-1)^{\nu}\sum_{n=1}^{\infty}\frac{(B-A)(1-\delta)}{\Psi n}y_n\overline{z}^n.$$

Then

$$\sum_{n=2}^{\infty}\Phi_n|a_n|+\sum_{n=1}^{\infty}\Psi_n|b_n| = (B-A)(1-\delta)\sum_{n=2}^{\infty}x_n+(B-A)(1-\delta)\sum_{n=1}^{\infty}y_n$$

$$= (B-A)(1-\delta)(1-x_1)\leqslant (B-A)$$

and so $f\in THS_q^{\nu}(\lambda,\delta,A,B)$. Conversely, if $f\in THS_q^{\nu}(\lambda,\delta,A,B)$, then

$$|a_n|\leqslant\frac{(B-A)(1-\delta)}{\Phi_n}\qquad and\qquad |b_n|\leqslant\frac{(B-A)(1-\delta)}{\Psi_n},$$

$$x_n=\frac{\Phi_n}{(B-A)(1-\delta)}|a_n|\quad (n=2,3...)$$

and

$$y_n=\frac{\Psi_n}{(B-A)(1-\delta)}|b_n|\quad (n=1,2...).$$

Then note by Theorem 2.3, $0\leqslant x_n\leqslant 1$ and $0\leqslant y_n\leqslant 1\quad (n=1,2,...)$. We define

$$x_1=1-\sum_{n=2}^{\infty}x_n-\sum_{n=1}^{\infty}y_n$$

and note that by Theorem 2.3, $x_1\geqslant 0$. Consequently, we obtain

$$f(z)=\sum_{n=1}^{\infty}(x_nh_n(z)+y_ng_n(z))$$

as required.

□

Now we show that $THS_q^{\nu}(\lambda,\delta,A,B)$ is closed under convex combinations of its members.

THEOREM 2.6. *The class $THS_q^{\nu}(\lambda,\delta,A,B)$ is closed under convex combination.*

PROOF. For $j=1,2,3,...$ let $f_j\in THS_q^{\nu}(\lambda,\delta,A,B)$, where f_j is given by

$$f_j(z)=z-\sum_{n=2}^{\infty}|a_{n_j}|z^n+(-1)^{\nu}\sum_{n=1}^{\infty}|b_{n_j}|\overline{z}^n.$$

Then by (2.3),

$$\sum_{n=1}^{\infty}(\Phi_n|a_{n_j}|+\Psi_n|b_{n_j}|)\leqslant 2(B-A)(1-\delta).$$

For $\sum_{j=1}^{\infty}\lambda_j=1$, $0\leqslant\lambda_j\leqslant 1$, convex combination of f_j may be written as

$$\sum_{j=1}^{\infty}\lambda_jf_j(z)=z-\sum_{n=2}^{\infty}\left(\sum_{j=1}^{\infty}\lambda_j|a_{n_j}|\right)z^n+(-1)^{\nu}\sum_{n=1}^{\infty}\left(\sum_{j=1}^{\infty}\lambda_j|b_{n_j}|\right)\overline{z}^n.$$

Then by (2.3),

$$\sum_{n=1}^{\infty}\left(\Phi_n\sum_{j=1}^{\infty}\lambda_j|a_{n_j}|+\Psi_n\sum_{j=1}^{\infty}\lambda_j|b_{n_j}|\right)=\sum_{j=1}^{\infty}\lambda_j\left(\sum_{n=1}^{\infty}\Phi_n|a_{n_j}|+\Psi_n|b_{n_j}|\right)$$
$$\leqslant 2(B-A)(1-\delta)\sum_{j=1}^{\infty}\lambda_j$$
$$=2(B-A)(1-\delta).$$

This is condition required by (2.3) and so $\sum_{j=1}^{\infty}\lambda_j f_j(z)\in THS_q^{\nu}(\lambda,\delta,A,B)$. □

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4. Author contributions

This study was produced from first author Adnan Canbulat's doctoral thesis. Adnan Canbulat made the calculations. Second author F. Müge Sakar was the supervisor and checked all the calculations. Third author Bilal Şeker revised the manuscript.

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