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SEMI $B-$ALGEBRAS

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Abstract. In this paper, we introduce a novel concept termed a semi $B-$algebra, which acts as an extension of the existing $B-$algebra. Our investigation delves into the derivation of essential arithmetical properties unique to semi $B-$algebras, coupled with diverse characterizations shedding light on various aspects of this particular algebraic structure.

1. Introduction

Emil L. Post's work in 1921 established the foundation for a comprehensive theory of propositions by introducing the Post lattice, which systematically categorizes closed classes of Boolean functions. His inquiry extended to the practical applications of Post's lattices in many-valued logical systems, confirming the finite basis of each class. The Post lattice subsequently became a valuable tool for exploring the intricacies of Boolean circuits and propositional formulas. In addition to its foundational significance, Post algebra has extensive applications in multiple-valued logic circuits, digital signal processing, formal verification, cryptography, and data analysis.

George Epstein contributed significantly to the exploration of Post algebraic characteristics by introducing B-algebras, BL-algebras, and P-algebras. These algebraic systems, each with unique properties, have found applications in diverse fields, including biological systems and neural science. Notably, B-algebras and BL-algebras have demonstrated significance in computer science by providing abstractions for specific facets of Post algebras.

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On the other hand, an A Heyting algebra, named after Arend Heyting, is a bounded lattice A that is equipped with a binary operation denoted as $x \rightarrow y$ of implication. In Heyting algebras, the implication operation $x \rightarrow y$ is located within the distributive lattice A. However, in B-algebras, $x \Rightarrow y$ is situated in the Boolean center B of the distributive lattice A, denoted by $x \Rightarrow y$. This distinction highlights the specific location of the implication operation within the broader mathematical structure and emphasizes the nuanced behavior of logical operations in Heyting algebras and B-algebras within the context of distributive lattices. The implication operator $\Rightarrow$ plays a significant role in both programming and logic and is a vital tool for articulating conditional statements.

In this context, we embark on an exploration of the concept of semi $B-$algebra, drawing inspiration from the foundational work of Post and Epstein. Our investigation delves into the derivation of fundamental arithmetical properties inherent to semi $B-$algebras, while also presenting diverse characterizations of these algebraic structures. These identified properties not only enrich the theoretical understanding of semi $B-$algebras but also open avenues for potential applications in the realms of logic and computer science.

2. Preliminaries

In this section, we revisit essential notations and foundational results crucial for ensuring the self-contained nature of this paper.

Definition 2.1. *[**1**] An algebra $(A, \vee, \wedge)$ of type $(2, 2)$ is called a lattice if it satisfies the following identities.*

(i) *Idempotency: $x \wedge x = x$ and $x \vee x = x$.*
(ii) *Commutativity: $x \wedge y = y \wedge x$ and $x \vee y = y \vee x$.*
(iii) *Associativity: $(x \wedge y) \wedge z = x \wedge (y \wedge z)$ and $(x \vee y) \vee z = x \vee (y \vee z)$.*
(iv) *Absorption laws: $x \wedge (x \vee y) = x$ and $x \vee (x \wedge y) = x$.*

Definition 2.2. *[**1**] A uniquely bounded complemented distributive lattice $(A, \vee, \wedge, ', 0, 1)$ is called a Boolean algebra. In other words a Boolean algebra is a Boolean lattice in which $0, 1$ and $'$(complementation) are also considered to be fundamental operations.*

For an in-depth understanding of lattice theory, readers are directed to the works of [**1**] and [**6**].

3. Semi $B-$algebras

As previously noted, during the examination of Post algebra, G. Epstein and A. Horn introduced the concept of $B-$algebra, elucidated in detail in [**4**]

Definition 3.1. *[4] Let A be a distributive lattice with $0, 1$, and let B represent its Birkhoff center. An algebra $(A, \vee, \wedge, \Rightarrow, 0, 1)$ is termed a $B-$algebra if, for every $x, y \in A$, there exists a greatest element $b \in B$ such that $x \wedge b \leqslant y$. In this context, the complemented element b is denoted by $x \Rightarrow y$.*

Furthermore, if, for any $x \in A$, there exists a greatest element $b \in B$ satisfying $1 \wedge b \leqslant x$, then A is identified as a pseudo-supplemented lattice. The element b is denoted as $x!$, and it is referred to as the pseudo-supplement of x. Notably, when x belongs to B, it holds that $x = x!$.

For additional properties of $B-$algebra, readers are directed to [**4**].

Subsequently, we move forward to establish two characterization theorems for a $B-$algebra.

THEOREM 3.1. *Let A be a distributive lattice with $0, 1$ and B, its Birkhoff center of A. Then A is a $B-$algebra if and if only if it satisfying the following conditions: $\forall x, y, z \in A$ and $b \in B$*

(i) $(x \Rightarrow x) = 1$
(ii) $\{(x \Rightarrow b) \wedge b\} = b$
(iii) $\{x \wedge (x \Rightarrow b)\} = x \wedge b$
(iv) $\{z \Rightarrow (x \wedge y)\} = \{(z \Rightarrow x) \wedge (z \Rightarrow y)\}$
(v) $\{(x \vee y) \Rightarrow z)\} = \{(x \Rightarrow z) \wedge (y \Rightarrow z)\}$

PROOF. Suppose A is a $B-$algebra.

(i) Since $x \wedge 1 \leqslant x$ implies $1 = (x \Rightarrow x)$.
(ii) Since $b \leqslant (x \Rightarrow b)$, we get $(x \Rightarrow b) \wedge b = b$
(iii) Since $x \wedge (x \Rightarrow b) \leqslant b$, we get $(x \wedge (x \Rightarrow b)) \leqslant x \wedge b$
Since $b \leqslant (x \Rightarrow b)$, we get $x \wedge b \leqslant (x \wedge (x \Rightarrow b))$.
Therefore $x \wedge (x \Rightarrow b) = x \wedge b$.
(iv) Since $\{z \wedge (z \Rightarrow x) \wedge (z \Rightarrow y)\} \leqslant (x \wedge y)$,
we get $\{(z \Rightarrow x) \wedge (z \Rightarrow y)\} \leqslant (z \Rightarrow (x \wedge y))$.
On the other hand, $z \wedge (z \Rightarrow (x \wedge y)) \leqslant (x \wedge y) \leqslant x$,
we get $(z \Rightarrow (x \wedge y)) \leqslant (z \Rightarrow x)$.
Similarly, we get $(z \Rightarrow (x \wedge y)) \leqslant (z \Rightarrow y)$
and hence $(z \Rightarrow (x \wedge y)) \leqslant \{(z \Rightarrow x) \wedge (z \Rightarrow y)\}$.
Similarly, we can prove $\{(x \vee y) \Rightarrow z)\} = \{(x \Rightarrow z) \wedge (y \Rightarrow z)\}$.
On the other hand, conditions (i) to (v) hold. The proof is evident and is consequently omitted.

□

We now proceed to establish an additional characterization theorem for a $B-$algebra.

THEOREM 3.2. *Let A be a distributive lattice with $0, 1$ and B, its Birkhoff center of A. Then A is a $B-$algebra if it satisfying the following conditions: $\forall x, y \in A$ and $b \in B$*

(i) $((x \wedge y) \Rightarrow x) = 1$
(ii) $\{x \wedge (x \Rightarrow b)\} = x \wedge b$
(iii) $[b \wedge (x \Rightarrow y)] = [b \wedge \{(b \wedge x) \Rightarrow (b \wedge y)\}]$

PROOF. The proof is evident and is therefore omitted. □

In the following, we introduce the concept of a semi $B-$algebra and systematically derive its inherent properties.

DEFINITION 3.2. *Let A be a distributive lattice with $0, 1$ and B, the set of all complemented elements of A. Then, an algebra $(A, \vee, \wedge, \Rightarrow, 0, 1)$ said to be a semi $B-$algebra if it satisfying the following conditions: $\forall x, y \in A, b \in B$,*

B_1 : $(x \Rightarrow x) = 1$
B_2 : $\{x \wedge (x \Rightarrow b)\} = x \wedge b$
B_3 : $[b \wedge (x \Rightarrow y)] = [b \wedge ((b \wedge x) \Rightarrow (b \wedge y))]$

Throughout this section, we denote by A a semi $B-$algebra with Birkhoff center B," unless specified otherwise.

EXAMPLE 3.1. *Let $A = \{0, 1\}$ be two element chain. Define a binary operation $\Rightarrow$ on A as follows*

$\Rightarrow$	0	1
0	1	1
1	0	1

A fulfills all the conditions of both a $B-$algebra and a semi $B-$algebra.

EXAMPLE 3.2. *Let $A = \{0, 1\}$ be two element chain. Define a binary operation $\Rightarrow$ on A as follows*

$\Rightarrow$	0	1
0	1	0
1	0	1

In this context, A is identified as a semi $B-$algebra, distinct from being a $B-$algebra.

In the forthcoming theorems, we present fundamental arithmetical properties inherent to a semi $B-$algebra, which will be instrumental in characterizing this algebraic structure.

THEOREM 3.3. *Let $x \in A, b, c \in B$ and $b \leqslant x$. Subsequently, the following relationships hold:*

(i) $b \wedge (x \Rightarrow c) = b \wedge c$
(ii) $b \leqslant (x \Rightarrow 1)$
(iii) $b \leqslant (b \Rightarrow 1)$
(iv) $b \leqslant (b \Rightarrow x)$
(v) $b \leqslant (x \Rightarrow b)$

PROOF. (i) $b \wedge (x \Rightarrow c) = b \wedge \{(b \wedge x) \Rightarrow (b \wedge c)\}$ (by B_3 of the definition)
$= b \wedge \{b \Rightarrow (b \wedge c)\}$
$= b \wedge c$ (by B_2 of the definition).

The proofs for (ii), (iii), (iv), and (v) are straightforward and are thus omitted. □

THEOREM 3.4. *Let $x, y, z, w \in A, b \in B$. Then, we the following*

(i) *if $b \leqslant x$ and $b \leqslant y$, then $b \leqslant (x \Rightarrow y)$*
(ii) $(x \Rightarrow y) \wedge b = \{(x \wedge b) \Rightarrow (y \wedge b)\} \wedge b$
(iii) $\{(x \wedge y) \Rightarrow (z \wedge w)\} \wedge b = \{(y \wedge x) \Rightarrow (w \wedge z)\} \wedge b$
(iv) $b \leqslant ((b \wedge x) \Rightarrow x)$
(v) $(x \wedge b) \leqslant (x \Rightarrow b)$

PROOF. (i) Let $b \leqslant x$ and $b \leqslant y$. Now

$$\begin{aligned} b \wedge (x \Rightarrow y) &= b \wedge \{(b \wedge x) \Rightarrow (b \wedge y)\} \\ &= \{b \wedge (b \Rightarrow b)\} \\ &= b. \end{aligned}$$

Therefore $b \leqslant (x \Rightarrow y)$.

(ii), (iii), and (iv) are consequences of B_3 in the definition.

(v) $$\begin{aligned} \{(x \wedge b) \wedge (x \Rightarrow b)\} &= \{b \wedge [x \wedge (x \Rightarrow b)]\} \\ &= b \wedge x \wedge b \text{ (by } B_2 \text{ of the definition)} \\ &= x \wedge b. \end{aligned}$$

Hence $(x \wedge b) \leqslant (x \Rightarrow b)$

□

THEOREM 3.5. *Let $x, y \in A, b \in B$. Then, we the following*

(i) $x \leqslant b \leqslant y$ *implies* $b \wedge (x \Rightarrow y) = b \wedge (x \Rightarrow b)$
(ii) $\{b \wedge (0 \Rightarrow 1)\} = \{b \wedge (0 \Rightarrow b)\}$
(iii) $b \wedge (0 \Rightarrow 1) = b \wedge (0 \Rightarrow b)$
(iv) $b \wedge (x \Rightarrow 1) = b \wedge ((x \wedge b) \Rightarrow b)$
(v) $b \leqslant ((b \wedge x) \Rightarrow x) \Rightarrow b$
(vi) $b \leqslant \{b \Rightarrow [x \Rightarrow (b \wedge x)]\}$
(vii) $[b \wedge (0 \Rightarrow x)] = \{b \wedge (0 \Rightarrow (b \wedge x))\}$
(viii) $(y \Rightarrow 0) \leqslant (x \Rightarrow 0)$ *if and only if* $x \wedge (y \Rightarrow 0) = 0$
(ix) $b \wedge x = 0$ *if and only if* $b \leqslant (x \Rightarrow 0)$

PROOF. Let $x, y \in A, b \in B$. (i) to (iv) are consequences of B_3 in the definition.

(v) $$\begin{aligned} b \wedge \{((b \wedge x) \Rightarrow x) \Rightarrow b\} &= b \wedge \{[b \wedge ((b \wedge x) \Rightarrow x)] \Rightarrow (b \wedge b)\} \\ &= b \wedge \{[b \wedge (b \wedge b \wedge x) \Rightarrow (b \wedge x)] \Rightarrow b\} \\ &= b \wedge \{[b \wedge ((b \wedge x) \Rightarrow (b \wedge x))] \Rightarrow b\} \\ &= b \wedge \{[b \wedge 1] \Rightarrow b\} \\ &= b \wedge \{b \Rightarrow b\} \\ &= b \end{aligned}$$

and hence $b \leqslant ((b \wedge x) \Rightarrow x) \Rightarrow b$.

Likewise, we establish the proofs for (vi) and (vii).

(viii). Suppose $(y \Rightarrow 0) \leqslant (x \Rightarrow 0)$.

Now $$\begin{aligned} [x \wedge (y \Rightarrow 0)] &= [x \wedge (y \Rightarrow 0) \wedge (x \Rightarrow 0)] \\ &= [x \wedge (x \Rightarrow 0) \wedge (y \Rightarrow 0)] \\ &= [x \wedge 0 \wedge (y \Rightarrow 0)] \end{aligned}$$

$[x \wedge (y \Rightarrow 0)] = 0$

Suppose $x \wedge (y \Rightarrow 0) = 0$.

Now $(y \Rightarrow 0) \wedge (x \Rightarrow 0) = (y \Rightarrow 0) \wedge \{[(y \Rightarrow 0) \wedge x] \Rightarrow [(y \Rightarrow 0) \wedge 0]\}$
$= (y \Rightarrow 0) \wedge [0 \Rightarrow 0]$ $(\because x \wedge (y \Rightarrow 0) = 0)$
$= (y \Rightarrow 0) \wedge 1$

$\therefore (y \Rightarrow 0) \wedge (x \Rightarrow 0) = (y \Rightarrow 0)$

(ix). Suppose $b \wedge x = 0$. Now

$\{b \wedge (x \Rightarrow 0)\} = \{b \wedge (x \Rightarrow (b \wedge x))\} = \{b \wedge [(b \wedge x) \Rightarrow (b \wedge b \wedge x))]\}$
$= \{b \wedge [(b \wedge x) \Rightarrow (b \wedge x))]\}$
$= b.$

Thus $b \leqslant (x \Rightarrow 0)$.

On the other hand, suppose $b \leqslant (x \Rightarrow 0)$.

Then $b \wedge x = b \wedge (x \Rightarrow 0) \wedge x = b \wedge 0 = 0$ and hence $b \wedge x = 0$.

□

THEOREM 3.6. *Let $x \in A$ and $b, y \in B$. Then, we the following*

(i) $b \leqslant (x \Rightarrow y)$ *implies* $b \wedge x \leqslant y$
(ii) $(x \Rightarrow y) = 1$ *implies* $x \leqslant y$
(iii) $b \leqslant \{(b \Rightarrow y) \Rightarrow y\}$
(iv) $[b \Rightarrow (y \wedge 1)] = 1$, *implies* $b \leqslant y$

PROOF. (i). Suppose $b \leqslant (x \Rightarrow y)$.

Now $b \wedge x \wedge y = \{b \wedge (x \Rightarrow y)\}$ $(\because x \wedge (x \Rightarrow y) = x \wedge y)$
$= b \wedge x$ $(\because b \leqslant (x \Rightarrow y))$

$\therefore b \wedge x \leqslant y$

(ii) follows from (i). (iii) follows from B_3 of the definition.

(iv) $b = b \wedge 1 = b \wedge [b \Rightarrow (y \wedge 1)]$ $(\because b \Rightarrow (y \wedge 1) = 1)$
$= b \wedge [(b \wedge b) \Rightarrow (b \wedge y \wedge 1)]$
$= b \wedge (b \Rightarrow (b \wedge y))$
$= b \wedge y$ $(\because b \wedge (b \Rightarrow (b \wedge y)) = b \wedge y)$

and hence $b \leqslant y$.

□

REMARK 3.1. The statements $b \leqslant (x \Rightarrow y)$ if and only if $x \wedge b \leqslant y$ and $1 = (x \Rightarrow y)$ if and only if $x \leqslant y$ that hold in $B-$algebras do not, in general, hold in semi $B-$algebras.

THEOREM 3.7. *Let $x \in A, b, c \in B$. Then, we have the following*

(i) $b = c$ *if and only if* $(b \Rightarrow c) \wedge (c \Rightarrow b) = 1$
(ii) $b = c$ *if and only if* $(b \vee c) \Rightarrow (b \wedge c) = 1$
(iii) *if* $b \leqslant x$, *then* $(x \Rightarrow b) \leqslant (b \Rightarrow x)$
(iv) $(c \Rightarrow (b \wedge c)) \leqslant ((b \wedge c) \Rightarrow c)$
(v) $(x \Rightarrow b) \leqslant (x \Rightarrow (b \wedge x))$
(vi) $(c \Rightarrow (b \wedge c)) \leqslant (b \Rightarrow (b \wedge c))$

PROOF. (i) $b = c$. Then $(b \Rightarrow c) \wedge (c \Rightarrow b) = 1$. On the other hand, suppose $(b \Rightarrow c) \wedge (c \Rightarrow b) = 1$. Then $b = b \wedge 1 = [b \wedge (b \Rightarrow c) \wedge (c \Rightarrow b)]$

$$= b \wedge c \wedge (c \Rightarrow b)$$
$$= b \wedge c.$$

Thus $b \leqslant c$. Similarly, we can prove that $c \leqslant b$ and hence $b = c$.
Likewise, we establish the proof (ii).

(iii) Suppose $b \leqslant x$. Then

$$\begin{aligned}(x \Rightarrow b) \wedge (b \Rightarrow x) &= (x \Rightarrow b) \wedge \{((x \Rightarrow b) \wedge b) \Rightarrow ((x \Rightarrow b) \wedge x)\} \\ &= (x \Rightarrow b) \wedge [b \wedge x \wedge (x \Rightarrow b)] \Rightarrow [x \wedge (x \Rightarrow b)] \ (\because b \leqslant x) \\ &= (x \Rightarrow b) \wedge [(b \wedge x) \Rightarrow (b \wedge x)] \\ &= (x \Rightarrow b) \wedge 1 \\ &= (x \Rightarrow b).\end{aligned}$$

Thus $(x \Rightarrow b) \leqslant (b \Rightarrow x)$.

(iv)
$$\begin{aligned}[c \Rightarrow (b \wedge c)] \wedge [(b \wedge c) \Rightarrow c] &= [c \Rightarrow (b \wedge c)] \wedge \{[(c \Rightarrow (b \wedge c)) \wedge (b \wedge c)] \Rightarrow [(c \Rightarrow (b \wedge c)) \wedge c]\} \\ &= [c \Rightarrow (b \wedge c)] \wedge \{(b \wedge c) \Rightarrow (b \wedge c)\} \\ &= (c \Rightarrow (b \wedge c)) \wedge 1 \\ &= (c \Rightarrow (b \wedge c))\end{aligned}$$

Therefore $(c \Rightarrow (b \wedge c)) \leqslant ((b \wedge c) \Rightarrow c)$

(v)
$$\begin{aligned}(x \Rightarrow b) \wedge (x \Rightarrow (b \wedge x)) &= \{(x \Rightarrow b) \wedge [((x \Rightarrow b) \wedge x) \Rightarrow ((x \Rightarrow b) \wedge (b \wedge x))]\} \\ &= \{(x \Rightarrow b) \wedge [(x \wedge b) \Rightarrow (x \wedge b)]\} \\ &= (x \Rightarrow b).\end{aligned}$$

Hence $(x \Rightarrow b) \leqslant (x \Rightarrow (b \wedge x))$.

Likewise, we establish the proofs for (vi).

□

THEOREM 3.8. *Let A is a semi B-algebra. Then the following conditions are equivalent $\forall b, c \in B$,*

(i) $(b \Rightarrow c) = (c \Rightarrow b)$
(ii) $(b \Rightarrow 1) = b$
(iii) $[b \wedge (c \Rightarrow b)] = b \wedge c$

PROOF. (i) implies (ii)
Suppose $(b \Rightarrow c) = (c \Rightarrow b)$.
Now
$$\begin{aligned}(b \Rightarrow 1) &= (1 \Rightarrow b) \\ &= 1 \wedge (1 \Rightarrow b) \\ &= 1 \wedge b \\ &= b\end{aligned}$$

(ii) implies (iii)
Suppose $(b \Rightarrow 1) = b$.
Now
$$\begin{aligned}[b \wedge (c \Rightarrow b)] &= [b \wedge \{(b \wedge c) \Rightarrow (b \wedge b)\}] \\ &= [b \wedge ((b \wedge c) \Rightarrow b)] \\ &= [b \wedge ((b \wedge c) \Rightarrow (c \wedge 1))] \\ &= [b \wedge (c \Rightarrow 1)] \\ &= b \wedge c \quad (\because (c \Rightarrow 1) = c\ )\end{aligned}$$

(iii) implies (i)
Suppose $[b \wedge (c \Rightarrow b)] = b \wedge c$.

Now $(b \Rightarrow c) \wedge (c \Rightarrow b) = (b \Rightarrow c) \wedge \{[(b \Rightarrow c) \wedge c] \Rightarrow [(b \Rightarrow c) \wedge b]\}$
$= (b \Rightarrow c) \wedge \{((b \wedge c) \Rightarrow (b \wedge c)\}$
$= (b \Rightarrow c)$.

Thus $(b \Rightarrow c) \leqslant (c \Rightarrow b)$.

Likewise, we establish $(b \Rightarrow c) \leqslant (c \Rightarrow b)$ and consequently $(b \Rightarrow c) = (c \Rightarrow b)$. □

THEOREM 3.9. *Let $x, y \in A$ and $b \in B$,*

(i) $b \wedge (x \Rightarrow y) = b \wedge \{(b \wedge x) \Rightarrow y\}$
(ii) $b \wedge (x \Rightarrow y) = b \wedge \{x \Rightarrow (b \wedge y)\}$

PROOF. (i) $b \wedge \{(b \wedge x) \Rightarrow y\} = b \wedge \{(b \wedge (b \wedge x)) \Rightarrow (b \wedge y)\}$
$= b \wedge \{(b \wedge x) \Rightarrow (b \wedge y)\}$
$= b \wedge (x \Rightarrow y)$

(ii) $b \wedge \{x \Rightarrow (b \wedge y)\} = \{b \wedge [(b \wedge x) \Rightarrow (b \wedge b \wedge y)]\}$
$= \{b \wedge [(b \wedge x) \Rightarrow (b \wedge y)]\}$
$= b \wedge (x \Rightarrow y)$

□

In the ensuing two theorems, we present axiomatizations for semi $B-$algebras.

THEOREM 3.10. *Let A be a distributive lattice with $0, 1$ with Birkhoff center B. Then A is a semi B-algebra if and only if it fulfills the following conditions for all $x, y \in A$ and $b \in B$*

(i) $x \wedge (x \Rightarrow b) = x \wedge b$
(ii) $(x \Rightarrow x) = 1$
(iii) $b \wedge (x \Rightarrow y) = b \wedge \{(b \wedge x) \Rightarrow y\}$
(iv) $b \wedge (x \Rightarrow y) = b \wedge \{x \Rightarrow (b \wedge y)\}$

PROOF. Assume that A is a semi $B-$algebra. Then, A complies with all specified conditions. Conversely, the given conditions are satisfied.

$b \wedge (x \Rightarrow y) = b \wedge \{x \Rightarrow (b \wedge y)\}$ (by condition (iv))
$= b \wedge \{(b \wedge x) \Rightarrow (b \wedge y)\}$ (by condition (iii)) □

Now, we present an alternative characterization of a semi $B-$algebra.

THEOREM 3.11. *Let A be a distributive lattice with $0, 1$ with Birkhoff center B. Then A is a semi B-algebra if and only if it satisfies the following conditions: $\forall x, y \in A, b \in B$*

(i) $x \wedge (x \Rightarrow b) = x \wedge b$
(ii) $b \wedge (x \Rightarrow y) = b \wedge \{(b \wedge x) \Rightarrow (b \wedge y)\}$
(iii) $b \leqslant [(b \wedge x) \Rightarrow x]$

PROOF. Assume that A is a semi $B-$algebra. Then, A complies with all specified conditions. Conversely, the given conditions are satisfied.

Put $b = 1$ in condition (iii), we get $1 \leqslant [(1 \wedge x) \Rightarrow x]$
implies $1 \leqslant (x \Rightarrow x)$
implies $1 = (x \Rightarrow x)$.

Therefore $(x \Rightarrow x) = 1$ and hence A is a semi $B-$algebra. □

In conclusion, we finalize this paper with the following theorem

THEOREM 3.12. *Let A be a semi B-algebra. Then the following are equivalent* $\forall x, y, z \in A$

i. A *is a* $B-$*algebra*
ii. $x \leqslant y$ *implies* $(x \Rightarrow y) = 1$
iii. $x \leqslant y$ *implies* $(x \Rightarrow z) \geqslant (y \Rightarrow z)$
iv. $x \leqslant y$ *implies* $z \Rightarrow x \leqslant z \Rightarrow y$
v. $(x \vee y) \Rightarrow z \leqslant (x \Rightarrow z) \wedge (y \Rightarrow z)$
vi. $x \Rightarrow (y \wedge z) \leqslant (x \Rightarrow y) \wedge (x \Rightarrow z)$

PROOF. If A is a $B-$algebra, then A satisfies all conditions that is (ii) to (vi) conditions.
(ii)$\Longrightarrow$ (i)
Suppose $x \leqslant y$, implies $(x \Rightarrow y) = 1$.
Since $x \wedge y \leqslant x$, by our assumption, we get $(x \wedge y) \Rightarrow x = 1$. Thus A is a $B-$algebra.
(iii)$\Longrightarrow$ (ii)
Suppose $x \leqslant y$ implies $(x \Rightarrow z) \geqslant (y \Rightarrow z)$.
Since $x \leqslant y$, by our assumption, we get $(x \Rightarrow y) \geqslant (y \Rightarrow y) = 1$ and hence $(x \Rightarrow y) = 1$. Thus $(iii) \Longrightarrow (ii)$. Similarly, we prove $(iv) \Longrightarrow (ii)$.
$(v) \Longrightarrow (i)$
Suppose $(x \vee y) \Rightarrow z \leqslant (x \Rightarrow z) \wedge (y \Rightarrow z)$.

$$\begin{aligned} \text{Now}\quad 1 = (z \Rightarrow z) &= ((x \wedge z) \vee z) \Rightarrow z \ (\because z = [(x \wedge z) \vee z]) \\ &\leqslant [(x \wedge z) \Rightarrow z] \wedge [z \Rightarrow z] \\ &\leqslant [(x \wedge z) \Rightarrow z] \wedge 1 \\ &\leqslant [(x \wedge z) \Rightarrow z] \end{aligned}$$

Thus $(x \wedge z) \Rightarrow z = 1$. Hence A is a $B-$algebra.
$(vi) \Longrightarrow (i)$
Suppose $x \Rightarrow (y \wedge z) \leqslant (x \Rightarrow y) \wedge (x \Rightarrow z)$.

$$\begin{aligned} \text{Now } 1 &= (y \wedge z) \Rightarrow (y \wedge z) \\ &\leqslant [(y \wedge z) \Rightarrow y] \wedge [(y \wedge z) \Rightarrow z] \end{aligned}$$

implies $((y \wedge z) \Rightarrow y) = 1$ and hence A is a $B-$algebra. □

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