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CURRENT ASSESSMENT UPON $\alpha-\left(\tilde{F},\hbar\right)-$GERAGHTY CONTRACTION VIA APPLICABILITY

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Abstract. The encouragement of presented writing is to acquire various theorems for $\alpha-\ (\mathcal{F},\hbar)-$Geraghty contraction type mappings having a mixed monotone property in partial order on metric spaces. An instance is preferred to display the straightness of the obtained findings. Further, we propel the applicability of our conclusion in finding the solution for a two-point boundary value problem. Our results not only extend and generalize several results in the fixed point theory literature but also unify most of them.

1. Introduction and preliminaries

Banach [**9**] constituted a pivotal fixed point (FP) theorem referred to as the "Banach Contraction Principle" (BCP) which plays an essential role in mathematical science and is seen as the primary source of metric FP theory. Concerning the implementation potency of theory, several authors have extended, improved, and generalized FP theory by presenting novel contractive states and substituting complete metric space (CMS) via numerous abstract spaces; see e.g. [**21**], [**12**], [**1**], [**16**], [**22**], [**19**].

Geraghty [**13**] notedly touched on one of the riveting generalizations of BCP in the context of CMS via support function. After that, the authors [**7**] obtained the findings of [**13**] in the setting of a partially ordered CMS. Kadelburg et al. [**17**] studied common coupled FP theorems for Geraghty-type contraction mappings $(G-tcm)$ involving mixed monotone property (mmp). Yolacan&Kır [**25**] presented

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the existence of a coupled FP for $\alpha-(G-tcm)$ by way of a partially ordered CMS and also gave some applications to a directed graph. Hammad et al. [**14**] established novel resultants about FP results with the concept of $\beta-\phi-(G-tcm)$ in partially ordered CMS. Handa [**15**] considered $(G-tcm)$ and investigated some FP theorem for $G-$non$-$decreasing mappings on partially ordered CMS.

Ansari et al. [**4**] discussed some FP results for various contractions via ordered CMS. Ansari et al. [**5**] showed several FP theorems for a kind of contractive maps in a partial order on CMS using compatible mappings. Babu & Kumar [**8**] proved common FP of $(\mathcal{F}-\hbar)-\varphi-\psi-\phi-$weakly contractive maps on CMS. Mahmood et al. [**20**] obtained interesting consequences for $(\mathcal{F},\hbar)$ pair of up-class functions via $\alpha-\psi-$contractive mappings.

The goal of the available writing is to present FP theorems for $\alpha-(\mathcal{F},\hbar)-(G-tcm)$ in ordered CMS. Eventually, we deal with the applicability of our conclusions in the probe fields of the two-point boundary value problem.

In the cause of integrity, we place some base definitions and attention-grabbing results in the literature on the topic.

Definition 1.1. *[**23**] Let $T:\vartheta^2\to\vartheta$ and $\alpha:\vartheta^2\times\vartheta^2\to R^+$ be endowed maps. T is $(\alpha)-$admissible mapping if*
$\alpha\left(\left(p,q\right),\left(m,n\right)\right)\geqslant 1\Rightarrow\alpha\left(\left(T\left(p,q\right),T\left(q,p\right)\right),\left(T\left(m,n\right),T\left(n,m\right)\right)\right)\geqslant 1$
for $\forall m,n,p,q\in\vartheta$.

In the sequel, we shall take $\alpha\left(\left(p,q\right),\left(m,n\right)\right)=\alpha_{pqmn}$ for purify.

Definition 1.2. *[**10**] An element $(p,q)\in\vartheta^2$ is a coupled FP of $T:\vartheta^2\to\vartheta$ if*

$$T\left(q,p\right)=q \text{ and } T\left(p,q\right)=p.$$

The partially order $\leqslant$ in ϑ could be promoted on ϑ^2 hereinbelow:

$$q\geqslant n \text{ and } p\leqslant m \text{ iff } \left(p,q\right),\ \left(m,n\right)\in\vartheta^2,\ \left(m,n\right)\geqslant\left(p,q\right).$$

Definition 1.3. *[**10**] Let $(\vartheta,\leqslant)$ be a partial order set and $T:\vartheta^2\to\vartheta$ be a map. Here T is called to hold (mmp) if*

$$p_1,p_2\in\vartheta,\quad p_2\geqslant p_1\quad \textit{implies}\quad T\left(p_2,q\right)\geqslant T\left(p_1,q\right)$$

and

$$q_1,q_2\in\vartheta,\quad q_1\leqslant q_2\quad \textit{implies}\quad T\left(p,q_2\right)\leqslant T\left(p,q_1\right)$$

for $\forall p,q\in\vartheta$.

Definition 1.4. *[**6**] $\hbar:\left(R^+\right)^2\to R$ is sub-class of type I if*

$$p\geqslant 1\Longrightarrow\hbar\left(1,q\right)\leqslant\hbar\left(p,q\right)\ \textit{ for } \forall q\in R^+.$$

example 1.1. *[**6**] Let the function $\hbar$ be a sub-class of type I. Here, for $p,q\in R^+$*
(a) $\hbar\left(p,q\right)=pq$;
(b) $\hbar\left(p,q\right)=2^{-1}q\left(1+p\right)$;
(c) $\hbar\left(p,q\right)=\left(l+q\right)^p$, $1<l$.

DEFINITION 1.5. *[**6**] Let $\hbar,\mathcal{F}:\left(R^{+}\right)^{2}\to R$ be two maps. $(\mathcal{F},\hbar)$ is an up-class of type I, if the function $\hbar$ is a sub-class of type I, also supplies:*
(a) $0\leqslant w\leqslant 1\Longrightarrow\mathcal{F}(w,v)\leqslant\mathcal{F}(1,v)$,
(b) $\hbar(1,q)\leqslant\mathcal{F}(1,v)\Longrightarrow q\leqslant v$,
for $\forall w,v,q\in R^{+}$.

EXAMPLE 1.2. *[**6**] Let $(\mathcal{F},\hbar)$ is an up-class of type I. Then*
(a) $\hbar(p,q)=pq$, $\mathcal{F}(w,v)=wv$;
(b) $\hbar(p,q)=(p+1)\,2^{-1}q$, $\mathcal{F}(w,v)=wv$;
(c) $\hbar(p,q)=(l+q)^{p}$, $1<l$, $\mathcal{F}(w,v)=wv$;
for $w\in[0,1]$ *and* $p,q,v\in R^{+}$.

2. Main results

Ω is the collection of functions $\beta:[0,\infty)\to[0,1)$ with $\beta(w_n)\to 1\Rightarrow w_n\to 0$.

DEFINITION 2.1. *Let $\left(\vartheta^{2},\leqslant,d\right)$ be an order MS, $T:\vartheta^{2}\to\vartheta$ and $\alpha:\vartheta^{2}\times\vartheta^{2}\to[0,\infty)$ be given mappings. T is an $\alpha-(\mathcal{F},\hbar)-(G-tcm)$ if*

$$\hbar\left(\alpha_{pqmn},\frac{d\left(T(p,q),T(m,n)\right)+d\left(T(q,p),T(n,m)\right)}{2}\right)\leqslant\mathcal{F}\left(\beta\left(\frac{d(p,m)+d(q,n)}{2}\right),\left(\frac{d(p,m)+d(q,n)}{2}\right)\right)\tag{2.1}$$

for all $(m,n),(p,q)\in\vartheta^{2}$ via $(m,n)\geqslant(p,q)$ where $\beta\in\Omega$ and $(\mathcal{F},\hbar)$ is an up-class of type I.

THEOREM 2.1. *Let $(\vartheta,\leqslant,d)$ be an ordered CMS, $T:\vartheta^{2}\to\vartheta$ be $(\alpha)-$admissible and $\alpha-(\mathcal{F},\hbar)-(G-tcm)$ through (mmp). Given that the undermentioned states hold:*
(a) there exists p_0, $q_0\in\vartheta$ such that

$$\alpha_{T(p_0,q_0)T(q_0,p_0)p_0q_0}\geqslant 1 \text{ and } \alpha_{q_0p_0T(q_0,p_0)T(p_0,q_0)}\geqslant 1,$$

(b) (i) T is continuous, or
(ii) $(\vartheta,\leqslant,d)$ is regular and, if $\{p_n\}$, $\{q_n\}\subset\vartheta$ such that $\alpha_{p_{n+1}q_{n+1}p_nq_n}\geqslant 1$ and $\alpha_{q_np_nq_{n+1}p_{n+1}}\geqslant 1$ and $q_n\to q,p_n\to p$ for all $q,p\in\vartheta$, then $\alpha_{pqp_nq_n}\geqslant 1$ and $\alpha_{q_np_nqp}\geqslant 1$ for all $n\geqslant 1$.
If there exists p_0, $q_0\in\vartheta$ such that $p_0\leqslant T(p_0,q_0)$ and $q_0\geqslant T(q_0,p_0)$, then T has a coupled FP.

PROOF. Constitute $\{p_n\},\{q_n\}\subset\vartheta$ via $T(p_n,q_n)=p_{n+1}$ and $T(q_n,p_n)=q_{n+1}$ for all $n\geqslant 1$. We shall show that

$$p_{n+1}\geqslant p_n \text{ and } q_{n+1}\leqslant q_n,\ \forall n\geqslant 1.\tag{2.2}$$

Due to hyp., by $p_0\leqslant T(p_0,q_0)=p_1$ and $q_0\geqslant T(q_0,p_0)=q_1$. Next, assume that (2.2) provides with for $n\in N$. Using (mmp), we have

$$p_{n+1}=T(p_n,q_n)\leqslant T(p_{n+1},q_n)\leqslant T(p_{n+1},q_{n+1})=p_{n+2},\tag{2.3}$$

$$q_{n+1}=T(q_n,p_n)\geqslant T(q_{n+1},p_n)\geqslant T(q_{n+1},p_{n+1})=q_{n+2}.\tag{2.4}$$

Beside, from (a), we obtain

$$\alpha_{T(p_0,q_0)T(q_0,p_0)(p_0,q_0)} = \alpha_{p_1q_1p_0q_0} \geqslant 1,$$

in connection with $(\alpha) - \textit{admissibility}$ of T

$$\alpha_{T(p_1,q_1)T(q_1,p_1)T(p_0,q_0)T(q_0,p_0)} = \alpha_{p_2q_2p_1q_1} \geqslant 1,$$

by means of inductive, we get

$$\alpha_{p_{n+1}q_{n+1}p_nq_n} \geqslant 1,\ \forall n \geqslant 1. \tag{2.5}$$

Invariably, by use of (a) and $(\alpha) - \textit{admissibility}$ of T

$$\alpha_{q_np_nq_{n+1}p_{n+1}} \geqslant 1, \forall n \geqslant 1. \tag{2.6}$$

Suppose that if $(p_n, q_n) = (p_{n+1}, q_{n+1})$ for some n, then

$$p_n = T(p_n, q_n) \text{ and } q_n = T(q_n, p_n),$$

we are done. Assume now that $(p_n, q_n) \neq (p_{n+1}, q_{n+1})$ for all $n \geqslant 1$. Owing to $(p_{n+1}, q_{n+1}) \geqslant (p_n, q_n)$, using (2.1) and (2.5), we find out

$$\begin{aligned}
& \hbar\left(1, \frac{d(p_{n+1}, p_n) + d(q_{n+1}, q_n)}{2}\right) \\
= \quad & \hbar\left(1, \frac{d(T(p_n, q_n), T(p_{n-1}, q_{n-1})) + d(T(q_n, p_n), T(q_{n-1}, p_{n-1}))}{2}\right) \\
\leqslant \quad & \hbar\left(\begin{array}{c} \alpha_{p_nq_np_{n-1}q_{n-1}} \\ \times 2^{-1} \times \left(\begin{array}{c} d(T(p_n, q_n), T(p_{n-1}, q_{n-1})) \\ +d(T(q_n, p_n), T(q_{n-1}, p_{n-1})) \end{array}\right) \end{array}\right) \\
\leqslant \quad & \mathcal{F}\left(\beta\left(\frac{d(p_n, p_{n-1}) + d(q_n, q_{n-1})}{2}\right), \left(\frac{d(p_n, p_{n-1}) + d(q_n, q_{n-1})}{2}\right)\right) \\
\Longrightarrow &
\end{aligned}$$

$$\begin{aligned}
\frac{d(p_{n+1}, p_n) + d(q_{n+1}, q_n)}{2} \quad & \leqslant \quad \beta\left(\frac{d(p_n, p_{n-1}) + d(q_n, q_{n-1})}{2}\right) \\
& \qquad \times \left(\frac{d(p_n, p_{n-1}) + d(q_n, q_{n-1})}{2}\right) \\
& < \quad \left(\frac{d(p_n, p_{n-1}) + d(q_n, q_{n-1})}{2}\right).
\end{aligned} \tag{2.7}$$

Set $\omega_n := d(p_n, p_{n+1}) + d(q_n, q_{n+1})$, then sequence $\{\omega_n\}$ is decreasing. Hence, there is some $\omega_n \geqslant 0$ such that $\lim_{n\to\infty} \omega_n = \omega$. Allege that $\omega = 0$. Assume, contrariantly, that $\omega > 0$, as it is by (2.7), we get

$$\begin{aligned}
\frac{\omega_n}{\omega_{n-1}} \quad & \leqslant \quad \beta\left(2^{-1} \times \omega_{n-1}\right) < 1, \\
& \Longrightarrow \\
\lim_{n\to\infty} \beta\left(2^{-1} \times \omega_{n-1}\right) \quad & = \quad 1.
\end{aligned}$$

Herefrom,

$$d(p_{n-1}, p_n) \to 0 \text{ and } d(q_{n-1}, q_n) \to 0 \text{ as } n \to \infty \text{ due to } \beta \in \Omega \tag{2.8}$$

or

$$\lim_{n\to\infty}\{\omega_{n-1}\}=0, \tag{2.9}$$

which is a contradiction. Thence $\omega=0$, viz

$$\lim_{n\to\infty}\{\omega_{n}\}=0. \tag{2.10}$$

Now, we show that $\{p_n\}$ and $\{q_n\}$ are Cauchy sequences (Cs). Let's say, contrarily, that at the least one of $\{p_n\}$ or $\{q_n\}$ are not (Cs). At that rate, there is an $\varepsilon>0$ for which we could obtain sub-sequences $\left\{p_{a(l)}\right\}$, $\left\{p_{b(l)}\right\}$ of $\{p_n\}$ and $\left\{q_{a(l)}\right\}$, $\left\{q_{b(l)}\right\}$ of $\{q_n\}$ by way of $l\leqslant b(l)<a(l)$ such that

$$d\left(p_{a(l)-1},p_{b(l)}\right)+d\left(q_{a(l)-1},q_{b(l)}\right)<\varepsilon \text{ and } d\left(p_{a(l)},p_{b(l)}\right)+d\left(q_{a(l)},q_{b(l)}\right)\geqslant\varepsilon. \tag{2.11}$$

Let $\xi_l := d\left(p_{a(l)},p_{b(l)}\right)+d\left(q_{a(l)},q_{b(l)}\right)$. From (2.10), (2.11) and using triangle inequality, we have

$$\lim_{l\to\infty}\xi_l=\varepsilon. \tag{2.12}$$

We obtain, using the triangle inequality,

$$\xi_l\leqslant\omega_{a(l)}+\omega_{b(l)}+d\left(p_{a(l)+1},p_{b(l)+1}\right)+d\left(q_{a(l)+1},q_{b(l)+1}\right).$$

By (2.5), $\alpha_{p_{a(l)}q_{a(l)}p_{b(l)}q_{b(l)}}\geqslant 1$. Since $b(l)<a(l)$, $\left(p_{b(l)},q_{b(l)}\right)\leqslant\left(p_{a(l)},q_{a(l)}\right)$ and from (2.1),

$$\begin{aligned}
&\hbar\left(1,\frac{d\left(p_{a(l)+1},p_{b(l)+1}\right)+d\left(q_{a(l)+1},q_{b(l)+1}\right)}{2}\right)\\
=\;&\hbar\left(1,2^{-1}\times\left(\begin{array}{c} d\left(T\left(p_{a(l)},q_{a(l)}\right),T\left(p_{b(l)},q_{b(l)}\right)\right)\\ +d\left(T\left(q_{a(l)},p_{a(l)}\right),T\left(q_{b(l)},p_{b(l)}\right)\right)\end{array}\right)\right)\\
\leqslant\;&\hbar\left(\begin{array}{c}\alpha_{p_{a(l)}q_{a(l)}p_{b(l)}q_{b(l)}}\\ \times\left(\frac{d\left(T\left(p_{a(l)},q_{a(l)}\right),T\left(p_{b(l)},q_{b(l)}\right)\right)+d\left(T\left(q_{a(l)},p_{a(l)}\right),T\left(q_{b(l)},p_{b(l)}\right)\right)}{2}\right)\end{array}\right)\\
\leqslant\;&\mathcal{F}\left(\begin{array}{c}\beta\left(\frac{d\left(p_{a(l)},p_{b(l)}\right)+d\left(q_{a(l)},q_{b(l)}\right)}{2}\right),\\ \left(\frac{d\left(p_{a(l)},p_{b(l)}\right)+d\left(q_{a(l)},q_{b(l)}\right)}{2}\right)\end{array}\right)\\
=\;&\mathcal{F}\left(\beta\left(\frac{d\left(p_{a(l)},p_{b(l)}\right)+d\left(q_{a(l)},q_{b(l)}\right)}{2}\right),\left(\frac{\xi_l}{2}\right)\right)\\
\Longrightarrow&
\end{aligned}$$

$$d\left(p_{a(l)+1},p_{b(l)+1}\right)+d\left(q_{a(l)+1},q_{b(l)+1}\right)\leqslant\beta\left(\frac{d\left(p_{a(l)},p_{b(l)}\right)+d\left(q_{a(l)},q_{b(l)}\right)}{2}\right)\xi_l.$$

Hence,

$$\xi_l\leqslant\omega_{a(l)}+\omega_{b(l)}+\beta\left(\frac{d\left(p_{a(l)},p_{b(l)}\right)+d\left(q_{a(l)},q_{b(l)}\right)}{2}\right)\xi_l.$$

Then we have

$$\frac{\xi_l - \omega_{a(l)} - \omega_{b(l)}}{\xi_l} \leqslant \beta\left(\frac{d\left(p_{a(l)}, p_{b(l)}\right) + d\left(q_{a(l)}, q_{b(l)}\right)}{2}\right) < 1.$$

Using (2.10) and (2.12), we get

$$\beta\left(\frac{d\left(p_{a(l)}, p_{b(l)}\right) + d\left(q_{a(l)}, q_{b(l)}\right)}{2}\right) \to 1 \quad \text{as } l \to \infty.$$

We have

$$\lim_{l\to\infty} d\left(p_{a(l)}, p_{b(l)}\right) = 0 = \lim_{l\to\infty} d\left(q_{a(l)}, q_{b(l)}\right),$$

then

$$d\left(p_{a(l)}, p_{b(l)}\right) + d\left(q_{a(l)}, q_{b(l)}\right) \to 0 \text{ as } l \to \infty$$

which is a contradiction. This show that $\{p_n\}$ and $\{q_n\}$ are (Cs). As ϑ is a CMS, there consists $p, q \in \vartheta$ such that

$$p_n \to p \text{ and } q_n \to q \text{ when } n \to \infty. \tag{2.13}$$

Assume the hyp. (i) provides. Letting $n \to \infty$ in (2.3) and (2.4), from (2.13), we have

$$\begin{aligned} p &= \lim_{n\to\infty} p_{n+1} = \lim_{n\to\infty} T\left(p_n, q_n\right) \\ &= T\left(\lim_{n\to\infty} p_n, \lim_{n\to\infty} q_n\right) = T\left(p, q\right). \end{aligned}$$

Similarly, we show that $q = T\left(q, x\right)$.

Assume the hyp. (ii) is satisfied. Inasmuch as $p_n \to p$ is a nondecreasing sequence, we obtain that $p_n \leqslant p$ for all n and likewise $q_n \geqslant q$ for all n, namely, $(p_n, q_n) \leqslant (p, q)$. Moreover, by (2.5), (2.6) and assumption (ii), we figure out $\alpha_{pqp_nq_n} \geqslant 1$ and $\alpha_{q_np_nqp} \geqslant 1$.

By (2.1), we have

$$\begin{aligned} & \hbar\left(1, 2^{-1} \times \left(d\left(T\left(p_n, q_n\right), T\left(p, q\right)\right) + d\left(T\left(q_n, p_n\right), T\left(p, q\right)\right)\right)\right) \\ \leqslant\ & \hbar\left(\alpha_{p_nq_npq}, 2^{-1} \times \left(d\left(T\left(p_n, q_n\right), T\left(p, q\right)\right) + d\left(T\left(q_n, p_n\right), T\left(p, q\right)\right)\right)\right) \\ \leqslant\ & \mathcal{F}\left(\beta\left(\frac{d\left(p_n, p\right) + d\left(q_n, q\right)}{2}\right), \left(\frac{d\left(p_n, p\right) + d\left(q_n, q\right)}{2}\right)\right) \\ \Longrightarrow & \end{aligned}$$

$$\begin{aligned} & 2^{-1} \times \left(d\left(T\left(p_n, q_n\right), T\left(p, q\right)\right) + d\left(T\left(q_n, p_n\right), T\left(p, q\right)\right)\right) \\ \leqslant\ & \beta\left(\frac{d\left(p_n, p\right) + d\left(q_n, q\right)}{2}\right)\left(\frac{d\left(p_n, p\right) + d\left(q_n, q\right)}{2}\right) \\ \to\ & 0 \text{ as } n \to \infty. \end{aligned}$$

Therefore $p = T\left(p, q\right)$. Accordingly, one can show that $q = T\left(q, p\right)$. □

THEOREM 2.2. *Appending the hypostasis of Theorem 2.1, given that for* $\forall\,(p,q)$, $(w,v)\in\vartheta^2$, *there consists* $(m,n)\in\vartheta^2$ *such that* $\alpha_{pqmn}\geqslant 1$ *and* $\alpha_{wvmn}\geqslant 1$, *and also suppose that* (m,n) *is comparable to* (p,q) *and* (w,v). *Here* T *hold a unique coupled FP.*

PROOF. On account of Theorem 2.1, $T\neq\emptyset$. Assume (p,q) and (w,v) are coupled FP of T; that is; $p=T(p,q)$, $q=T(q,p)$, $w=T(w,v)$ and $v=T(v,w)$. By hypothesis, there consists $(m,n)\in\vartheta^2$ such that (m,n) is comparable to (p,q) and (w,v). We define sequences $\{m_l\}$, $\{n_l\}$ as follows

$$m_0=m,\ n_0=n,\ m_{l+1}=T(m_l,n_l)\ \text{ and }\ n_{l+1}=T(n_l,m_l)\ \text{ for all } l.$$

Seeing (m,n) is comparable with (p,q), it is simple to express that $p\leqslant m_1$ and $q\geqslant n_1$. Hence, $p\leqslant m_l$ and $q\geqslant n_l$ for all l. By presume, there exists $(m,n)\in\vartheta^2$ such that

$$\alpha_{pqmn}\geqslant 1 \text{ and } \alpha_{wvmn}\geqslant 1. \tag{2.14}$$

Owing to $(\alpha)-$*admissibility* of T, by (2.14), we infer

$$\alpha_{pqm_ln_l}\geqslant 1 \text{ and } \alpha_{qpn_lm_l}\geqslant 1, \tag{2.15}$$

$$\alpha_{wvm_ln_l}\geqslant 1 \text{ and } \alpha_{vwn_lm_l}\geqslant 1, \tag{2.16}$$

for all l. By (2.1), (2.5) and (2.6), we hold

$$\begin{aligned}
&\hbar\left(1,\frac{d(p,m_l)+d(q,n_l)}{2}\right)\\
=\ &\hbar\left(1,2^{-1}\times\left(d\left(T(p,q),T(m_{l-1},n_{l-1})\right)+d\left(T(q,p),T(n_{l-1},m_{l-1})\right)\right)\right)\\
\leqslant\ &\hbar\left(\begin{array}{c}\alpha_{pqm_{l-1}n_{l-1}},\\ \left(2^{-1}\times\left(d\left(T(p,q),T(m_{l-1},n_{l-1})\right)+d\left(T(q,p),T(n_{l-1},m_{l-1})\right)\right)\right)\end{array}\right)\\
\leqslant\ &\mathcal{F}\left(\beta\left(\frac{d(p,m_{l-1})+d(q,n_{l-1})}{2}\right),\left(\frac{d(p,m_{l-1})+d(q,n_{l-1})}{2}\right)\right)\\
\Rightarrow\ &
\end{aligned}$$

$$\begin{aligned}
\frac{d(p,m_l)+d(q,n_l)}{2}\ &\leqslant\ \beta\left(\frac{d(p,m_{l-1})+d(q,n_{l-1})}{2}\right)\\
&\quad\ \times\left(\frac{d(p,m_{l-1})+d(q,n_{l-1})}{2}\right)\\
&\leqslant\ \frac{d(p,m_{l-1})+d(q,n_{l-1})}{2}.
\end{aligned}\tag{2.17}$$

Set $\mu_l:=d(p,m_l)+d(q,n_l)$, then $\{\mu_l\}$ is decreasing and nonnegative. Therefore, there is some $\mu_l\geqslant 0$ such that $\lim_{n\to\infty}\mu_l=\mu$.

We argue that $\mu=0$. Supposing, contrarily, that $\mu>0$. By switching to subsequences, if required, we can pretend that $\lim_{n\to\infty}\beta\left(\frac{\mu_{l-1}}{2}\right)=\theta$ exists $\left(\text{since } 0\leqslant\beta\left(\frac{\mu_{l-1}}{2}\right)<1\right)$. Then by (2.17), taking $l\to\infty$, we have $\theta\mu=\mu$ and so $\theta=1$. We get

$$d(p,m_{l-1})\to 0 \text{ and } d(q,n_{l-1})\to 0 \text{ as } l\to\infty \text{ since } \beta\in\Omega,$$

viz,

$$d\left(p,m_{l-1}\right)+d\left(q,n_{l-1}\right)\to 0 \text{ when } l\to\infty$$

which is a contradiction. Thereby $\mu=0$, viz,

$$\lim_{l\to\infty}\left\{d\left(p,m_{l}\right)+d\left(q,n_{l}\right)\right\}=0.$$

It is implies

$$\lim_{l\to\infty}d\left(p,m_{l}\right)=0=\lim_{n\to\infty}d\left(q,n_{l}\right). \tag{2.18}$$

Similarly, we get

$$\lim_{l\to\infty}d\left(w,m_{l}\right)=0=\lim_{n\to\infty}d\left(v,n_{l}\right). \tag{2.19}$$

From (2.18) and (2.19), we have $p=w$ and $q=v$. □

EXAMPLE 2.1. *Let $\vartheta=[0,\infty)$, $d\left(p,q\right)=\left|p-q\right|$ for all $p,q\in\vartheta$ and partial order be defined by $p\leqslant q$. Then $(\vartheta,\leqslant,d)$ is regular and ordered CMS. Let $\beta\left(v\right)=\frac{\ln(1+v)}{v}$ $\in\Omega$ for all $v\geqslant 0$. Let a mapping $T:\vartheta^{2}\to\vartheta$ be defined by*

$$T\left(p,q\right)=\begin{cases}\frac{1}{2}\ln\left(\frac{1+p}{1+q}\right), & \text{if } p\geqslant q\\ 0, & \text{otherwise.}\end{cases}$$

Define $\alpha:\vartheta^{2}\times\vartheta^{2}\to\vartheta$ by

$$\alpha_{pqmn}=\begin{cases}1, & \text{if } p\geqslant m, q\leqslant n\\ 5^{-3}, & \text{otherwise.}\end{cases}$$

Farther, we define the mappings $\mathcal{F},\hbar:\left(R^{+}\right)^{2}\to R$ by

$$\hbar\left(p,q\right)=\begin{cases}pq, & \text{if } p\geqslant 1\\ p, & \text{otherwise}\end{cases}\quad\text{and } \mathcal{F}\left(pq\right)=pq.$$

Herein, $(\mathcal{F},\hbar)$ is an up-class of type I. Next, we show that T is an $\alpha-$ $(\mathcal{F},\hbar)-$ $(G-tcm)$.

*Following the lines of the solution of Example 2.4 in [**18**], we have*

$$\begin{aligned}
&\hbar\left(\alpha_{pqmn},2^{-1}\times\left(d\left(T\left(p,q\right),T\left(m,n\right)\right)+d\left(T\left(q,p\right),T\left(n,m\right)\right)\right)\right)\\
&\alpha_{pqmn}2^{-1}\times\left(d\left(T\left(p,q\right),T\left(m,n\right)\right)+d\left(T\left(q,p\right),T\left(n,m\right)\right)\right)\\
&\vdots\\
\leqslant\;&\ln\left(1+\frac{\left[\left|p-m\right|+\left|q-n\right|\right]}{2}\right)\\
=\;&\frac{\ln\left(1+\frac{\left[\left|p-m\right|+\left|q-n\right|\right]}{2}\right)}{\frac{\left[\left|p-m\right|+\left|q-n\right|\right]}{2}}\frac{\left[\left|p-m\right|+\left|q-n\right|\right]}{2}\\
\leqslant\;&\mathcal{F}\left(\beta\left(\frac{d\left(p,m\right)+d\left(q,n\right)}{2}\right),\left(\frac{d\left(p,m\right)+d\left(q,n\right)}{2}\right)\right)
\end{aligned}$$

holds for all $p\geqslant m$ and $q\leqslant n$. In addition of this,

$$\frac{\left[\left|p-m\right|+\left|q-n\right|\right]}{2}=\frac{d\left(p,m\right)+d\left(q,n\right)}{2}.$$

Therefore, all states of Theorem 2.1 are provided. Also, the point $(0,0)$ *is the coupled FP of* T.

3. Application

We appeal our consequences to prove in existence & uniqueness of solving the two-point boundary value problem

$$\left\{\begin{array}{ll} -\frac{d^2p}{dt^2}(v)=\varrho(v,p(v),p(v)), & v\in[0,1], \\ p(0)=p(1)=0, & \end{array}\right. \tag{3.1}$$

where $\varrho:[0,1]\times R\times R\to R$ is a continuous function.

The Green function affiliated with (3.1) is endowed with

$$G(v,w)=\left\{\begin{array}{ll} v(1-w), & 0\leqslant v\leqslant w\leqslant 1 \\ w(1-v), & 0\leqslant w\leqslant v\leqslant 1. \end{array}\right.$$

We reckon with $\vartheta=C(I,R)$ of continuous functions identified $I=[0,1]$. Let

$$d(p,q)=\|p-q\|_\infty=\sup_{t\in I}|p(t)-q(t)|,\ \ \forall p,q\in\vartheta.$$

Then (ϑ,d) is a CMP.

Let $\kappa:[0,\infty)\to[0,\infty)$ be a function involving circumstances below:

(a) κ is increasing;

(b) for each $v>0$, $\kappa(v)<v$;

(c) $\beta(v)=\frac{\kappa(v)}{v}\in\Omega$.

For instance, $\kappa(v)=\ln(1+v)$ are in Ω.

We consider terms below:

(i) $\varrho:[0,1]\times R\times R\to R$ is a continuous function.

(ii) There consists $\rho:R^2\to R$ such that

$$0\leqslant\varrho(v,a,c)-\varrho(v,b,d)\leqslant 2\left[\ln(1+a-b)+\ln(1+d-c)\right],\ v\in I$$

for $\forall a,b,c,d\in R$ via $\rho(a,b)\geqslant 0$, $\rho(c,d)\geqslant 0$ and $a\geqslant b$, $c\leqslant d$.

(iii) There exists $p_1\in\vartheta$ such that

$$\rho\left(p_1(v),\left(\int_0^1 G(v,w)\,\varrho(w,p(w),q(w))\,dw\right)\right)\geqslant 0,\ t\in I.$$

(iv) $\rho(p(v),u(v))\geqslant 0$ for all $p,m\in\vartheta$ implies for $v\in I$,

$$\rho\left(\int_0^1 G(v,w)\,\varrho(w,p(w),q(w))\,dw,\int_0^1 G(v,w)\,\varrho(w,m(w),n(w))\,dw\right)\geqslant 0.$$

(v) There exists $(\gamma,\delta)\in\vartheta^2\times\vartheta^2$ solution to

$$\left\{\begin{array}{ll} -\frac{d^2\gamma}{dt^2}(v)\leqslant\varrho(v,\gamma(v),\delta(v)), & v\in[0,1] \\ -\frac{d^2\delta}{dt^2}(v)\geqslant\varrho(v,\delta(v),\gamma(v)), & v\in[0,1] \\ \gamma(0)=\gamma(1)=\delta(0)=\delta(1)=0. & \end{array}\right. \tag{3.2}$$

THEOREM 3.1. *Assume that circumstances* $(i)-(v)$ *are provided. In that case, (3.1) hold a unique solution in* ϑ^2.

Proof. It is considered that $p \in \vartheta^2$ is a solution of (3.1) iff $p \in \vartheta$ is a solution of the expression below,

$$p(v) = \int_0^1 G(v,w)\,\varrho(w,p(w))\,dw, \quad v \in I.$$

Define $T : \vartheta^2 \to \vartheta$ by

$$T(p,q)(v) = \int_0^1 G(v,w)\,\varrho(w,p(w),q(w))\,dw, \quad v \in I$$

for all $p,q \in \vartheta$.

From term (ii), it is obvious that T has the (mmp) in accordance with the partial order $\leqslant$ in ϑ.

Let $p,q,m,n \in \vartheta$ such that $p \geqslant m$ and $q \leqslant n$. By condition (ii), we derive

$$\begin{aligned}
& 2^{-1} \times (d(T(p,q),T(m,n)) + d(T(q,p),T(n,m))) \\
= \; & 2^{-1} \sup_{v\in I} |T(p,q)(v) - T(m,n)(v)| + 2^{-1} \sup_{t\in I} |T(q,p)(v) - T(n,m)(v)| \\
= \; & 2^{-1} \sup_{v\in I} \int_0^1 G(v,w)\,|\varrho(w,p(w),q(w)) - \varrho(w,m(w),n(w))|\,ds \\
& +2^{-1} \sup_{v\in I} \int_0^1 G(v,w)\,|\varrho(w,p(w),q(w)) - \varrho(w,n(w),m(w))|\,ds \\
\leqslant \; & 2^{-1} \sup_{v\in I} \int_0^1 2G(v,w)\,[\ln(1+p(w)-m(w)) + \ln(1+n(w)-q(w))] \\
& +2^{-1} \sup_{v\in I} \int_0^1 2G(v,w)\,[\ln(1+q(w)-n(w)) + \ln(1+m(w)-p(w))] \\
\leqslant \; & 2^{-1} \begin{pmatrix} [\ln(1+d(p,m)) + \ln(1+d(n,q))] \\ +[\ln(1+d(q,n)) + \ln(1+d(m,p))] \end{pmatrix} \left(\sup_{v\in I} \int_0^1 2G(v,w)\right) \\
\leqslant \; & 4^{-1} [[\ln(1+d(p,m)) + \ln(1+d(n,q))]] \\
\leqslant \; & 4^{-1} [\ln(1+d(p,m)+d(n,q)) + \ln(1+d(n,q)+d(p,m))] \\
= \; & 2^{-1} \ln(1+d(p,m)+d(n,q)) \\
\leqslant \; & \ln\left(1 + \frac{d(p,m)+d(n,q)}{2}\right) \\
= \; & \frac{\ln\left(1+\frac{d(p,m)+d(n,q)}{2}\right)}{\frac{d(p,m)+d(n,q)}{2}} \frac{d(p,m)+d(n,q)}{2}.
\end{aligned}$$

Put

$$\kappa\left(\frac{d(p,m)+d(n,q)}{2}\right) = \ln\left(\frac{d(p,m)+d(n,q)}{2} + 1\right)$$

and

$$\beta\left(\frac{d(p,m)+d(n,q)}{2}\right) = \frac{\kappa\left(\frac{d(p,m)+d(n,q)}{2}\right)}{\frac{d(p,m)+d(n,q)}{2}}.$$

Therefore, we have

$$\begin{aligned} & 2^{-1}\left(d\left(T\left(p,q\right),T\left(m,n\right)\right)+d\left(T\left(q,p\right),T\left(n,m\right)\right)\right) \\ \leqslant\ & \beta\left(\frac{d\left(p,m\right)+d\left(n,q\right)}{2}\right)\frac{d\left(p,m\right)+d\left(n,q\right)}{2}, \end{aligned}$$

for $p,q,m,n\in\vartheta$.

We define $\alpha:\vartheta^2\times\vartheta^2\to[0,\infty)$ by

$$\alpha_{pqmn}=\begin{cases} 1 & \text{if } \rho\left(T\left(p,q\right)\left(t\right),T\left(m,n\right)\left(t\right)\right)\geqslant 0,\ t\in I \\ 0 & \text{otherwise.}\end{cases}.$$

Then, for $\forall p,q,m,n\in\vartheta$, we own

$$\begin{aligned} & \alpha_{pqmn}2^{-1}\left(d\left(T\left(p,q\right),T\left(m,n\right)\right)+d\left(T\left(q,p\right),T\left(n,m\right)\right)\right) \\ \leqslant\ & \beta\left(\frac{d\left(p,m\right)+d\left(n,q\right)}{2}\right)\frac{d\left(p,m\right)+d\left(n,q\right)}{2}. \end{aligned}$$

From (iii), there exists $p_1\in\vartheta$ such that

$$\alpha\left(p_1\left(v\right),\left(\int_0^1 G\left(v,w\right)\varrho\left(w,p\left(w\right),q\left(w\right)\right)dw\right)\right)=1,\ v\in I.$$

Now, let $(\gamma,\delta)\in\vartheta^2\times\vartheta^2$ be a solution to (3.2). We claim that $\gamma\leqslant T\left(\gamma,\delta\right)$ and $\delta\geqslant T\left(\gamma,\delta\right)$. In fact,

$$-\frac{d^2\gamma}{dt^2}\left(w\right)\leqslant\varrho\left(w,\gamma\left(w\right),\delta\left(w\right)\right),\qquad w\in[0,1].$$

Multiplying by $G\left(v,w\right)$, we obtain

$$\int_0^1-\frac{d^2\gamma}{dt^2}\left(w\right)G\left(v,w\right)dw\leqslant T\left(\gamma,\delta\right)\left(v\right),\qquad v\in[0,1].$$

By $\gamma\left(0\right)=\gamma\left(1\right)=0$ and using an integration, we have

$$\gamma\left(t\right)\leqslant T\left(\gamma,\delta\right)\left(v\right),\qquad v\in[0,1].$$

Then, we get $\gamma\leqslant T\left(\gamma,\delta\right)$. Similarly one can prove that $\delta\geqslant T\left(\gamma,\delta\right)$.

By taking $\mathcal{F}\left(p,q\right)=\hbar\left(p,q\right)=pq$ in (2.1) and fulfilling Theorem 2.1 and 2.2, we deduce the existence of a uniqueness $(p,q)\in\vartheta^2$ solution to $p=T\left(p,q\right)$ and $q=T\left(q,p\right)$. □

The study of metric spaces plays the main role in most scientific fields both in applied and pure science such as computer science, biology, physics, and medicine, see [**11**], [**24**], [**2**], [**3**]. Therefore, we introduced and investigated some of novel FP theorems for $\alpha-\left(\mathcal{F},\hbar\right)-\left(G-tcm\right)$ having (mmp) in partially ordered metric spaces. Besides, we presented and verified these theorems for mappings satisfying the existence & unique of solving to the two-point boundary value problem. Inside of the future scope of the thought, the reader may infer, coupled coincidence and coupled common FP theorems on abstract space via a graph by considering auxiliary function.

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