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EXTENDED AND METRIC IDEALS OF A PSEUDO RING

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Abstract. In this paper, we introduce the notions of an extended ideal, a metric ideal, and a stable ideal in a pseudo ring. We prove that every non-zero maximal ideal M is an extended ideal of some non-empty subset of a pseudo ring R. Also we prove the set of all metric ideals form a complete distributive lattice. Further, we obtain the relationship between extended and metric ideals.

1. Introduction

In [**1**] Chajda and Länger initiated the study of pseudo rings, which are ring-like structures with a natural bijective relationship to MV-algebras. A lot of study has been made on an ideal theory in MV-algebras by many authors. For instance, maximal and essential ideals in MV-algebras by Hoo [**7**], obstinate ideals by Forouzesh et al [**6**], n-fold obstinate ideals, nodal and conodal ideals by Forouzesh [**3**, **5**], contraction and extension of ideals by Saidi and Borzooei [**8**], extended ideals by Forouzesh [**2**], expansion of ideals in MV-algebras by Foruzesh and Bedrood [**4**] are a few.

Keeping this in view, we initiate the study of extended and metric ideals in a pseudo ring. We obtain the basic properties of the ideals of a pseudo ring. Also, we prove that every non-zero maximal ideal is an extended ideal of some non-empty subset of a pseudo ring R. Also we prove the set of all metric ideals form a complete distributive lattice. Further, we obtain the relationship between extended and metric ideals. Throughout this paper, R stands for a pseudo ring.

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2. Preliminaries

We recall the following from Chajda and Länger in [**1**]

Definition 2.1. *[**1**] A pseudo ring is an algebra* $(R, +, \cdot, 1)$ *of type* $(2, 2, 0)$ *satisfying*

P_1. $(xy)z = x(yz)$,
P_2. $xy = yx$,
P_3. $x1 = x$,
P_4. $1 + (1 + x) = x$
P_5. $x0 = 0$,
P_6. $(1 + x(1 + y))(1 + y) = (1 + y(1 + x))(1 + x)$,
P_7. $1 + (1 + x(1 + y))(1 + y(1 + x)) = x + y$,

where 0 denotes the element $1 + 1$. Note that the following properties are consequence of the above definition.

Proposition 2.1. *[**1**] Let R be a pseudo-ring.Then*
1. $x(x + 1) = 0, \ \forall x \in R.$
2. $1 + y(1 + 0) = 1 + y, \ \forall y \in R$
3. $x + 0 = x.$

Definition 2.2. *[**1**] Define $x \leqslant y$ for any two elements $x, y \in R$ if and only if x and y satisfy the condition $(y + 1)x = 0$.*

3. Extended Ideals

Definition 3.1. *A subset I of R that satisfies the following conditions is an ideal I of R.*

(i) $0 \in I$
(ii) $1 + (x + 1)(y + 1) \in I$ *for every* $x, y \in I$.
(iii) $x \leqslant y$ *for every* $y \in I$ *implies that* $x \in I$.

Proposition 3.1. *Let $I \subseteq R$. I is an ideal of R if and only if the following holds:*

(i) $0 \in I$,
(ii) $1 + (x + 1)(y + 1) \in I$ *for every* $x, y \in I$,
(iii) $(y + 1)x, y \in I \Rightarrow \ x \in I$.

Proof. straightforward. □

Remark 3.1. If $x, y \in I$ then $x + y \in I$ for an ideal I of R.

Definition 3.2. *A proper ideal P of R is called prime ideal if for every $x, y \in R$, either $x(y + 1) \in P$ or $y(x + 1) \in P$.*

Definition 3.3. *Let R_1 and R_2 be two pseudo rings. A pseudo ring homomorphism is a mapping $\varphi : R_1 \longrightarrow R_2$ that meets the following conditions:*

1) $\varphi(1) = 1$,
2) $\varphi(x \cdot y) = \varphi(x) \cdot \varphi(y)$,
3) $\varphi(x + y) = \varphi(x) + \varphi(y)$ *for every* $x, y \in R_1$.

Note that $\varphi(0) = \varphi(1 + 1) = \varphi(1) + \varphi(1) = 1 + 1 = 0$.

Theorem 3.1. *Let R_1 and R_2 be two pseudo rings and a mapping $\varphi : R_1 \longrightarrow R_2$ be a pseudo ring homomorphism. Then the following properties hold:*

(a) *$ker\ \varphi$ is an ideal of R_1;*
(b) *If φ is a pseudo ring epimorphism, then $R_1/ker\ \varphi \cong R_2$;*
(c) *$\varphi(x) \leqslant \varphi(y)$ if and only if $x(y + 1) \in ker\ \varphi$;*
(d) *φ injective if and only if $ker\ \varphi = \{0\}$;*
(e) *$\varphi(x) \leqslant \varphi(y) \Rightarrow x \leqslant y$ if and only if φ is injective.*

Proof. Let R_1 and R_2 be two pseudo rings and a mapping $\varphi : R_1 \longrightarrow R_2$ be a pseudo ring homomorphism.

(a) $\varphi(0) = 0 \Rightarrow 0 \in ker\ \varphi$. Let $x, y \in ker\ \varphi$. This implies $\varphi(1 + (x + 1)(y + 1)) = 0 \Rightarrow 1 + (x + 1)(y + 1) \in ker\ \varphi$.
Let $(y+1)x, y \in ker\ \varphi$. It follows that $\varphi(x) = \varphi((0+1)x) = ((0+1)\varphi(x)) = (\varphi(y) + 1)\varphi(x) = \varphi((y + 1)x) = 0 \Rightarrow x \in ker\ \varphi$. As a result, $ker\ \varphi$ is an ideal of R_1.

(b) Define $\xi : R_1/ker\ \varphi \longrightarrow R_2$ by $\xi(\bar{x}) = \varphi(x)$.
Let $y \in R_2$. Since φ is epimorphism there is $a \in R_1$, such that $\varphi(a) = y$. Thus $\xi(\bar{a}) = \varphi(a) = y$.

$$\begin{aligned} ker\ \xi =& \{\bar{a} \in R_1/ker\ \varphi : \xi(\bar{a}) = 0\} \\ &= \{a \in R_1 : \varphi(a) = 0\} \\ &= \{a \in R_1 : a \in ker\ \varphi\} \\ &= ker\ \varphi. \end{aligned}$$

Therefor ξ is isomorphism from $R_1/ker\ \varphi$ to R_2.

(c) Suppose $\varphi(x) \leqslant \varphi(y)$. By Definition 2.2 and Definition 3.3, $x(1 + y) \in ker\ \varphi$. Conversely,suppose $x(y + 1) \in ker\ \varphi$. This implies $\varphi(x(1 + y)) = 0 \Rightarrow \varphi(x)(1 + \varphi(y)) = 0$. Thus by Definition 2.2, $\varphi(x) \leqslant \varphi(y)$.

(d) Assume $ker\ \varphi \neq \{0\}$. This implies there is $x \in ker\ \varphi$ such that $x \neq 0$. But, $\varphi(x) = 0 \Rightarrow \varphi(x) = \varphi(0)$. Since φ is injective $x = 0$. This is a contradiction. Therefore $ker\ \varphi = \{0\}$.

Conversely, suppose $ker\ \varphi = \{0\}$. Let $x, y \in R_1$ such that $\varphi(x) = \varphi(y)$. This means $\varphi(x) \leqslant \varphi(y)$ and $\varphi(y) \leqslant \varphi(x)$. From Definition 2.2, we have $\varphi(x)(1 + \varphi(y)) = 0 \Rightarrow \varphi(x(1 + y)) = 0 \Rightarrow x(1 + y) \in ker\ \varphi = \{0\} \Rightarrow x(1 + y) = 0$.
Therefore, $x \leqslant y$. Similarly $y \leqslant x$. Hence $x = y$

(e) Suppose $\varphi(x) \leqslant \varphi(y) \Rightarrow x \leqslant y$. Let $x \in ker\ \varphi$ and $x = a(1+b)$ for some $a, b \in R_1$. $\varphi(a)(1+\varphi(b)) = \varphi(x) = 0 \Rightarrow \varphi(a) \leqslant \varphi(b) \Rightarrow a \leqslant b \Rightarrow x = a(1+b) = 0$. Hence, $ker\ \varphi = \{0\}$.

□

THEOREM 3.2. *Let $\varphi : R_1 \to R_2$ is a pseudo ring epimorphism. If I is an ideal of R_1 and $ker(\varphi) \subseteq I$, then $\varphi(I)$ is an ideal of R_2.*

PROOF. It is noticeable that both the first and second requirements outlined in Definition 3.1 are met. Let $x \in R_2$ and $y \in \varphi(I)$ with $x \leqslant y$. Since φ is epimorphism,there are $a, b \in R_1$ such that $\varphi(a) = x,\ and\ \varphi(b) = y$. By Definition 2.2 and Definition 3.3, $a(b+1) \in ker(\varphi) \subseteq I \Rightarrow$ by Proposition 3.1, $a \in I$. Therefore $\varphi(I)$ is an ideal of R_2. □

COROLLARY 3.1. *If the map $\varphi : R_1 \to R_2$ is an epimorphism, then for any prime ideal P of R_1 with $ker(\varphi) \subseteq P$, $\varphi(P)$ is the prime ideal of R_2.*

DEFINITION 3.4. *Let I be an ideal of R and $X \subseteq R$, we define the extended ideal of I associated with X as follows: $\mathscr{E}_I(X) = \{x \in R : x(1+(a+1)x) \in I$, for all $a \in X\}$.*

EXAMPLE 3.1. *Let $A = \{\alpha, \beta\}$, consider the power set of A, $R = P(A)$ is a pseudo ring with the binary operation $D + C = (D' \cup C)' \cup (D \cup C')'$ and $D \cdot C = D \cap C$ for $C, D \in R$. Clearly $I_1 = \{\phi, \{\alpha\}\}$ and $I_2 = \{\phi, \{\beta\}\}$ are ideals of R. $\mathscr{E}_{I_1}(\{\beta\}) = I_1$ and $\mathscr{E}_{I_1}(\{\alpha\}) = R$. Similarly $\mathscr{E}_{I_2}(\{\beta\}) = R$ and $\mathscr{E}_{I_2}(\{\alpha\}) = I_2$.*

THEOREM 3.3. *Let I be an ideal of R and $X \subseteq R$, then $\mathscr{E}_I(X)$ is an ideal of R and $I \subseteq \mathscr{E}_I(X)$.*

PROOF. Let I be an ideal of R and $X \subseteq R$.

i. It is clearly that $0 \in \mathscr{E}_I(X)$.
ii. *Since* $b(1+(x+1)(y+1)b) \leqslant 1+(b(1+(x+1)b)+1)(b(1+(y+1)b)+1) \Rightarrow 1+(x+1)(y+1) \in \mathscr{E}_I(X),\ for\ any\ x, y \in \mathscr{E}_I(X)$.
iii. Let $x \leqslant y$ and $y \in \mathscr{E}_I(X) \Rightarrow x(1+(b+1)x) \leqslant y(1+(b+1)y)$, for all $b \in X \Rightarrow x(1+(b+1)x) \in I$

□

DEFINITION 3.5. *An ideal I is called stable with respect to a subset X of R if $I = \mathscr{E}_I(X)$.*

EXAMPLE 3.2. *In Example 3.1 $I_2 = \{\phi, \{\beta\}\}$ is stable with-respect to $X = \{\{\alpha\}\}$.*

PROPOSITION 3.2. *Let R_1 and R_2 be pseudo rings and $\varphi : R_1 \longrightarrow R_2$ be epimorphism such that $\varphi(H) = K$ where $H \subseteq R_1$ and $K \subseteq R_2$. Then the following statements hold.*

i. *If I is an ideal of R_2, then $\varphi^{-1}(I)$ is an ideal of R_1.*
ii. *If I is a stable relative to K, then $\varphi^{-1}(I)$ is stable relative to H.*

iii. *If I is a stable relative to H and ker $\varphi \subseteq I$, then $\varphi(I)$ is a stable relative to K.*

Proof. i. Let I be an ideal of R_2 and $a, b \in \varphi^{-1}(I)$. Clearly, $0 \in \varphi^{-1}(I)$. There are $x, y \in I$ such that $\varphi(a) = x$ and $\varphi(b) = y$. By Definition 3.3, $\varphi(1 + (a+1)(b+1)) \in I \Rightarrow 1 + (a+1)(b+1) \in \varphi^{-1}(I)$. Let $a \leqslant b$ and $b \in \varphi^{-1}(I)$. It follows $a(b+1) = 0 \Rightarrow \varphi(a)(\varphi(b)+1) = 0 \Rightarrow \varphi(a) \leqslant \varphi(b) \in I \Rightarrow a \in \varphi^{-1}(I)$. Thus, $\varphi^{-1}(I)$ is an ideal of R_1.

ii. By (i), $\varphi^{-1}(I)$ is an ideal of R_1. Suppose I is stable relative to K. Let $x \in \mathscr{E}_{\varphi^{-1}(I)}(H)$. $x \in \mathscr{E}_{\varphi^{-1}(I)}(H) \Leftrightarrow x(1 + (a+1)x) \in \varphi^{-1}(I)$, for all $a \in H \Leftrightarrow \varphi(x)(1 + (\varphi(a)+1)\varphi(x)) \in I$, for all $\varphi(a) \in K \Leftrightarrow \varphi(x) \in \mathscr{E}_I(K) \Leftrightarrow x \in \varphi^{-1}(\mathscr{E}_I(K))$. Therefore $\mathscr{E}_{\varphi^{-1}(I)}(H) = \varphi^{-1}(\mathscr{E}_I(K)) = \varphi^{-1}(I)$.

iii. Suppose I is stable relative to H and $ker\ \varphi \subseteq I$. Since φ is epimorphism, by Theorem 3.2, $\varphi(I)$ is an ideal of R_2. Let $x \in \varphi(\mathscr{E}_I(H))$. This implies there exists $s \in \mathscr{E}_I(H)$ such that $\varphi(s) = x \Rightarrow s(1+(a+1)s) \in I$, for all $a \in H \Rightarrow x(1+(\varphi(a)+1)x) = \varphi(s)(1+(\varphi(a)+1)\varphi(s)) = \varphi(s(1+(a+1)s)) \in \varphi(I)$ for all $\varphi(a) \in K \Rightarrow x \in \mathscr{E}_{\varphi(I)}(K)$. Thus, $\varphi(\mathscr{E}_I(H) \subseteq \mathscr{E}_{\varphi(I)}(K)$. Conversely, suppose $x \in \mathscr{E}_{\varphi(I)}(K)$. Since φ is epimorphism there exists $p \in R_1$ such that $\varphi(p) = x$. Then $x \in \mathscr{E}_{\varphi(I)}(K) \Rightarrow x(1+(b+1)x) \in \varphi(I)$, for all $b \in K \Rightarrow \varphi(p(1+(a+1)p)) = \varphi(t)$ for some $t \in I$, for all $a \in H \Rightarrow \varphi(p(1+(a+1)p) \cdot (t+1)) = 0$, for all $a \in H \Rightarrow p(1+(a+1)p) \cdot (t+1) \in ker(\varphi) \subseteq I$, for all $a \in H \Rightarrow p(1+(a+1)p) \in I$, for all $a \in H \Rightarrow p \in \mathscr{E}_I(H) \Rightarrow x \in \varphi(\mathscr{E}_I(H) \Rightarrow \mathscr{E}_{\varphi(I)}(K) \subseteq \varphi(\mathscr{E}_I(H) = \varphi(I)$. Therefore $\varphi(I)$ is stable relative to K.

□

Proposition 3.3. *For any $x, y \in R$, $y(1+(x+1)y) = x(1+(y+1)x)$.*

Proof. Let $x+1 = a$ and $y+1 = b$. By P_6 of Definition 2.1, we get $y(1+(x+1)y) = (b+1)(1+a(b+1)) = (a+1)(1+b(1+a)) = x(1+(y+1)x)$. □

Theorem 3.4. *Let I and J be the ideals of a pseudo ring R and X be any subset of R. Then each of the following holds:*

1. $\mathscr{E}_I(X) = R$ *if and only if* $X \subseteq I$;
2. *If* $I \subseteq J$, *then* $\mathscr{E}_I(X) \subseteq \mathscr{E}_J(X)$;
3. $X \subseteq \mathscr{E}_I(\mathscr{E}_I(X))$;
4. *If* $I \subseteq J$, *then* $\mathscr{E}_I(J) \cap J = I$;
5. *If* $X \subseteq Y$, *then* $\mathscr{E}_I(Y) \subseteq \mathscr{E}_I(X)$;
6. $\mathscr{E}_I(\mathscr{E}_I(X)) \cap \mathscr{E}_I(X) = I$.

Proof. 1. Suppose $\mathscr{E}_I(X) = R$ and $y \in X$. As $y \in R \Rightarrow y \in \mathscr{E}_I(X) \Rightarrow y(1+(a+1)y) \in I$, for all $a \in X$. In particular $y = y(1+(y+1)y) \in I$ (by Proposition 2.1), implies $X \subseteq I$. Conversely, suppose $X \subseteq I$ and $y \in R$. By Proposition 3.3, we have $x(1+(y+1)x) = y(1+(x+1)y) \leqslant x \in X \subseteq I \Rightarrow y \in \mathscr{E}_I(X) \Rightarrow \mathscr{E}_I(X) = R$.

2. Let $x \in \mathscr{E}_I(X) \Rightarrow x(1+(y+1)x) \in I \subseteq J$, for all $y \in X \Rightarrow x \in \mathscr{E}_J(X)$.

3. Let $x \in X$ and $y \in \mathscr{E}_I(X)$. By Proposition 3.3, $x(1+(y+1)x) = y(1+(x+1)y) \in I$, for all $y \in \mathscr{E}_I(X) \Rightarrow x \in \mathscr{E}_I(\mathscr{E}_I(X))$. Therefore $X \subseteq \mathscr{E}_I(\mathscr{E}_I(X))$.
4. Suppose $I \subseteq J$ and let $x \in \mathscr{E}_I(J) \cap J \Rightarrow x \in \mathscr{E}_I(J) \Rightarrow x = x(1+(x+1)x) \in I \Rightarrow \mathscr{E}_I(J) \cap J \subseteq I$. Conversely, observe that by Theorem 3.3 and hypothesis, $I \subseteq J$ and $I \subseteq \mathscr{E}_I(J) \Rightarrow I \subseteq \mathscr{E}_I(J) \cap J$. Therefore $I = \mathscr{E}_I(J) \cap J$.
5. Suppose $X \subseteq Y$. Let $a \in \mathscr{E}_I(Y) \Rightarrow a(1+(x+1)a) \in I$, for all $x \in Y \Rightarrow a(1+(x+1)a) \in I$, for all $x \in X$ as $X \subseteq Y$.
6. It follows from Theorem 3.3 and (4).

□

Proposition 3.4. *If P is a prime ideal, then*

i. $\mathscr{E}_P(X) = R$ *for* $X \subseteq P$ *and*
ii. P *is stable relative to* X *for* $X \nsubseteq P$.

Proof. i. Follows from Theorem 3.4 (1).

ii. Suppose there is $x_0 \in X$ but not in P with $x(1+(x_0+1)x) \in P$ where P is prime ideal of R. Since $x_0, x \in R$, by Definition 3.2 either $x(x_0+1) \in P$ or $x_0(x+1) \in P$. If $x(x_0+1) \in P$, then by Proposition 3.1 it follows that $x \in P$. If $x_0(x+1) \in P$, then by the same proposition $x_0 \in P$. This contradicts the fact that $x_0 \notin P$. Thus, from Theorem 3.3 we conclude that $\mathscr{E}_P(X) = P$.

□

Definition 3.6. *A proper ideal M of R is called a maximal if and only if $x \in I$ or $x+1 \in I$ but not both for every $x \in R$.*

Theorem 3.5. *If I is a maximal ideal and $\mathscr{E}_I(X)$ be a proper ideal of R, then $\mathscr{E}_I(X)$ is a maximal ideal of R and I is stable relative to X.*

Proof. Clearly $I \subseteq \mathscr{E}_I(X)$ (by assumption and by Theorem 3.3). Hence $\mathscr{E}_I(X)$ is maximal. Consequently $I = \mathscr{E}_I(X)$. □

Proposition 3.5. *Let $X \subseteq Y \subseteq R$ and I be a stable relative to X.Then I is stable ideal relative to Y.*

Proof. Since $X \subseteq Y$, from Theorem 3.4 (6) it follows that $\mathscr{E}_I(Y) \subseteq \mathscr{E}_I(X) = I$. Thus $\mathscr{E}_I(Y) = I$. □

Theorem 3.6. *Let $\{J_\alpha\}_{\alpha\in\Delta}$ be a family of ideals of R.Then the following hold:*

i. *if $\{J_\alpha\}_{\alpha\in\Delta}$ are totally ordered ideals and X is finite, then* $\bigcup_{\alpha\in\Delta} \mathscr{E}_{J_\alpha}(X) = \mathscr{E}_{\cup J_\alpha}(X)$.
ii. $\bigcap_{\alpha\in\Delta} \mathscr{E}_{J_\alpha}(X) = \mathscr{E}_{\cap J_\alpha}(X)$.

Proof. i. Let $\{J_\alpha\}_{\alpha\in\Delta}$ be a totally ordered ideal of R and X be finite subsets of R. It is clear that $\bigcup_{\alpha\in\Delta} J_\alpha$ is an ideal of R. Let $x \in \bigcup_{\alpha\in\Delta} \mathscr{E}_{J_\alpha}(X)$. Then there exist $\alpha_0 \in \Delta$ such that $x \in \mathscr{E}_{J_{\alpha_0}}(X)$. This implies $x(1+(a+1)x) \in J_{\alpha_0} \subseteq \bigcup_{\alpha\in\Delta} I_\alpha$, for all $a \in X \Rightarrow x \in \mathscr{E}_{\cup J_\alpha}(X)$. Thus, $\bigcup_{\alpha\in\Delta} \mathscr{E}_{J_\alpha}(X) \subseteq \mathscr{E}_{\cup J_\alpha}(X)$.

Conversely, suppose $x \in \mathscr{E}_{\cup J_\alpha}(X) \Rightarrow x(1+(a+1)x) \in \bigcup_{\alpha\in\Delta} J_\alpha$, for all $a \in X$. Since X is finite, there exist $\alpha_i \in \Delta$ such that $x(1+(a+1)x) \in J_{\alpha_i}$, for all $a \in X \Rightarrow x \in \mathscr{E}_{J_{\alpha_i}}(X) \subseteq \bigcup_{\alpha\in\Delta} \mathscr{E}_{J_\alpha}(X)$. Therefore, $\bigcup_{\alpha\in\Delta} \mathscr{E}_{J_\alpha}(X) = E_{\cup J_\alpha}(X)$.

ii. Let $x \in \bigcap_{\alpha\in\Delta} \mathscr{E}\ _{J_\alpha}(X)$. Then $x \in \bigcap_{\alpha\in\Delta} \mathscr{E}\ _{J_\alpha}(X) \Leftrightarrow x \in \mathscr{E}_{J_\alpha}$, for all $\alpha \in \Delta \Leftrightarrow x(1+(a+1)x) \in J_\alpha$, for all $x(1+(a+1)x \in \bigcap_{\alpha\in\Delta} J_\alpha \Leftrightarrow x \in \mathscr{E}_{\cap J_\alpha}(X)$.
Therefore, $\bigcap_{\alpha\in\Delta} \mathscr{E}\ _{J_\alpha}(X) = \mathscr{E}_{\cap J_\alpha}(X)$.

□

As a consequence, we get the following corollary.

Corollary 3.2. *The intersection of stable ideals with respect to a non-empty subset X of R is stable.*

Theorem 3.7. *Every non-zero maximal ideal M is an extended ideal of some non-empty subset of a pseudo ring R.*

Proof. Let $a \in R$. Suppose that $a \notin M$. Since M maximal ideal, by Definition 3.6, $a+1 \in M$. $y \in \mathscr{E}_M(\{a\}) \Rightarrow y(1+(a+1)y) \in M$. By Proposition 3.1, $y \in M$. Thus M is an extended ideal. □

Proposition 3.6. *Every ideal I of R is stable with respect to the set $X = \{1\}$.*

Proof. Let $X = \{1\}$ and suppose that I is an ideal of R. Then $\mathscr{E}_I(\{X\}) = \mathscr{E}_I(\{1\})$. Let $y \in \mathscr{E}_I(\{X\})$. Thus $y(1+(1+1)y) = y \in I \Rightarrow I = \mathscr{E}_I(\{X\})$. □

4. Metric ideal

We define the distance function on R as $\delta : R \times R \to R$ by $\delta(x,y) = x+y$

Proposition 4.1. *The following properties hold for all $x, y, z, t, r \in R$*

i. $\delta(x,y) = \delta(y,x)$.
ii. $\delta(x,0) = x$.
iii. $\delta(x,y) = 0$ *implies* $x = y$.
iv. $\delta(x,z) \leqslant \delta(x,y) + \delta(y,z)$.
v. $\delta(1+(x+1)(t+1), 1+(y+1)(r+1)) \leqslant 1 + (\delta(x,y)+1)(\delta(t,r)+1)$.

Define binary operation $'\circledast'$ by $x \circledast y = \delta(x,1)\cdot\delta(y,1)$. Since $'\cdot'$ is well defined and so is $'\circledast'$.

Definition 4.1. *Let H a non-empty subset of R. If for $x \in H$ with $x \circledast 0 \leqslant y \circledast 0 \Rightarrow y \in H$, then H is a metric ideal of R.*

Remark 4.1. Every ideal of R is a metric ideal, but the converse is not true in general (see the following example).

example 4.1. *Let $R = \{0, a, b, c, d, 1\}$, then the operations '+' and '·' on R is defined as follows:*

$+$	0	a	b	c	d	1
0	0	a	b	c	d	1
a	a	0	a	d	c	d
b	b	a	0	1	d	c
c	c	d	1	0	a	b
d	d	c	d	a	0	a
1	1	d	c	b	a	0

$\cdot$	0	a	b	c	d	1
0	0	0	0	0	0	0
a	0	0	0	0	0	a
b	0	b	b	0	a	b
c	0	0	0	c	c	c
d	0	0	a	c	c	d
1	0	a	b	c	d	1

We see that $R = (R, +, \cdot, 1)$ is a pseudo ring. Clearly all the ideals of R are metric ideal and the set $H = \{0, a\}$ is an example of metric ideal but not an ideal of R.

PROPOSITION 4.2. *A metric ideal H become an ideal of R,if for every $x, y \in H \Rightarrow (x \circledast y) \circledast 0 \in H$.*

PROOF.
i. Clearly $0 \in H$ as H is metric ideal.
ii. Let $x, y \in H$.
Consider $1+(x+1)(y+1) = 1+\delta(x,1)\cdot\delta(y,1) = \delta(1,0)(1+\delta(x,1)\delta(y,1)) = (x \circledast y) \circledast 0 \in H$.
iii. Let $x \in H$ and $y \leqslant x \Leftrightarrow x+1 \leqslant y+1 \Leftrightarrow (x+1)(0+1) \leqslant (y+1)(0+1) \Leftrightarrow x \circledast 0 \leqslant y \circledast 0$ and $x \in H \Rightarrow y \in H$

□

PROPOSITION 4.3. *Let R_1 and R_2 be pseudo ring , φ is a homomorphism from R_1 to R_2 and H is metric ideal of R_1 and K is metric ideal of R_2. Then following holds:*

i. *$\varphi(H)$ is a metric ideal if φ is bi-jective,*
ii. *$\varphi^{-1}(K)$ is metric ideal of R_1 if φ is onto.*

PROOF.
i. Suppose φ is surjective. Let $x \in \varphi(H)$ with $x \circledast 0 \leqslant y \circledast 0$. Since $y \in R_2$ there is t in R_1 such that $\varphi(t) = y$ and $a \in H \Rightarrow \varphi(a) = x \Rightarrow \varphi(a) \circledast 0 \leqslant \varphi(t) \circledast 0 \Rightarrow \varphi(a)+1 \leqslant \varphi(t)+1 \Rightarrow (\varphi(a)+1)\varphi(t) = 0 \Rightarrow \varphi((a+1)t) = 0$. Since φ is bi-jective $(a+1)t = 0 \Rightarrow a+1 \leqslant t+1 \Rightarrow a \circledast 0 \leqslant t \circledast 0 \Rightarrow t \in H \Rightarrow y \in \varphi(H)$.
ii. Suppose $a \in \varphi^{-1}(K)$ with $a \circledast 0 \leqslant b \circledast 0 \Rightarrow a+1 \leqslant b+1 \Rightarrow b \leqslant a$ and $\varphi(a) \in K \Rightarrow \varphi(b) \leqslant \varphi(a) \Rightarrow \varphi(a) \circledast 0 \leqslant \varphi(b) \circledast 0 \Rightarrow \varphi(b) \in K \Rightarrow b \in \varphi^{-1}(K)$.

□

For any non-empty subset X of R, the intersection of all metric ideal containing X denoted by $[X\rangle$ is called a metric ideal generated by X.

THEOREM 4.1. *Let X is non-empty subset of R, then $[X\rangle = \{y \in R : a \circledast 0 \leqslant y \circledast 0$ for some $a \in X\}$ is the metric ideal generated by X.*

PROOF. Let $W = \{y \in R : a \circledast 0 \leqslant y \circledast 0$ for some $a \in X\}$. Let $x \in X \Rightarrow x \leqslant x \Rightarrow x \circledast 0 \leqslant x \circledast 0 \Rightarrow X \subseteq W$.

Let $b \in W$ with $b \circledast 0 \leqslant y \circledast 0$. This implies that there is $x \in X$ such that $x \circledast 0 \leqslant b \circledast 0 \leqslant y \circledast 0 \Rightarrow y \in W$. Hence W is a metric ideal of R. To show that W is the smallest metric ideal containing X. Assume that K is any metric ideal containing X. Let $x \in W$. This implies there exists $a \in X$ such that $a \circledast 0 \leqslant x \circledast 0$. Since $X \subseteq K \Rightarrow a \in K$ with $a \circledast 0 \leqslant x \circledast 0 \Rightarrow x \in K \Rightarrow W \subseteq K$. Therefore W is the smallest metric ideal containing X, so $W = [X\rangle$. □

Corollary 4.1. *Let H be a metric ideal of R and $x \notin H$, then $[H \cup \{x\}\rangle = H \cup \{y \in R : x \circledast 0 \leqslant y \circledast 0\}$*

Observe that $[x\rangle = [\{x\}\rangle = \{y \in R : x \circledast 0 \leqslant y \circledast 0\}$ for any $x \in R$ is a metric ideal of R and if $x = 1$, then $[1\rangle = \{y \in R : 1 \circledast 0 \leqslant y \circledast 0\} = R$. Note that $\{0\}$ and R are trivial metric ideal of R.The following property represent the relation between an ideal generated by X and metric ideal generated by X.

Proposition 4.4. *Let X be any non-empty subset of R.Then $[X\rangle \subseteq \prec X \succ$ where $[X\rangle$ is metric ideal generated by X and $\prec X \succ$ is a ideal generated by X.*

Proof. Let $b \in [X\rangle$ implies there exist $a \in X$ such that $a \circledast 0 \leqslant b \circledast 0 \Rightarrow b \leqslant a \in X \subseteq \prec X \succ \Rightarrow b \in \prec X \succ$. □

The reverse direction of inclusion is not true in general.

example 4.2. *In Example 4.1 H is a metric ideal but not ideal ,the smallest ideal generated by H in R is $\prec H \succ = \{0, a, b\} \Rightarrow b \notin H = \{0, a\} = [H\rangle \Rightarrow \prec H \succ \nsubseteq [H\rangle$.*

Lemma 4.1. *If H and K are metric ideals, then so is $H \cup K$.*

Proof. Let $x \in H \cup K$ with $x \circledast 0 \leqslant y \circledast 0$, for some $y \in R$. It follows that $x \in H$ or $x \in K$ implies $y \in H$ or $y \in K$. Thus $y \in H \cup K$. □

example 4.3. *In Example 4.1 $I = \{0, c\}$ and $H = \{0, a\}$ are metric ideals of R, so $K = I \cup H = \{0, a, c\}$ is a metric ideal.*

Observe that K in Example 4.3 is not an ideal of R of Example 4.1. Because $1 + (a + 1)(c + 1) = 1 + d \cdot b = 1 + a = d \notin K$

Theorem 4.2. *Let $\mathfrak{I}(R)$ be set of all metric ideal of R, then*

i. *$\mathfrak{I}(R) \cup \{\phi\}$ is a topology on R.*
ii. *$(\mathfrak{I}(R), \cup, \cap)$ is a bounded distributive complete lattice.*

Proof. Let $\mathfrak{I}(R)$ be set of all metric ideal of R,

i. Clearly $\phi, R \in \tau$. Let $x \in \bigcup_{\alpha \in \Delta} A_\alpha$ with $x \circledast 0 \leqslant y \circledast 0$.This implies, there is $A_{\alpha_0} \in \{A_\alpha\}_{\alpha \in \Delta}$ such that $x \in A_{\alpha_0} \subseteq \bigcup_{\alpha \in \Delta} A_\alpha \Rightarrow \bigcup_{\alpha \in \Delta} A_\alpha$ is a metric ideal of R. It is straightforward that the finite intersection of metric ideal of R is a metric ideal.
ii. Clearly $\mathfrak{I}(R)$ with the specified operations ($\cap$ and $\cup$) is a lattice. From Lemma 4.1, we observe that every subset of $\mathfrak{I}(R)$ has a least upper bound. Let $A, B \in \mathfrak{I}(R)$. If $x \in A \cap B$ with $x \circledast 0 \leqslant y \circledast 0$ for some $y \in R$, then $x \in A$ and $x \in B$. It follows that $y \in A \cap B$. Hence $A \cap B \in \mathfrak{I}(R)$. Thus every subset of $\mathfrak{I}(R)$ has a greatest lower bound.Thus R is the top and

$\{0\}$ is the bottom element of $\Im(R)$. It is straightforward that for every $A,\ B,\ C \in \Im(R)$,

$$A \cap (B \cup C) = (A \cap B) \cup (A \cap C).$$

Therefore $\Im(R)$ is a complete distributive lattice.

□

THEOREM 4.3. *Let H be a metric ideal of R,then for any $x \in H, \Rightarrow x^n \in H$ for all $n \in \mathbb{Z}^+$.*

PROOF. Let H be a metric ideal of R and suppose $x \in H$. We know that $x^n \leqslant x \Rightarrow x + 1 \leqslant x^n + 1 \Rightarrow x \circledast 0 \leqslant x^n \circledast 0$ for all $n \in \mathbb{Z}^+$. □

The converse of Theorem 4.3 does not hold in general but the following proposition will give when the converse becomes true.

PROPOSITION 4.5. *Let $H \subseteq R$ with $0 \in H$. Then H is metric ideal of R if the following conditions are satisfied*

i. *Every element of H is of finite order.*
ii. *For $x \in H$ with $x \circledast 0 \leqslant y \circledast 0$, there is $z \in H$ such that $x \circledast 0 \leqslant y \circledast 0 \leqslant z \circledast 0 \Rightarrow y \in H$.*

PROOF. suppose $H \subseteq R$ with $x, 0 \in H \Rightarrow x^n = 0$ for some $n \in \mathbb{Z}^+$. Let $x \circledast 0 \leqslant y \circledast 0 \Rightarrow y \leqslant x \Rightarrow y^n \leqslant x^n \Rightarrow y^n = 0 \in H \Rightarrow x \circledast 0 \leqslant y \circledast 0 \leqslant y^n \circledast 0 \Rightarrow y \in H$. □

THEOREM 4.4. *If N is set of all nilpotent elements of R, then N is a metric ideal.*

PROOF. Let N be set of all nilpotent elements of R. Let $x \in N$ with $x \circledast 0 \leqslant y \circledast 0 \Rightarrow y \leqslant x \Rightarrow y^n \leqslant x^n$ for all $n \in \mathbb{Z}^+ \Rightarrow y^n = 0 \Rightarrow y \in N$. □

From Remark 4.1 and Theorem 3.7 we conclude that every stable ideal is a metric ideal and every extended ideal is metric ideal, but the converse is not true in general. We can note that the ideal $\{0\}$ is neither stable nor extended ideal but it is a metric ideal.

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