BULLETIN OF THE INTERNATIONAL MATHEMATICAL VIRTUAL INSTITUTE
ISSN (p) 2303-4874, ISSN (o) 2303-4955
www.imvibl.org /JOURNALS / BULLETIN
Bull. Int. Math. Virtual Inst., **13**(3)(2023), 517–527
DOI: 10.7251/BIMVI2303517A

Former
BULLETIN OF THE SOCIETY OF MATHEMATICIANS BANJA LUKA
ISSN 0354-5792 (o), ISSN 1986-521X (p)

F-KENMOTSU MANIFOLDS WITH GENERALIZED SYMMETRIC METRIC CONNECTION

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Abstract. The motto of the present paper is to study the generalized symmetric metric connection on f-Kenmotsu manifolds. We discuss the properties of f-Kenmotsu manifolds with generalized symmetric metric connection. Further, we investigate conservative pseudo-projective curvature tensor and conservative quasi-conformal curvature tensor on f-Kenmotsu manifolds.

1. Introduction

The concept of almost contact metric manifolds was introduced and studied by Kenmotsu in 1972 [**10**]. This concept is conceded as Kenmotsu manifolds. The concept of f-Kenmotsu manifold (f-KM) (almost contact metric manifold which is normal and locally conformal almost cosymplectic) was studied by Olszaka and Rosca [**19**]. Continuation of their study proposes a geometric representation of f-KM.

A linear connection on a (semi-)Riemannian manifold M is a generalized symmetric connection if its torsion tensor T is presented as follows:

$$T(Y_1, Y_2) = \alpha[u(Y_2)Y_1 - u(Y_1)Y_2] + \beta[u(Y_2)\varphi Y_1 - u(Y_1)\varphi Y_2], \tag{1.1}$$

for every vector fields $Y_1, Y_2 \in M$, where φ is viewed as a tensor of type (1,1), α and β are smooth functions on M and u is a 1-form connected with the vector field which has a non-vanishing smooth non-null unit. Also, the connection is a generalized

2010 *Mathematics Subject Classification.* Primary 53D10.
Key words and phrases. f-Kenmotsu manifolds, generalized symmetric metric connection, pseudo-projective curvature tensor, conservative quasi-conformal curvature tensor.
Communicated by Dusko Bogdanic.

metric when a Riemannian metric g in M is available as $\bar{\nabla} g = 0$; otherwise, it is non-metric. We say that

(1) β-quarter-symmetric connection if $\alpha = 0$, $\beta \neq 0$;
(2) α-semi-symmetric connection if $\alpha \neq 0$, $\beta = 0$.

The generalized symmetric connection reduces to a semi-symmetric and quarter-symmetric, respectively, when $(\alpha, \beta) = (1, 0)$ and $(\alpha, \beta) = (0, 1)$. Thus, a generalizing semi-symmetric and quarter-symmetric connections paves the way for a generalized symmetric metric connection (GSMC). These two connections are of extreme implication for the research of geometry and applications in physics. For instance, Pahan et al. [**20**] have investigated the generalized Robertson-Walker space-time with respect to a quarter-symmetric connection. Many authors have investigated the geometrical and physical aspects of different spaces (see [**4, 6, 7, 9, 11**–**13, 21, 24, 41**]). Some related developments can be found in [**1**–**3, 5, 8, 16**–**18, 22, 25**–**39**].

2. Preliminaries

Let M be a differentiable manifold of $(2n+1)$-dimension bestowed with a $(1,1)$ tensor field φ, a contravariant vector field ξ, a 1-form η and f-KM with GSMC g, which satisfies

$$\varphi^2 Y_1 = -Y_1 + \eta(Y_1)\xi, \quad \eta(\xi) = 1, \tag{2.1}$$

$$g(\varphi Y_1, \varphi Y_2) = g(Y_1, Y_2) - \eta(Y_1)\eta(Y_2), \tag{2.2}$$

$$\nabla_{Y_1}\xi = fY_1 - \eta(Y_1)\xi, \quad (\nabla_{Y_1}\varphi)(Y_1) = fg(\varphi Y_1, Y_2)\xi - \eta(Y_2)\varphi Y_1, \tag{2.3}$$

for every vector fields $Y_1, Y_2 \in M$, where ∇ is the Levi-Civita connection with respect to the GSMC g. Such manifolds $(M, \varphi, \xi, \eta, g)$ is called an f-KM with GSMC (See [**14, 15**]).

The following are provided for an f-KM with GSMC.

$$\varphi\xi = 0, \quad \eta(\varphi Y_1) = 0, rank\varphi = n - 1. \tag{2.4}$$

If we take $\Phi(Y_1, Y_2) = g(\varphi Y_1, Y_2)$ for every vector fields $Y_1, Y_2 \in M$, then the Φ is a symmetric $(0,2)$ tensor field [**14**]. Therefore, if η is closed on an f-KM with GSMC then we compel

$$(\nabla_{Y_1}\eta)Y_2 = \Phi(Y_1, Y_2), \Phi(Y_1, \xi) = 0, \tag{2.5}$$

for all vector fields Y_1, Y_2 on M (See [**14**], [**40**]).

An f-KM provides the following relations [**19**]:

$$\begin{aligned} R(Y_1, Y_2)Y_3 = &\left(\frac{r}{2} + 2f^2 + 2f'\right)(Y_1 \wedge Y_2)Y_3 \\ &- \left(\frac{r}{2} + 3f^2 + 3f'\right)[\eta(Y_1)(\xi \wedge Y_2)Y_3 + \eta(Y_2)(Y_1 \wedge \xi)Y_3], \end{aligned} \tag{2.6}$$

$$R(\xi, Y_2)Y_3 = -(f^2 + f')[g(Y_2, Y_3)\xi - \eta(Y_3)Y_2], \tag{2.7}$$

$$R(Y_1, Y_2)\xi = -(f^2 + f')[\eta(Y_2)Y_1 - \eta(Y_1)Y_2], \tag{2.8}$$

$$S(Y_1, \xi) = -2(f^2 + f')\eta(Y_1). \tag{2.9}$$

Since ξ is a killing vector and S, r remain invariant under it, we have

$$L_\xi \bar{S} = 0 \tag{2.10}$$

and

$$L_\xi \bar{r} = 0. \tag{2.11}$$

By Weingarten and Gauss formulae, we have

$$\nabla_{Y_1} Y_2 = \nabla'_{Y_1} Y_2 + h(Y_1, Y_2) \ , \ \forall Y_1, Y_2 \in \Gamma(TM'), \tag{2.12}$$

and

$$\nabla_{Y_1} N = -A_N Y_1 + \nabla_{\frac{1}{Y_1}} N \ , \ \forall N \in \Gamma(T^\perp M'). \tag{2.13}$$

3. Generalized symmetric metric connection on f-Kenmotsu manifolds

In sight of $\bar{\nabla}$ as a linear connection and ∇ as a Levi-Civita connection of f-KM with a GSMC in such a way that

$$\bar{\nabla}_{Y_1} Y_2 = \nabla_{Y_1} Y_2 + H(Y_1, Y_2), \tag{3.1}$$

for all vector fields Y_1, Y_2. The succeeding is $\bar{\nabla}$ is generalized symmetric connection of ∇ and H is noticed as a tensor of type $(1, 2)$:

$$H(Y_1, Y_2) = \frac{1}{2}[T(Y_1, Y_2) + T'(Y_1, Y_2) + T'(Y_2, Y_1)], \tag{3.2}$$

where T is noticed as the torsion tensor of $\bar{\nabla}$ and

$$g(T'(Y_1, Y_2), W_1) = g(T(W_1, Y_1), Y_2). \tag{3.3}$$

Owing to (1.1) and (3.3), it yields

$$T'(Y_1, Y_2) = \alpha[\eta(Y_1)Y_2 - g(Y_1, Y_2)\xi] - \beta[\eta(Y_1)\varphi Y_2 - g(\varphi Y_1, Y_2)\xi]. \tag{3.4}$$

Employing (1.1), (3.2) and (3.4), we get

$$H(Y_1, Y_2) = \alpha \{\eta(Y_2)Y_1 - g(Y_1, Y_2)\xi\} - \beta\eta(Y_1)\varphi Y_2. \tag{3.5}$$

We have the following result:

LEMMA 3.1. *For an f-KM with GSMC, $\overline{\nabla}$ of type (α, β) is specified by*

$$\bar{\nabla}_{Y_1} Y_2 = \nabla_{Y_1} Y_2 + \alpha[\eta(Y_2)Y_1 - g(Y_1, Y_2)\xi] - \beta\eta(Y_1)\varphi Y_2. \tag{3.6}$$

If $(\alpha, \beta) = (1, 0)$ and $(\alpha, \beta) = (0, 1)$ are chosen, the GSMC is diminished to a semi-symmetric metric and a quarter-symmetric metric one as carried in the succeeding:

$$\bar{\nabla}_{Y_1} Y_2 = \nabla_{Y_1} Y_2 + \eta(Y_2)Y_1 - g(Y_1, Y_2)\xi$$
$$\bar{\nabla}_{Y_1} Y_2 = \nabla_{Y_1} Y_2 - \eta(Y_1)\varphi Y_2.$$

From (2.3), (2.5) and (3.6), we have the following result:

Proposition 3.1. *When M is an f-KM with generalized metric connection, we get following relations:*

$$(\bar{\nabla}_{Y_1}\varphi)Y_2 = (f+\alpha)[g(\varphi Y_1, Y_2)\xi - \eta(Y_2)\varphi Y_1],$$
$$\bar{\nabla}_{Y_1}\xi = (f+\alpha)[Y_1 - \eta(Y_1)\xi],$$
$$(\bar{\nabla}_{Y_1}\eta)Y_2 = (f+\alpha)[g(Y_1, Y_2) - \eta(Y_1)\eta(Y_2)],$$

for every $Y_1, Y_2 \in \Gamma(TM)$.

4. Curvature tensor

Consider a $(2n+1)$-dimensional f-KM M. Then the succeeding can interpret the curvature tensor $\bar{R}$ of the generalized metric connection $\bar{\nabla}$ on M.

$$\bar{R}(Y_1, Y_2)Y_3 = \bar{\nabla}_{Y_1}\bar{\nabla}_{Y_2}Y_3 - \bar{\nabla}_{Y_2}\bar{\nabla}_{Y_1}Y_3 - \bar{\nabla}_{[Y_1,Y_2]}Y_3. \tag{4.1}$$

Using Proposition 3.1 through (3.6) and (4.1), we get

$$\begin{aligned}\bar{R}(Y_1, Y_2)Y_3 =& R(Y_1, Y_2)Y_3 + (f+\alpha)\{\alpha[2g(Y_1, Y_3)Y_2 - 2g(Y_2, Y_3)Y_1 \\ & - \eta(Y_1)\eta(Y_3)Y_2 + \eta(Y_2)\eta(Y_3)Y_1 - \eta(Y_2)g(Y_1, Y_3)\xi \\ & + \eta(Y_1)g(Y_2, Y_3)\xi] - \beta[g(\varphi Y_1, Y_3)\eta(Y_2)\xi \\ & - \eta(Y_2)\eta(Y_3)\varphi Y_1 - g(\varphi Y_2, Y_3)\eta(Y_1)\xi + \eta(Y_1)\eta(Y_3)\varphi Y_2]\} \\ & + \alpha^2[g(Y_2, Y_3)Y_1 - g(Y_1, Y_3)Y_2]. \end{aligned} \tag{4.2}$$

Thus, we have the following result:

Lemma 4.1. *For a a $(2n+1)$-dimensional f-KM M, we have*

(4.3)
$$\left.\begin{aligned} \bar{R}(Y_1, Y_2)\xi &= [(f^2+f') + \alpha f][\eta(Y_1)Y_2 - \eta(Y_2)Y_1] - \beta[\eta(Y_1)\varphi Y_2 - \eta(Y_2)\varphi Y_1], \\ \bar{R}(\xi, Y_2)Y_3 &= -[(f^2+f') + \alpha f][g(Y_2, Y_3)\xi \\ & \qquad -\eta(Y_3)Y_2] + \beta[g(\varphi Y_2, Y_3)\xi - \eta(Y_3)\varphi Y_2], \\ \bar{R}(\xi, Y_2)\xi &= -[(f^2+f') + \alpha f][\eta(Y_2)\xi - Y_2], \end{aligned}\right\}$$

for every $Y_1, Y_2, Y_3 \in \Gamma(TM)$.

By succeeding the Ricci tensor $\bar{S}$ and the scalar curvature $\bar{r}$ of an f-KM is presented with GSMC as follows:

$$\bar{S}(Y_1, Y_2) = \sum_{i=1}^{n} g(\bar{R}(v_i, Y_1)Y_2, v_i),$$
$$\bar{r} = \sum_{i=1}^{n} \bar{S}(v_i, v_i),$$

in which $Y_1, Y_2 \in \Gamma(TM)$ and $\{v_1, v_2, ..., v_n\}$ is viewed as an orthonormal frame. Then by using (2.3) and (4.2), we have

$$\begin{aligned}\bar{S}(Y_2, Y_3) =& S(Y_2, Y_3) + (f+\alpha)[\alpha[(1-4n)g(Y_2, Y_3) + (2n-1)\eta(Y_2)\eta(Y_3)] \\ & + \beta g(\varphi Y_2, Y_3)] + \alpha^2 2ng(Y_2, Y_3). \end{aligned} \tag{4.4}$$

Owing of specific that Ricci tensor S of the Levi-civita connection is symmetric, (4.4) is bestowing:

Corollary 4.1. *If M is an $(2n+1)$-dimensional f-KM is presented with GSMC $\bar{\nabla}$, then Ricci tensor $\bar{S}$ with respect to the GSMC $\bar{\nabla}$ is symmetric.*

Using (2.1), (2.4) and (2.9) in the equation (4.4), we get the following result:

Theorem 4.1. *If M is an $(2n+1)$-dimensional f-Kenmotsu manifold is presented with GSMC $\bar{\nabla}$, then*

$$\bar{S}(Y_2,\xi) = -2[(f^2+f'+n\alpha f)]\eta(Y_2). \tag{4.5}$$

5. f-Kenmotsu manifold with generalized symmetric metric connection satisfying $\bar{R}(Y_1,Y_2)\cdot\bar{S}=0$

Let $\bar{R}(Y_1,Y_2)\cdot\bar{S}=0$ on an $(2n+1)$-dimensional f-Kenmotsu manifold with GSMC M, for any $Y_1,Y_2,Y_3,X_4\in\Gamma(TM)$. Then we have

$$\bar{S}(\bar{R}(Y_1,Y_2)Y_3,X_4)+\bar{S}(Y_3,\bar{R}(Y_1,Y_2)X_4)=0. \tag{5.1}$$

Setting $Y_3=\xi$ and $Y_1=\xi$ in (5.1), we have

$$\bar{S}(\bar{R}(\xi,Y_2)\xi,X_4)+\bar{S}(\xi,\bar{R}(\xi,Y_2)X_4)=0. \tag{5.2}$$

Making use of (4.3) and (4.5) in (5.2), it yields

$$\begin{aligned}&[(f^2+f')+\alpha f]\bar{S}(Y_2,X_4)-\beta\bar{S}(\varphi Y_2,X_4)\\&=2\beta[(f^2+f')+n\alpha f]g(\varphi Y_2,X_4)-[(f^2+f')+n\alpha f]^2 g(Y_2,X_4)\end{aligned} \tag{5.3}$$

Putting $Y=\varphi Y$ in (5.3) and making use of (4.5), we obtain

$$\begin{aligned}&[(f^2+f')+\alpha f]\bar{S}(\varphi Y_2,X_4)-\beta\bar{S}(Y_2,X_4)+2\beta[(f^2+f')+n\alpha f]\eta(Y_2)\eta(X_4)\\&=2\beta[(f^2+f')+n\alpha f][g(Y_2,X_4)+\eta(Y_2)\eta(X_4)]\\&-2[(f^2+f')+n\alpha f]g(\varphi Y_2,\varphi X_4)\end{aligned} \tag{5.4}$$

From (5.3) and (5.4), we obtain

$$(1-\beta^2)\bar{S}(Y_2,X_4)=2[(f^2+f')+n\alpha f][\beta-(f^2+f')+n\alpha f]g(Y_2,X_4) \tag{5.5}$$

Consequently, for $\alpha\neq 0,\beta=0$ and $\alpha=0,\beta\neq 0,1$, we arrive at the following:

$$\bar{S}(Y_2,X_4)=-2[(f^2+f')+n\alpha f]^2 g(Y_2,X_4), \tag{5.6}$$

and

$$(1-\beta^2)\bar{S}(Y_2,X_4)=2(f^2+f')[\beta-(f^2+f')]g(Y_2,X_4). \tag{5.7}$$

Thus, we have the following result:

Theorem 5.1. *Let M be an $(2n+1)$-dimensional f-KM endowed with a GSMC $\bar{\nabla}$. If M is Ricci semi-symmetric with respect to $\bar{\nabla}$. Then we have the following:*

1. *M is generalized Einstein manifold with respect to the GSMC of type (α,β).*
2. *M is an Einstein manifold with respect to the GSMC of type $(\alpha,0)$.*

(3) *M is an Einstein manifold with respect to the GSMC of type* $(0, \beta)$, $(\beta \neq 1)$.

6. Conservative pseudo-projective curvature tensor on f-Kenmotsu manifold with generalized symmetric metric connection

The pseudo-projective curvature tensor on an f-KM with GSMC $\bar{P}$ on a manifold of dimension $(2n+1)$ is defined by [**23**]

$$\begin{aligned}\bar{P}(Y_1, Y_2)Y_3 =& a\bar{R}(Y_1, Y_2)Y_3 + b[\bar{S}(Y_2, Y_3)Y_1 - \bar{S}(Y_1, Y_3)Y_2] \\ & - \frac{\bar{r}}{2n+1}(\frac{a}{2n} + b)[g(Y_2, Y_3)Y_1 - g(Y_1, Y_3)Y_2]. \end{aligned} \tag{6.1}$$

Taking covariant derivative of (6.1), we get

$$\begin{aligned}(\nabla_{W_1}\bar{P})(Y_1, Y_2)Y_3 =& a(\nabla_{W_1}\bar{R})(Y_1, Y_2)Y_3 + b[(\nabla_{W_1}\bar{S})(Y_2, Y_3)Y_1 \\ & - (\nabla_{W_1}\bar{S})(Y_1, Y_3)Y_2] \\ & - \frac{d\bar{r}(W_1)}{2n+1}\left(\frac{a}{2n} + b\right)[g(Y_2, Y_3)Y_1 - g(Y_1, Y_3)Y_2]. \end{aligned} \tag{6.2}$$

Contraction of (6.2), we have

$$\begin{aligned}(div\bar{P})(Y_1, Y_2)Y_3 =& a(div\bar{R})(Y_1, Y_2)Y_3 + b[(\nabla_{Y_1}\bar{S})(Y_2, Y_3) - (\nabla_{Y_2}\bar{S})(Y_1, Y_3)] \\ & - \left(\frac{a+2nb}{(2n+1)2n}\right)[g(Y_2, Y_3)d\bar{r}(Y_1) - g(Y_1, Y_3)d\bar{r}(Y_2)]. \end{aligned} \tag{6.3}$$

If the pseudo-projective curvature tensor is conservative f-Kenmotsu manifold with GSMC i.e., $div\bar{P} = 0$, then (6.3) becomes

$$\begin{aligned}&(a+b)[(\nabla_{Y_1}\bar{S})(Y_2, Y_3) - (\nabla_{Y_2}\bar{S})(Y_1, Y_3)] \\ &= \left(\frac{a+2nb}{(2n+1)2n}\right)[g(Y_2, Y_3)d\bar{r}(Y_1) - g(Y_1, Y_3)d\bar{r}(Y_2)]. \end{aligned} \tag{6.4}$$

From (2.10) and (2.11), we get, respectively

$$(\nabla_{\xi}\bar{S})(Y_2, Y_3) = -\bar{S}(\nabla_{Y_2}\xi, Y_3) - \bar{S}(Y_2, \nabla_{Y_3}\xi) \quad \text{and} \quad d\bar{r}(\xi) = 0. \tag{6.5}$$

Putting $Y_1 = \xi$ in (6.4) and using (6.5), we get

$$\bar{S}(Y_2, \nabla_{Y_3}\xi) + \nabla_{Y_2}\bar{S}(\xi, Y_3) - \bar{S}(\xi, \nabla_{Y_2}Y_3) = \left(\frac{a+2nb}{(2n+1)2n}\right)[\eta(Y_3)d\bar{r}(Y_2)], \tag{6.6}$$

which by virtue of (2.1), (2.4), (2.3) and (4.5) reduces to

$$\begin{aligned}&(a+b)\{f[\bar{S}(Y_2, Y_3) + 2[(f^2 + f') + n\alpha f]\eta(Y_2)\eta(Y_3)] \\ &\quad - 2[(f^2 + f') + n\alpha f][\eta(\nabla_{Y_2}Y_3) - \nabla_{Y_2}\eta(Y_3)] \\ &= \left(\frac{a+2nb}{(2n+1)2n}\right)\eta(Y_3)d\bar{r}(Y_2)\}. \end{aligned} \tag{6.7}$$

Putting $Y_3 = \varphi Y_3$ in (6.7) and using (2.1), we get

$$f(a+b)\{\bar{S}(Y_2, \varphi Y_3) - 2[(f^2 + f') + n\alpha f]g(\varphi Y_2, Y_3)\} = 0. \tag{6.8}$$

Setting $Y_3 = \varphi Y_3$ in (6.8) and using (2.1), we have

$$\bar{S}(Y_2, Y_3) = -2[(f^2 + f') + n\alpha f]g(Y_2, Y_3), \quad f(a+b) \neq 0. \tag{6.9}$$

From contraction of (6.9), we get

$$\bar{r} = -2(2n+1)[(f^2 + f') + n\alpha f]. \tag{6.10}$$

By using (6.9) and (6.10) in (6.1), we get

$$\bar{P}(Y_1, Y_2)Y_3 = a\bar{R}(Y_1, Y_2)Y_3 + \frac{a}{n}[(f^2+f')+n\alpha f][g(Y_2, Y_3)Y_1 - g(Y_1, Y_3)Y_2]. \tag{6.11}$$

Taking covariant derivative and contraction yields

$$(div\bar{P})(Y_1, Y_2)Y_3 = a(div\bar{R})(Y_1, Y_2)Y_3. \tag{6.12}$$

But $(div\bar{P}) = 0$ and so the above equation takes the form

$$(div\bar{R})(Y_1, Y_2)Y_3 = 0 \quad if \quad a \neq 0. \tag{6.13}$$

Hence we have the following theorem:

THEOREM 6.1. *Suppose that pseudo-projective curvature tensor in f-KM M with GSMC is conservative. Then*

(1) *M is Einstein space and hence of constant Sclr.curvtr if $a + b \neq 0$,*
(2) *M is $(div\bar{R}) = 0$ if $a \neq 0$. ie., curvature tensor is conservative.*

7. Conservative quasi-conformal curvature tensor on f-Kenmotsu manifold with generalized symmetric metric connection

The Quasi-Conformal Curvature tensor on f-KM with GSMC $\bar{C}$ on a manifold of $(2n+1)$-dimension (see [**23**]) is defined by

$$\begin{aligned}\bar{C}(Y_1, Y_2)Y_3 = {} & a\bar{R}(Y_1, Y_2)Y_3 + b[\bar{S}(Y_2, Y_3)Y_1 - \bar{S}(Y_1, Y_3)Y_2 \\ & + g(Y_2, Y_3)\bar{Q}Y_1 - g(Y_1, Y_3)\bar{Q}Y_2] \\ & - \frac{\bar{r}}{2n+1}\left(\frac{a}{2n} + 2b\right)[g(Y_2, Y_3)Y_1 - g(Y_1, Y_3)Y_2].\end{aligned} \tag{7.1}$$

Taking covariant derivative of (7.1), we get

$$\begin{aligned}[(\nabla_{W_1}\bar{C})(Y_1, Y_2)Y_3, X_4] = {} & a[(\nabla_{W_1}\bar{R})(Y_1, Y_2)Y_3, X_4] + b[(\nabla_{W_1}\bar{S})(Y_2, Y_3)g(Y_1, X_4) \\ & - (\nabla_{W_1}\bar{S})(Y_1, Y_3)g(Y_2, X_4)] + g(Y_2, Y_3)(\nabla_{W_1}\bar{S})(Y_1, X_4) \\ & - g(Y_1, Y_3)(\nabla_{W_1}\bar{S})(Y_2, X_4) \\ & - \frac{d\bar{r}(W_1)}{2n+1}\left(\frac{a}{2n} + 2b\right)[g(Y_2, Y_3)g(Y_1, X_4) \\ & - g(Y_1, Y_3)g(Y_2, X_4)].\end{aligned} \tag{7.2}$$

Contraction of (7.2), we obtain

$$\begin{aligned}(div\bar{P})(Y_1, Y_2)Y_3 = {} & a(div\bar{R})(Y_1, Y_2)Y_3 + b[(\nabla_{Y_1}\bar{S})(Y_2, Y_3) - (\nabla_{Y_2}\bar{S})(Y_1, Y_3) \\ & + g(Y_2, Y_3)d\bar{r}(Y_1) - g(Y_1, Y_3)d\bar{r}(Y_2)] \\ & - \left(\frac{a + 4nb}{(2n+1)2n}\right)[g(Y_2, Y_3)d\bar{r}(Y_1) - g(Y_1, Y_3)d\bar{r}(Y_2)].\end{aligned} \tag{7.3}$$

If the quasi-conformal curvature tensor is conservative i.e., $div\bar{C} = 0$, then (7.3) becomes

$$\begin{aligned}(a+b)[(\nabla_{Y_1}\bar{S})(Y_2,Y_3) - (\nabla_{Y_2}\bar{S})(Y_1,Y_3)]\\ = \left(b - \frac{a+4nb}{(2n+1)2n}\right)[g(Y_2,Y_3)d\bar{r}(Y_1) - g(Y_1,Y_3)d\bar{r}(Y_2)].\end{aligned} \tag{7.4}$$

From (2.10) and (2.11), we get respectively

$$(\nabla_{\xi}\bar{S})(Y_2,Y_3) = -\bar{S}(\nabla_{Y_2}\xi, Y_3) - \bar{S}(Y_2, \nabla_{Y_3}\xi) \quad \text{and} \quad d\bar{r}(\xi) = 0. \tag{7.5}$$

Putting $Y_1 = \xi$ in (7.4) and using (7.5), we get

$$\bar{S}(Y_2,\nabla_{Y_3}\xi) + \nabla_{Y_2}\bar{S}(\xi,Y_3) - \bar{S}(\xi,\nabla_{Y_2}Y_3) = \left(\frac{a+2nb}{(2n+1)2n}\right)[\eta(Y_3)d\bar{r}(Y_2)], \tag{7.6}$$

which by virtue of (2.1),(2.3) and (4.5) reduces to

$$\begin{aligned}(a+b)\{f[\bar{S}(Y_2,Y_3) + 2[(f^2+f') + n\alpha f]\eta(Y_2)\eta(Y_3)]\\ - 2[(f^2+f') + n\alpha f][\eta(\nabla_{Y_2}Y_3) - \nabla_{Y_2}\eta(Y_3)]\\ = \left(b - \frac{a+4nb}{(2n+1)2n}\right)\eta(Y_3)d\bar{r}(Y_2)\}.\end{aligned} \tag{7.7}$$

Putting $Y_3 = \varphi Y_3$ in (7.7) and using (2.1), we have

$$f(a+b)\{\bar{S}(Y_2,\varphi Y_3) + 2[(f^2+f') + n\alpha f]g(Y_2,\varphi Y_3)\} = 0. \tag{7.8}$$

Setting $Y_3 = \varphi Y_3$ in (7.8) and using (2.1), we get

$$\bar{S}(Y_2,Y_3) = -2[(f^2+f') + n\alpha f]g(Y_2,Y_3), \quad f(a+b) \neq 0. \tag{7.9}$$

From contraction of (7.9), we get

$$\bar{r} = -2(2n+1)[(f^2+f') + n\alpha f]. \tag{7.10}$$

By using (7.9) and (7.10) in (7.1), we get

$$\begin{aligned}\bar{P}(Y_1,Y_2)Y_3 = a\bar{R}(Y_1,Y_2)Y_3 + [a + 4b(n-1)][(f^2+f')\\ + n\alpha f][g(Y_2,Y_3)Y_1 - g(Y_1,Y_3)Y_2].\end{aligned} \tag{7.11}$$

Taking the covariant derivative and contracting, we have

$$(div\bar{C})(Y_1,Y_2)Y_3 = a(div\bar{R})(Y_1,Y_2)Y_3. \tag{7.12}$$

But $(div\bar{C}) = 0$ and so the above equation takes the form

$$(div\bar{R})(Y_1,Y_2)Y_3 = 0, \quad \text{if} \quad a \neq 0. \tag{7.13}$$

Hence we can state the following

THEOREM 7.1. *Suppose that the quasi-conformal curvature tensor in f-KM with GSMC M is conservative. Then*

(1) *M is an Einstein space and of constant scalar curvature, if $a + b \neq 0$;*
(2) *M with $(div\bar{R}) = 0$ i.e., curvature tensor is conservative, if $a \neq 0$.*

Acknowledgements

The authors would like to thank the referees for their valuable comments which helped to improve the manuscript.

References

1. P. G. Angadi, G. S. Shivaprasanna, G. Somashekhara and P. Siva Kota Reddy, Ricci-Yamabe Solitons on Submanifolds of Some Indefinite Almost Contact Manifolds, *Adv. Math., Sci. J.*, **9**(11) (2020), 10067–10080.
2. P. G. Angadi, G. S. Shivaprasanna, G. Somashekhara and P. Siva Kota Reddy, Ricci Solitons on (LCS)-Manifolds under D-Homothetic Deformation, *Italian Journal of Pure & Applied Mathematics*, **46** (2021), 672-683.
3. P. G. Angadi, G. S. Shivaprasanna, R. Rajendra, P. Siva Kota Reddy and G. Somashekhara, η-Yamabe soliton on 3-dimensional α-Para Kenmotsu manifold, *Chinese Journal of Mathematical Sciences*, 1(1) (2021), 29–37.
4. P. G. Angadi, P. Siva Kota Reddy, G. S. Shivaprasanna and G. Somashekhara, On Weakly Symmetric Generalized (k, μ)-Space Forms, *Proc. Jangjeon Math. Soc.*, 25(2) (2022), 133-144.
5. P. G. Angadi, R. Rajendra, P. Siva Kota Reddy, G. S. Shivaprasanna and G. Somashekhara, $*$-Ricci Tensor on Generalized Sasakian-Space-Form, *International J. Math. Combin.*, 4 (2023), to appear.
6. S. K. Chaubey and U. C. De, Characterization of the Lorentzian para-Sasakian manifolds admitting a quarter-symmetric non-metric connection, *SUT J. Math.*, 55(1) (2019), 53–67.
7. S. K. Chaubey and U. C. De, Lorentzian para-Sasakian manifolds admitting a new type of quarter-symmetric non-metric ξ-connection, *Int. Electron. J. Geom.*, 12(2) (2019), 250–259.
8. S. Girish Babu, R. Rajendra, P. Siva Kota Reddy and N. Pavani, (ϵ)-Kenmotsu Manifold admitting Schouten-Van Kampen Connection, *Journal of AppliedMath*, 1(2) (2023), 1-7.
9. D. H. Jin, Lightlike hypersurfaces of an indefinite Kaehler manifold with a symmetric metric connection of type (l, m), *Bull. Korean Math. Soc.*, 53(4) (2016), 1171–1184.
10. K. Kenmotsu, A class of almost contact Riemannian manifolds, *Tohoku Math. J. (2)*, 24 (1972), 93–103.
11. J. W. Lee. C. W. Lee and D. W. Yoon, Inequalities for generalized δ-Casorati curvatures of submanifolds in real space forms endowed with a semi-symmetric metric connection, *Rev. Un. Mat. Argentina*, 57(2) (2016), 53–62.
12. C. W. Lee, J. W. Lee, G. E. Vilcu and D. W. Yoon, Optimal inequalities for the Casorati curvatures of submanifolds of generalized space forms endowed with semi-symmetric metric connections, *Bull. Korean Math. Soc.*, 52(5) (2015), 1631–1647.
13. Y. B. Maralabhavi and G. S. Shivaprasanna, On pseudo-projective curvature tensor in LP-Sasakian manifolds, *Int. Math. Forum*, 7(21-24) (2012), 1121–1128.
14. K. Matsumoto, On Lorentzian almost paracontact manifolds, *Bull. Yamagata Univ. Natur. Soc.*, 12 (1989), 151-156.
15. I. Mihai and R. Rosca, On Lorentzian P-Sasakian manifolds, In Classical analysis (Kazimierz Dolny, 1991), *World Sci. Publ.*, River Edge, NJ, 1992, 155–169.
16. H. G. Nagaraja, Dipansha Kumari and P. Siva Kota Reddy, Submanifolds of (k, μ)- Contact Metric Manifold as Ricci Solitons, *Proc. Jangjeon Math. Soc.*, 24(1) (2021), 11-19.
17. R. T. Naveen Kumar, P. Siva Kota Reddy, Venkatesha and Khaled A. A. Alloush, $(LCS)_n$-Manifold Endowed With Torseforming Vector Field and Concircular Curvature Tensor, *International J. Math. Combin.*, 3 (2022), 38-47.
18. R. T. Naveen Kumar, P. Siva Kota Reddy, Venkatesha and M. Sangeetha, Certain Results on (k, μ)-Contact Metric Manifold endowed with Concircular Curvature Tensor, *Commun. Math. Appl.*, 14(1) (2023), 215-225.
19. Z. Olszak and R. Rosco, Normal locally conformal almost cosymplectic manifolds, *Publ. Math. Debrecen*, 39(3 & 4) (1991), 315–323.

20. S. Pahan, B. Pal and A. Bhattacharyya, On Einstein warped products with a quarter-symmetric connection, *Int. J. Geom. Methods Mod. Phys.*, 14(4) (2017), Article Id 1750050, 15 Pages.
21. S. Pahan, B. Pal and A. Bhattacharyya, On Ricci flat warped products with a quarter-symmetric connection, *J. Geom.*, 107(3) (2016), 627–634.
22. B. Phalaksha Murthy, R. T. Naveen Kumar, P. Siva Kota Reddy and Venkatesha, On $N(k)$-Contact Metric Manifold endowed with Pseudo-quasi-conformal Curvature Tensor, *Adv. Math., Sci. J.*, 10(4) (2021), 1969–1982.
23. B. Prasad A pseudo projective curvature tensor on a Riemannian manifold, *Bull. Calcutta Math. Soc.*, 94(3) (2002), 163–166.
24. Q. Qu and Y. Wang, Multiply warped products with a quarter-symmetric connection, *J. Math. Anal. Appl.*, 431(2) (2015), 955–987.
25. G. Somashekhara, N. Pavani and P. Siva Kota Reddy, Invariant Sub-manifolds of LP-Sasakian Manifolds with Semi-Symmetric Connection, *Bull. Math. Anal. Appl.*, **12**(2) (2020), 35–44.
26. G. Somashekhara, S. Girish Babu and P. Siva Kota Reddy, C-Bochner Curvature Tensor under D-Homothetic Deformation in LP-Sasakian Manifold, *Bull. Int. Math. Virtual Inst.*, **11**(1) (2021), 91–98.
27. G. Somashekhara, S. Girish Babu and P. Siva Kota Reddy, Indefinite Sasakian Manifold with Quarter-Symmetric Metric Connection, *Proc. Jangjeon Math. Soc.*, **24**(1) (2021), 91–98.
28. G. Somashekhara, P. Siva Kota Reddy, N. Pavani and G. J. Manjula, η-Ricci-Yamabe Solitons on Submanifolds of some Indefinite almost Contact Manifolds, *J. Math. Comput. Sci.*, **11**(3) (2021), 3775–3791.
29. G. Somashekhara, S. Girish Babu and P. Siva Kota Reddy, Conformal Ricci Soliton in an Indefinite Trans-Sasakian manifold, *Vladikavkaz Math. J*, **23**(3) (2021), 43–49.
30. G. Somashekhara, S. Girish Babu and P. Siva Kota Reddy, Ricci Solitons and Generalized Weak Symmetries under D-Homothetically Deformed LP-Sasakian Manifolds, *Italian Journal of Pure & Applied Mathematics*, **46** (2021), 684–695.
31. G. Somashekhara, S. Girish Babu and P. Siva Kota Reddy, Conformal η-Ricci Solitons in Lorentzian Para-Sasakian Manifold Admitting Semi-Symmetric Metric Connection, *Italian Journal of Pure & Applied Mathematics*, **46** (2021), 1008–1019.
32. G. Somashekhara, P. Siva Kota Reddy, K. Shivashankara and N. Pavani, Slant Sub-manifolds of Generalized Sasakian-Space-Forms, *Proc. Jangjeon Math. Soc.*, **25**(1) (2022), 83–88.
33. G. Somashekhara, P. Siva Kota Reddy and N. Pavani, Semi-invariant Submanifolds of Generalized Sasakian-Space-Forms, *International J. Math. Combin.*, **2** (2022), 47-55.
34. G. Somashekhara, S. Girish Babu, P. Siva Kota Reddy and K. Shivashankara, On LP-Sasakian Manifolds admitting Generalized Symmetric Metric Connection, *Proc. Jangjeon Math. Soc.*, **25**(3) (2022), 287–296.
35. G. Somashekhara, R. Rajendra, G. S. Shivaprasanna, P. Siva Kota Reddy, Savithri Shashidhar and Khaled A. A. Alloush, Generalised Sasakian-space-form in Submanifolds, *International J. Math. Combin.*, **3** (2022), 69–81.
36. G. Somashekhara, S. Girish Babu and P. Siva Kota Reddy, η-Ricci soliton in an indefinite trans-Sasakian manifold admitting semi-symmetric metric connection, *Bol. Soc. Parana. Mat. (3)*, **41** (2023), 1–9.
37. G. Somashekhara, R. Rajendra, G. S. Shivaprasanna and P. Siva Kota Reddy, Pseudo Parallel and Generalized Ricci Pseudo Parallel Invariant Submanifolds of a Generalized Sasakian Space Form, *Proc. Jangjeon Math. Soc.*, **26**(1) (2023), 69–78.
38. G. Somashekhara, S. Girish Babu, R. Rajendra, P. Siva Kota Reddy and Khaled A. A. Alloush, Lorentzian β-Kenmotsu Manifolds admitting Semi-symmetric Non-Metric Connection, *Bull. Int. Math. Virtual Inst.*, 13(1) (2023), 51-62.
39. P. Somashekhara, R. T. Naveen Kumar, P. Siva Kota Reddy, Venkatesha and Khaled A. A. Alloush, Pseudo Projective Curvature Tensor on Generalized Sasakian Space Forms, *Proc. Jangjeon Math. Soc.*, 26(3) (2023), 243-251.

40. G. S. Shivaprasanna, Md. Samiul Haque and G. Somashekhara, η-Ricci soliton on f-Kenmotsu manifolds, *Journal of Physics: Conference Series*, 1543(1) (2020), 012007.
41. G. S. Shivaprasanna, Y. B. Maralabhavi and G. Somashekhara, On Semi-Symmetric Metric Connection in a Generalized (k, μ) Space Forms, *International Journal of Mathematics Trends and Technology*, 9(3) (2014), 173-188.

Received by editors 29.7.2023; Revised version 22.12.2023; Available online 31.12.2023.

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