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PADOVAN POLYNOMIALS MATRIX

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Abstract. In this paper, we explore the Padovan numbers and polynomials, and define the Padovan polynomials matrix. We obtain its Binet-like formula and a sum formula. Subsequently, we derive the Padovan polynomials matrix series. Additionally, we establish the generating and exponential generating functions for the Padovan polynomials matrix.

1. Introduction

Special number sequences play an important role in mathematics and the applied sciences. Moreover, some special number sequences, such as Fibonacci, Lucas, Pell, Jacobsthal, Padovan and Perrin sequences have many applications in art, music, photography, architecture, painting, engineering, geometryi, and others. Almost all applications are related to the golden and plastic ratios. It is well-known that the golden ratio is defined as the limit of the ratio of two consecutive Fibonacci numbers, and equals to

$$\frac{1+\sqrt{5}}{2} \approx 1.618034.$$

It finds applications in engineering, physics, architecture, arts, and more. Similarly, the ratio of two consecutive Padovan or Perrin numbers approaches

$$\sqrt[3]{\frac{1}{2}+\frac{1}{6}\sqrt{\frac{23}{3}}}+\sqrt[3]{\frac{1}{2}-\frac{1}{6}\sqrt{\frac{23}{3}}} \approx 1.324718,$$

which is called the plastic ratio. The plastic ratio is discovered by Dom Hans van der Laan (1904-1991) in 1928 shortly after he had abandoned his architectural studies, differs from ratios like the golden ratio in several fundamental ways. He

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described applications to architecture and illustrated the use of the plastic number in many buildings. Furthermore, the plastic number is the unique real root of the characteristic equation of the Padovan numbers sequence

$$t^3 - t - 1 = 0.$$

If its roots are denoted by α, β and γ then the following equalities can be derived

$$\alpha + \beta + \gamma = 0,$$
$$\alpha\beta + \alpha\gamma + \beta\gamma = -1,$$
$$\alpha\beta\gamma = 1.$$

More information for Padovan numbers is available in [**2**–**5**, **17**–**19**, **22**], and for the plastic constant in [**6**, **15**, **24**].

Kılıç and Stanica [**11**] considered k sequences of generalized order$-k$ linear recurrences with arbitrary initial conditions and coefficients, and they gave their generalized Binet formulae and generating functions. They also obtained a new matrix method to derive explicit formulas for the sums of terms of the k sequences. We note that the Padovan sequence and its generalizations can be derived with a generalized $k-$order linear recurrence sequence $G_n - \sum_{j=1}^{k} G_{n-j} = 0$. Here, we only focus on the Padovan numbers.

The Padovan sequence $\{P_n\}_{n\geqslant 0}$ is defined by the third order recurrence

$$P_{n+3} = P_{n+1} + P_n$$

with the initial conditions $P_0 = 1$, $P_1 = 0$ and $P_2 = 1$. The first few values of this sequence are given as follows

n	-2	-1	0	1	2	3	4	5	6	7	8	9	10	11	12	13	...
P_n	0	0	1	0	1	1	1	2	2	3	4	5	7	9	12	16	...

Moreover, the Binet-like formula for the Padovan sequence is

$$P_n = a\alpha^n + b\beta^n + c\gamma^n \tag{1.1}$$

where,

$$a = \frac{\beta\gamma + 1}{(\alpha-\beta)(\alpha-\gamma)}, \quad b = \frac{\alpha\gamma + 1}{(\beta-\alpha)(\beta-\gamma)}, \quad c = \frac{\alpha\beta + 1}{(\gamma-\alpha)(\gamma-\beta)}.$$

From [**17**, **22**], the following relations are valid

$$P_{-n-3} = P_n^2 - P_{n+1}P_{n-1}, \tag{1.2}$$

$$P_n = P_{m-1}P_{n-m} + P_{m+1}P_{n-m+1} + P_m P_{n-m+2}. \tag{1.3}$$

In [**17**], the Padovan numbers have the Q_P-matrix

$$Q_P = \begin{bmatrix} 0 & 1 & 0 \\ 0 & 0 & 1 \\ 1 & 1 & 0 \end{bmatrix}$$

such that

$$Q_P^n = \begin{bmatrix} P_{n-3} & P_{n-1} & P_{n-2} \\ P_{n-2} & P_n & P_{n-1} \\ P_{n-1} & P_{n+1} & P_n \end{bmatrix}.$$

By using (1.2) and (1.3), the determinant of the Padovan matrix above gives an identity as follows

$$P_{n-3}P_{-n-3} + P_{n-1}P_{-n-2} + P_{n-2}P_{-n-1} = 1.$$

That is, the determinant of Q_P^n is 1. It is denoted by $|Q_P^n| = 1$.
The Fibonacci polynomials are considered, and some investigations are given by Hoggatt and Bicknell in [**8**]. Machenry [**14**] provided the generalized Fibonacci and Lucas polynomials and some connections with some multiplicative arithmetic functions.Various studies on Fibonacci polynomials can be found in [**1**, **9**, **20**, **21**].
The Padovan polynomials sequence $\{P_n(x)\}_{n\geqslant 0}$ is defined (see [**7**, **10**, **16**, **23**]) by a third order recurrence

$$P_{n+3}(x) = xP_{n+1}(x) + P_n(x)$$

with the initial conditions $P_0(x) = 1$, $P_1(x) = 0$ and $P_2(x) = x$. To simplify notation, take $P_n(x) = \mathcal{P}_n$. The first few values of this sequence are given as follows

n	-2	-1	0	1	2	3	4	5	6	7	8	$\ldots$
$\mathcal{P}_n$	0	0	1	0	x	1	x^2	$2x$	x^3+1	$3x^2$	x^4+3x	$\ldots$

The recurrence above involves the characteristic equation

$$\mu^3 - x\mu - 1 = 0.$$

If its roots are denoted by α_x, β_x and γ_x then the following equalities can be derived

$$\alpha_x + \beta_x + \gamma_x = 0,$$

$$\alpha_x\beta_x + \alpha_x\gamma_x + \beta_x\gamma_x = -x,$$

$$\alpha_x\beta_x\gamma_x = 1.$$

The Binet-like formula for the Padovan polynomial sequence is

$$\mathcal{P}_n = a_x\alpha_x^n + b_x\beta_x^n + c_x\gamma_x^n \tag{1.4}$$

where,

$$a_x = \frac{\beta_x\gamma_x + x}{(\alpha_x-\beta_x)(\alpha_x-\gamma_x)}, b_x = \frac{\alpha_x\gamma_x + x}{(\beta_x-\alpha_x)(\beta_x-\gamma_x)}, c_x = \frac{\alpha_x\beta_x + x}{(\gamma_x-\alpha_x)(\gamma_x-\beta_x)}.$$

2. Main results

In the present work, we have obtained the following main results.

PROPOSITION 2.1. *Let $\mathcal{P}_n$ be n-th Padovan polynomial. Then,*

$$\mathcal{P}_{-n-3} = \mathcal{P}_n^2 - \mathcal{P}_{n+1}\mathcal{P}_{n-1}. \tag{2.1}$$

PROOF. We establish this using principle of mathematical induction. Since,

$$\mathcal{P}_0^2 - \mathcal{P}_1\mathcal{P}_{-1} = 1 - 0 = \mathcal{P}_{-3}$$
$$\mathcal{P}_1^2 - \mathcal{P}_2\mathcal{P}_0 = 0 - x = \mathcal{P}_{-4}$$
$$\mathcal{P}_2^2 - \mathcal{P}_3\mathcal{P}_1 = x^2 - 0 = \mathcal{P}_{-5}$$

The result is true when $n = 0, 1, 2$. Suppose it is true for all positive integers $n \leqslant k$. Then,

$$\begin{aligned}\mathcal{P}_{k+1}^2 - \mathcal{P}_{k+2}\mathcal{P}_k &= x^2\mathcal{P}_{k-1}^2 + 2x\mathcal{P}_{k-1}\mathcal{P}_{k-2} + \mathcal{P}_{k-2}^2 - x^2\mathcal{P}_k\mathcal{P}_{k-2} - x\mathcal{P}_k\mathcal{P}_{k-3} \\ &\quad - x\mathcal{P}_{k-1}\mathcal{P}_{k-2} - \mathcal{P}_{k-1}\mathcal{P}_{k-3} \\ &= x^2\mathcal{P}_{-k-2} - x\mathcal{P}_{-k} + \mathcal{P}_{-k-1} \\ &= x^2\mathcal{P}_{-k-2} + x\mathcal{P}_{-k-3} - x\mathcal{P}_{-k-3} - x\mathcal{P}_{-k} + \mathcal{P}_{-k-1} \\ &= x\mathcal{P}_{-k} - x\mathcal{P}_{-k-3} - x\mathcal{P}_{-k} + \mathcal{P}_{-k-1} \\ &= \mathcal{P}_{-k-4}.\end{aligned}$$

Thus, by the strong version of principle of mathematical induction, the formula holds for all positive integers $n \geqslant 3$. □

We utilize the following relation, as provided in [**17**].

$$\mathcal{P}_n = \mathcal{P}_{m-1}\mathcal{P}_{n-m} + \mathcal{P}_{m+1}\mathcal{P}_{n-m+1} + \mathcal{P}_m\mathcal{P}_{n-m+2} \tag{2.2}$$

It is well known from [**23**] that the generating function for the Padovan polynomials is

$$\sum_{n=0}^{\infty} \mathcal{P}_n t^n = \frac{1}{1 - xt^2 - t^3}.$$

Now, we establish the exponential generating function for the Padovan polynomials as follows:

THEOREM 2.1. *The exponential generating function for the Padovan polynomials is*

$$E_{\mathcal{P}} = \sum_{n=1}^{\infty} \frac{\mathcal{P}_n}{n!} t^n = a_x e^{\alpha_x t} + b_x e^{\beta_x t} + c_x e^{\gamma_x t}$$

PROOF. We know that

$$e^{\alpha_x t} = \sum_{n=1}^{\infty} \frac{\alpha_x^n}{n!} t^n,$$

$$e^{\beta_x t} = \sum_{n=1}^{\infty} \frac{\beta_x^n}{n!} t^n,$$

$$e^{\gamma_x t} = \sum_{n=1}^{\infty} \frac{\gamma_x^n}{n!} t^n.$$

Let's multiply each side of the first equation by a_x, the second equation by b_x and the third equation by c_x . By adding all equalities above side by side, we obtain the following equality.

$$a_x e^{\alpha_x t} + b_x e^{\beta_x t} + c_x e^{\gamma_x t} = \sum_{n=1}^{\infty} \frac{a_x \alpha_x^n + b_x \beta_x^n + c_x \gamma_x^n}{n!} t^n = \sum_{n=1}^{\infty} \frac{\mathcal{P}_n}{n!} t^n.$$

□

THEOREM 2.2. *The series for the Padovan polynomials is*

$$S_{\mathcal{P}}(x) = \sum_{n=0}^{\infty} \frac{\mathcal{P}_n}{t^{n+1}} = \frac{t^2}{t^3 - xt - 1}.$$

PROOF. Let

$$S_{\mathcal{P}}(x) = \sum_{n=0}^{\infty} \frac{\mathcal{P}_n}{t^{n+1}} = \frac{\mathcal{P}_0}{t} + \frac{\mathcal{P}_1}{t^2} + \frac{\mathcal{P}_2}{t^3} + \cdots + \frac{\mathcal{P}_n}{t^{n+1}} + \ldots$$

be series of the Padovan polynomials. Multiply this function every side by t^3 such as

$$t^3 S_{\mathcal{P}}(x) = \mathcal{P}_0 t^2 + \mathcal{P}_1 t + \mathcal{P}_2 + \cdots + \frac{\mathcal{P}_n}{t^{n-2}} + \ldots$$

and that is multiplied every side with $-xt$ such as

$$-xt S_{\mathcal{P}}(x) = -\mathcal{P}_0 x - \frac{\mathcal{P}_1 x}{t} - \frac{\mathcal{P}_2 x}{t^2} - \cdots - \frac{\mathcal{P}_n x}{t^n} - \ldots$$

and that is multiplied every side with -1 negative such as

$$-S_{\mathcal{P}}(x) = -\frac{\mathcal{P}_0}{t} - \frac{\mathcal{P}_1}{t^2} - \frac{\mathcal{P}_2}{t^3} - \cdots - \frac{\mathcal{P}_n}{t^{n+1}} - \ldots$$

Then, we write

$$\begin{aligned}(t^3 - xt - 1) S_{\mathcal{P}}(x) &= \mathcal{P}_0 t^2 + \mathcal{P}_1 t + \mathcal{P}_2 - \mathcal{P}_0 x + (\mathcal{P}_3 - \mathcal{P}_1 x - \mathcal{P}_0) \tfrac{1}{t} \\ &\quad - \cdots + (\mathcal{P}_{n+2} - \mathcal{P}_n x - \mathcal{P}_{n-1}) \tfrac{1}{t^n} + \ldots\end{aligned}$$

Now using values of the Padovan polynomials and $\mathcal{P}_{n+2} - \mathcal{P}_n x - \mathcal{P}_{n-1} = 0$. So, we obtain

$$S_{\mathcal{P}}(x) = \frac{t^2}{t^3 - xt - 1}.$$

Thus, the proof is completed. □

By employing analogous methods outlined in [**3**], we can demonstrate the following result.

THEOREM 2.3. *The partial sum of the Padovan Polynomials sequence is*

$$\sum_{i=0}^{n} \mathcal{P}_i = \frac{\mathcal{P}_{n+2} + \mathcal{P}_{n+1} + \mathcal{P}_n - 1}{x}, \quad n \geqslant 0.$$

Proof. We know that

$$\mathcal{P}_n = x\mathcal{P}_{n-2} + \mathcal{P}_{n-3}$$

So, applying to the identity above, we deduce that

$$\begin{aligned}
\mathcal{P}_3 &= x\mathcal{P}_1 + \mathcal{P}_0,\\
\mathcal{P}_4 &= x\mathcal{P}_2 + \mathcal{P}_1,\\
\mathcal{P}_5 &= x\mathcal{P}_3 + \mathcal{P}_2,\\
&\dots\\
\mathcal{P}_{n-2} &= x\mathcal{P}_{n-4} + \mathcal{P}_{n-5},\\
\mathcal{P}_{n-1} &= x\mathcal{P}_{n-3} + \mathcal{P}_{n-4},\\
\mathcal{P}_n &= x\mathcal{P}_{n-2} + \mathcal{P}_{n-3}.
\end{aligned}$$

Summing the both of sides of the identities above, we obtain

$$\mathcal{P}_{n-2}+\mathcal{P}_{n-1}+\mathcal{P}_n = x\left(\mathcal{P}_1 + \mathcal{P}_2 + \mathcal{P}_3 + \ldots + \mathcal{P}_{n-4} + \mathcal{P}_{n-3} + \mathcal{P}_{n-2}\right) + \mathcal{P}_0 + \mathcal{P}_1 + \mathcal{P}_2.$$

□

3. Padovan polynomials matrix

We explore the properties of the Padovan polynomials in connection with the Padovan polynomials matrix formula. Thus, we aim to derive new relations for the Padovan polynomials matrices. The Padovan polynomials matrix, denoted as $Q_P(x)$, is generated by a 3rd-order matrix.

$$Q_P(x) = \begin{bmatrix} 0 & 1 & 0 \\ 0 & 0 & 1 \\ 1 & x & 0 \end{bmatrix}$$

and the *n-th* powers of $Q_P(x)$ polynomials matrix is given

$$Q_P^n(x) = \begin{bmatrix} \mathcal{P}_{n-3} & \mathcal{P}_{n-1} & \mathcal{P}_{n-2} \\ \mathcal{P}_{n-2} & \mathcal{P}_n & \mathcal{P}_{n-1} \\ \mathcal{P}_{n-1} & \mathcal{P}_{n+1} & \mathcal{P}_n \end{bmatrix}. \tag{3.1}$$

In the books by Koshy [**12, 13**], various properties of the Fibonacci numbers are presented. In this work, we extend similar properties to the Padovan polynomials.

Proposition 3.1. *The relations are valid:*

1. $|Q_P^3(x) + xQ_P(x) + I| = 8.$
2. *If* $|Q_P(x) - \mu I| = 0$, *then* $\mu = \alpha_x, \beta_x, \gamma_x$
3. $|Q_P^n(x)| = 1.$

Proof.

1.

$$Q_P^3(x) + xQ_P(x) + I = \begin{bmatrix} 1 & x & 0 \\ 0 & 1 & x \\ x & x^2 & 1 \end{bmatrix} + \begin{bmatrix} 0 & x & 0 \\ 0 & 0 & x \\ x & x^2 & 0 \end{bmatrix} + \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix}$$
$$= \begin{bmatrix} 2 & 2x & 0 \\ 0 & 2 & 2x \\ 2x & 2x^2 & 2 \end{bmatrix} = 2\begin{bmatrix} 1 & x & 0 \\ 0 & 1 & x \\ x & x^2 & 1 \end{bmatrix}.$$

So,

$$|Q_P^3(x) + xQ_P(x) + I| = 8$$

2.

$$|Q_P(x) - \mu I| = \begin{vmatrix} -\mu & 1 & 0 \\ 0 & -\mu & 1 \\ 1 & x & -\mu \end{vmatrix} = \mu^3 - x\mu - 1 = 0.$$

3. By using (2.1) and (2.2). We have

$$\begin{vmatrix} \mathcal{P}_{n-3} & \mathcal{P}_{n-1} & \mathcal{P}_{n-2} \\ \mathcal{P}_{n-2} & \mathcal{P}_{n} & \mathcal{P}_{n-1} \\ \mathcal{P}_{n-1} & \mathcal{P}_{n+1} & \mathcal{P}_{n} \end{vmatrix} = \mathcal{P}_{n-3}\mathcal{P}_{-n-3} + \mathcal{P}_{n-1}\mathcal{P}_{-n-2} + \mathcal{P}_{n-2}\mathcal{P}_{-n-1} = 1.$$

□

THEOREM 3.1. *The Binet-like formula for the sequence* $Q_P^n(x)$ *is*

$$Q_P^n(x) = a_x\underline{\alpha_x}\alpha_x^n + b_x\underline{\beta_x}\beta_x^n + c_x\underline{\gamma_x}\gamma_x^n, \qquad n \geqslant 0 \tag{3.2}$$

where

$$\underline{\alpha_x} = \begin{bmatrix} \alpha_x^{-3} & \alpha_x^{-1} & \alpha_x^{-2} \\ \alpha_x^{-2} & 1 & \alpha_x^{-3} \\ \alpha_x^{-1} & \alpha_x & 1 \end{bmatrix}, \ \underline{\beta_x} = \begin{bmatrix} \beta_x^{-3} & \beta_x^{-1} & \beta_x^{-2} \\ \beta_x^{-2} & 1 & \beta_x^{-3} \\ \beta_x^{-1} & \beta_x & 1 \end{bmatrix} \text{ and } \underline{\gamma_x} = \begin{bmatrix} \gamma_x^{-3} & \gamma_x^{-1} & \gamma_x^{-2} \\ \gamma_x^{-2} & 1 & \gamma_x^{-3} \\ \gamma_x^{-1} & \gamma_x & 1 \end{bmatrix}.$$

PROOF. From the definition of *n-th* Padovan polynomials matrix $Q_P^n(x)$ in (3.1) and Binet-like formula for the *n-th* Padovan polynomials $\mathcal{P}_n$ in (1.4), we write

$$Q_P^n(x) = \begin{bmatrix} \mathcal{P}_{n-3} & \mathcal{P}_{n-1} & \mathcal{P}_{n-2} \\ \mathcal{P}_{n-2} & \mathcal{P}_{n} & \mathcal{P}_{n-1} \\ \mathcal{P}_{n-1} & \mathcal{P}_{n+1} & \mathcal{P}_{n} \end{bmatrix}$$

or

$$Q_P^n(x) = a_x\begin{bmatrix} \alpha_x^{-3} & \alpha_x^{-1} & \alpha_x^{-2} \\ \alpha_x^{-2} & 1 & \alpha_x^{-1} \\ \alpha_x^{-1} & \alpha_x & 1 \end{bmatrix}\alpha_x^n + b_x\begin{bmatrix} \beta_x^{-3} & \beta_x^{-1} & \beta_x^{-2} \\ \beta_x^{-2} & 1 & \beta_x^{-3} \\ \beta_x^{-1} & \beta_x & 1 \end{bmatrix}\beta_x^n + c_x\begin{bmatrix} \gamma_x^{-3} & \gamma_x^{-1} & \gamma_x^{-2} \\ \gamma_x^{-2} & 1 & \gamma_x^{-3} \\ \gamma_x^{-1} & \gamma_x & 1 \end{bmatrix}\gamma_x^n$$

Hence, we have

$$Q_P^n(x) = a_x\underline{\alpha_x}\alpha_x^n + b_x\underline{\beta_x}\beta_x^n + c_x\underline{\gamma_x}\gamma_x^n$$

Thus, the proof is completed. □

THEOREM 3.2. *The generating function for the sequence $Q_P^n(x)$ is*

$$G_{Q_P}(x) = \sum_{n=1}^{\infty} Q_P^n(x)t^n = \frac{t}{1-xt^2-t^3}\begin{bmatrix} t^2 & 1 & t \\ t & xt+t^2 & 1 \\ 1 & x+t & xt+t^2 \end{bmatrix}.$$

PROOF. Let

$$G_{Q_P}(x) = \sum_{n=1}^{\infty} Q_P^n(x)t^n = Q_P(x)t + Q_P^2(x)t^2 + Q_P^3(x)t^3 + \cdots + Q_P^n(x)t^n + \ldots$$

be generating function of the Padovan polynomials matrix. Multiply this function every side by $-xt^2$ such as

$$-xt^2 G_{Q_P}(x) = -xQ_P(x)t^3 - xQ_P^2(x)t^4 - xQ_P^3(x)t^5 - \cdots - xQ_P^n(x)t^{n+2} - \ldots$$

and that is multiplied every side by $-t^3$ such as

$$-t^3 G_{Q_P}(x) = -Q_P(x)t^4 - Q_P^2(x)t^5 - Q_P^3(x)t^6 - \cdots - Q_P^n(x)t^{n+3} - \ldots$$

Then, we write

$$\begin{aligned}(1-xt^2-t^3)G_{Q_P}(x) = \quad & Q_P(x)t + Q_P^2(x)t^2 + (Q_P^3(x) - xQ_P(x))t^3 \\ & (Q_P^4(x) - xQ_P^2(x) - Q_P(x))t^4 + \ldots \\ & +(Q_P^{n+3}(x) - xQ_P^{n+1}(x) - Q_P^n(x))t^{n+3} + \ldots\end{aligned}$$

Now using (3.1), we obtain $Q_P^{n+3}(x) - xQ_P^{n+1}(x) - Q_P^n(x) = 0$. So

$$G_{Q_P}(x) = \frac{t}{1-xt^2-t^3}\begin{bmatrix} t^2 & 1 & t \\ t & xt+t^2 & 1 \\ 1 & x+t & xt+t^2 \end{bmatrix}.$$

Thus, the proof is completed. □

THEOREM 3.3. *The exponential generating function for the sequence $Q_P^n(x)$ is*

$$E_{Q_P}(x) = \sum_{n=1}^{\infty} \frac{Q_P^n(x)}{n!}t^n = a_x\underline{\alpha_x}e^{\alpha_x t} + b_x\underline{\beta_x}e^{\beta_x t} + c_x\underline{\gamma_x}e^{\gamma_x t}.$$

PROOF. We know that

$$e^{\alpha_x t} = \sum_{n=1}^{\infty} \frac{\alpha_x^n}{n!}t^n, \quad e^{\beta_x t} = \sum_{n=1}^{\infty} \frac{\beta_x^n}{n!}t^n, \quad e^{\gamma_x t} = \sum_{n=1}^{\infty} \frac{\gamma_x^n}{n!}t^n$$

Let's multiply each side of the first equality by $a_x\underline{\alpha_x}$, the second equality by $b_x\underline{\beta_x}$ and the third equality by $c_x\underline{\gamma_x}$. Then, we added all equalities. So, the following equality is obtained.

$$a_x\underline{\alpha_x}e^{\alpha_x t} + b_x\underline{\beta_x}e^{\beta_x t} + c_x\underline{\gamma_x}e^{\gamma_x t} = \sum_{n=1}^{\infty} \frac{a_x\underline{\alpha_x}\alpha_x^n + b_x\underline{\beta_x}\beta_x^n + c_x\underline{\gamma_x}\gamma_x^n}{n!}t^n = \sum_{n=1}^{\infty} \frac{Q_P^n(x)}{n!}t^n$$

□

THEOREM 3.4. *The series for the sequence $Q_P^n(x)$ is*

$$S_{Q_P}(x)=\sum_{n=1}^{\infty}\frac{Q_P^n(x)}{t^{n+1}}=\frac{1}{t^3-xt-1}\begin{bmatrix}\frac{1}{t} & t & 1\\ 1 & x+\frac{1}{t} & t\\ t & 1+xt & x+\frac{1}{t}\end{bmatrix}.$$

PROOF. Let

$$S_{Q_P}(x)=\sum_{n=1}^{\infty}\frac{Q_P^n(x)}{t^{n+1}}=\frac{Q_P(x)}{t^2}+\frac{Q_P^2(x)}{t^3}+\frac{Q_P^3(x)}{t^4}+\cdots+\frac{Q_P^n(x)}{t^{n+1}}+\ldots$$

be series of the Padovan polynomials matrix. Multiply this function every side by t^3, $-xt$ and -1, respectively, such as

$$t^3S_{Q_P}(x)=Q_P(x)t+Q_P^2(x)+\frac{Q_P^3(x)}{t}+\cdots+\frac{Q_P^n(x)}{t^{n-2}}+\ldots$$

$$-xtS_{Q_P}(x)=-x\frac{Q_P(x)}{t}-x\frac{Q_P^2(x)}{t^2}-x\frac{Q_P^3(x)}{t^3}-\cdots-x\frac{Q_P^n(x)}{t^n}-\ldots$$

$$-S_{Q_P}(x)=-\frac{Q_P(x)}{t^2}-\frac{Q_P^2(x)}{t^3}-\frac{Q_P^3(x)}{t^4}-\cdots-\frac{Q_P^n(x)}{t^{n+1}}-\ldots$$

Then, we write

$$\begin{aligned}(t^3-xt-1)S_{Q_P}(x) &= Q_P(x)t+Q_P^2(x)+\left(Q_P^3(x)-xQ_P(x)\right)\tfrac{1}{t}\\ &\quad+\left(Q_P^4(x)-xQ_P^2(x)-Q_P(x)\right)\tfrac{1}{t^2}+\ldots\\ &\quad+\left(Q_P^{n+2}(x)-xQ_P^n(x)-Q_P^{n-1}(x)\right)\tfrac{1}{t^n}+\ldots\end{aligned}$$

Now using (3.1), we obtain $Q_P^{n+2}(x)-xQ_P^n(x)-Q_P^{n-1}(x)=0$. So

$$S_{Q_P}(x)=\frac{1}{t^3-xt-1}\begin{bmatrix}\frac{1}{t} & t & 1\\ 1 & x+\frac{1}{t} & t\\ t & 1+xt & x+\frac{1}{t}\end{bmatrix}.$$

Thus, the proof is completed. □

THEOREM 3.5. *The partial sum for the sequence $Q_P^n(x)$ is*

$$\sum_{i=1}^{n}Q_P^i(x)=\frac{1}{x}\left(Q_P^{n+2}(x)+Q_P^{n+1}(x)+Q_P^n(x)-\begin{bmatrix}1 & 1 & 1\\ 1 & 1+x & 1\\ 1 & 1+x & 1+x\end{bmatrix}\right).$$

PROOF. We know that

$$Q_P^n(x)=xQ_P^{n-2}(x)+Q_P^{n-3}(x)$$

So, applying to the identity above, we deduce that

$$\begin{aligned} Q_P^3(x) &= xQ_P^1(x) + Q_P^0(x), \\ Q_P^4(x) &= xQ_P^2(x) + Q_P^1(x), \\ Q_P^5(x) &= xQ_P^3(x) + Q_P^2(x), \\ &\dots, \\ Q_P^{n-2}(x) &= xQ_P^{n-4}(x) + Q_P^{n-5}(x), \\ Q_P^{n-1}(x) &= xQ_P^{n-3}(x) + Q_P^{n-4}(x), \\ Q_P^{n}(x) &= xQ_P^{n-2}(x) + Q_P^{n-3}(x). \end{aligned}$$

By summing the identities above side by side, we obtain

$$\begin{aligned} Q_P^{n-2}(x) + Q_P^{n-1}(x) + Q_P^{n}(x) &= x(Q_P^1(x) + Q_P^2(x) + \cdots + Q_P^{n-3}(x) + Q_P^{n-2}(x)) \\ &\quad + Q_P^0(x) + Q_P^1(x) + Q_P^2(x). \end{aligned}$$

Hence, we get the desired result. □

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