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COMMON FIXED POINT THEOREMS IN S-METRIC SPACES INVOLVING CONTROL FUNCTION

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Abstract. In this paper, we prove some common fixed point theorems in S-metric spaces for a pair of weakly compatible self-mappings using control function and give some consequences of the established result. We also furnish some examples to demonstrate the validity of the results. The results presented in this paper generalize, extend and enrich several results in the existing literature.

1. Introduction

The Banach contraction principle [**3**] is the most celebrated fixed point theorem and has been generalized in various directions. Fixed point problems for contractive mappings in metric spaces and generalized metric spaces with a partial order have been studied by many authors (see, for example, Agarwal *et al.* [**2**], Ćirić *et al.* [**4**], Van Dung *et al.* [**5**]). Fixed point theory has an application in many areas such as chemistry, physics, biology, computer science and many branches of mathematics. The famous Banach contraction mapping principle states that every self mapping $\mathcal{T}$ defined on a complete metric space (X, d) satisfying the condition:

$$d(\mathcal{T}(x), \mathcal{T}(y)) \leqslant k\, d(x, y), \tag{1.1}$$

for all $x, y \in X$, where $k \in (0, 1)$ is a constant, has a unique fixed point and for every $x_0 \in X$ a sequence $\{\mathcal{T}^n x_0\}_{n \geqslant 1}$ is convergent to the fixed point.

Generalizing the Banach contraction principle, Jungck [**8**] initiated the study of common fixed point for a pair of commuting mappings satisfying contractive

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type conditions. In 1982, Sessa [**31**] introduced a weaker concept of commutativity, which is generally known as weak commutativity and proved some interesting results on the existence of common fixed points for a pair of self maps. He also showed that weak commuting mappings are commuting but the converse need not to be true. Later, Jungck [**9**] generalized the concept of weak commutativity by introducing the notion of compatible mappings which is more general than weakly commuting mappings and showed that weak commuting maps are compatible but converse need not be true. In 1996, Jungck [**10**] generalized the concept of compatibility by introducing weakly compatible mappings.

In literature, there are many generalizations of the metric space exists. One of the generalizations of the metric space is the generalized metric space or S-metric space given by Sedghi et al. [**27**].

In 2012, Sedghi *et al.* [**27**] introduced the concept of a S-metric space which is different from other spaces and proved fixed point theorems in such spaces. They also give some examples of a S-metric space which shows that the S-metric space is different from other spaces. They built up some topological properties in such spaces and proved some fixed point theorems in the framework of S-metric spaces. For more details regarding this space we refer [**6**, **11**, **12**, **14**–**16**, **26**–**30**].

Recently, many number of authors have published many papers on S-metric spaces in different directions (see, e.g., [**13**, **17**–**25**, **32**, **33**] and many others).

In this paper, we prove some common fixed point theorems in S-metric spaces for a pair of weakly compatible self-mappings using control function and give some consequences of the established result. We also give some examples to demonstrate the validity of the results. Our results generalize, extend and enrich several results from the existing literature.

2. Preliminaries

In this section, we need some auxiliary results, basic definitions and lemmas to prove our main results.

DEFINITION 2.1. *([**27**]) Let X be a nonempty set and let $S\colon X^3 \to [0,+\infty)$ be a function satisfying the following conditions for all u, v, w, $t \in X$:*

$(S1)$ $S(u,v,w) = 0$ if and only if $u = v = w$;

$(S2)$ $S(u,v,w) \leqslant S(u,u,t) + S(v,v,t) + S(w,w,t)$.

Then the function S is called an S-metric on X and the pair (X,S) is called an S-metric space or simply SMS.

EXAMPLE 2.1. *([**27**]) Let $X = \mathbb{R}^n$ and $\|\cdot\|$ a norm on X, then $S(u,v,w) = \|v + w - 2u\| + \|v - w\|$ is an S-metric on X.*

EXAMPLE 2.2. *([**27**]) Let X be a nonempty set and d be an ordinary metric on X. Then $S(u,v,w) = d(u,w) + d(v,w)$ for all $u,v,w \in X$ is an S-metric on X.*

EXAMPLE 2.3. *([**28**]) Let $X = \mathbb{R}$ be the real line. Then $S(u,v,w) = |u - w| + |v - w|$ for all $u,v,w \in \mathbb{R}$ is an S-metric on X. This S-metric on X is called the usual S-metric on X.*

DEFINITION 2.2. *Let (X,S) be an S-metric space. For $\varepsilon > 0$ and $u \in X$ we define the open ball $\mathcal{B}_S(u,\varepsilon)$ and closed ball $\mathcal{B}_S[u,\varepsilon]$ with center u and radius ε as follows, respectively:*

$$(B_1) \qquad \mathcal{B}_S(u,\varepsilon) = \{v \in X : S(v,v,u) < \varepsilon\},$$

$$(B_2) \qquad \mathcal{B}_S[u,\varepsilon] = \{v \in X : S(v,v,u) \leqslant \varepsilon\}.$$

EXAMPLE 2.4. *([**28**]) Let $X = \mathbb{R}$. Denote $S(u,v,w) = |v+w-2u| + |v-w|$ for all $u,v,w \in \mathbb{R}$. Then*

$$\begin{aligned} \mathcal{B}_S(1,2) &= \{v \in \mathbb{R} : S(v,v,1) < 2\} = \{v \in \mathbb{R} : |v-1| < 1\} \\ &= \{v \in \mathbb{R} : 0 < v < 2\} = (0,2), \end{aligned}$$

and

$$\begin{aligned} \mathcal{B}_S[2,4] &= \{v \in \mathbb{R} : S(v,v,2) \leqslant 4\} = \{v \in \mathbb{R} : |v-2| \leqslant 2\} \\ &= \{v \in \mathbb{R} : 0 \leqslant v \leqslant 4\} = [0,4]. \end{aligned}$$

DEFINITION 2.3. *([**27**], [**28**]) Let (X,S) be an S-metric space and $A \subset X$.*

(Θ_1) *The subset A is said to be an open subset of X, if for every $x \in A$ there exists $c > 0$ such that $\mathcal{B}_S(x,c) \subset A$.*

(Θ_2) *A sequence $\{a_n\}$ in X converges to $a \in X$ if $S(a_n,a_n,a) \to 0$ as $n \to +\infty$, that is, for each $\varepsilon > 0$, there exists $n_0 \in \mathbb{N}$ such that for all $n \geqslant n_0$ we have $S(a_n,a_n,a) < \varepsilon$. We denote this by $\lim_{n\to+\infty} a_n = a$ or $a_n \to a$ as $n \to +\infty$.*

(Θ_3) *A sequence $\{a_n\}$ in X is called a Cauchy sequence if $S(a_n,a_n,a_m) \to 0$ as $n,m \to +\infty$, that is, for each $\varepsilon > 0$, there exists $n_0 \in \mathbb{N}$ such that for all $n,m \geqslant n_0$ we have $S(a_n,a_n,a_m) < \varepsilon$.*

(Θ_4) *The S-metric space (X,S) is called complete if every Cauchy sequence in X is convergent.*

(Θ_5) *Let τ be the set of all $A \subset X$ having the property that for every $x \in A$, A contains an open ball centered in x. Then τ is a topology on X (induced by the S-metric space).*

(Θ_6) *A nonempty subset A of X is S-closed if closure of A coincides with A.*

DEFINITION 2.4. *Let X be a non-empty set and $f,g\colon X \to X$ be two self mappings of X. Then a point $z \in X$ is called a*

(Ξ_1) *fixed point of operator f if $f(z) = z$;*

(Ξ_2) *common fixed point of f and g if $f(z) = g(z) = z$.*

DEFINITION 2.5. *([**27**]) Let (X,S) be an S-metric space. A mapping $\mathcal{F}\colon X \to X$ is said to be a contraction if there exists a constant $0 \leqslant h < 1$ such that*

$$S(\mathcal{F}u,\mathcal{F}v,\mathcal{F}w) \leqslant h\, S(u,v,w), \tag{2.1}$$

for all $u,v,w \in X$.

REMARK 2.1. ([**27**]) If the S-metric space (X,S) is complete and $\mathcal{F}\colon X \to X$ is a contraction mapping, then $\mathcal{F}$ has a unique fixed point in X.

Definition 2.6. *([**27**]) Let (X, S) and (X', S') be two S-metric spaces. A function $Q\colon X \to X'$ is said to be continuous at a point $x_0 \in X$ if for every sequence $\{a_n\}$ in X with $S(a_n, a_n, x_0) \to 0$, $S'(Q(a_n), Q(a_n), Q(x_0)) \to 0$ as $n \to +\infty$. We say that Q is continuous on X if Q is continuous at every point $x_0 \in X$.*

Definition 2.7. *([**1**]) Let $\mathcal{A}$ and $\mathcal{B}$ be single valued self-mappings on a set X. If $z = \mathcal{A}u = \mathcal{B}u$ for some $u \in X$, then u is called a coincidence point point of $\mathcal{A}$ and $\mathcal{B}$, and z is called a point of coincidence of $\mathcal{A}$ and $\mathcal{B}$.*

Definition 2.8. *([**9**]) Let $\mathcal{A}$ and $\mathcal{B}$ be single valued self-mappings on a set X. Mappings $\mathcal{A}$ and $\mathcal{B}$ are said to be commuting if $\mathcal{A}\mathcal{B}u = \mathcal{B}\mathcal{A}u$ for all $u \in X$.*

Definition 2.9. *([**10**]) Let $\mathcal{A}$ and $\mathcal{B}$ be single valued self-mappings on a set X. Mappings $\mathcal{A}$ and $\mathcal{B}$ are said to be weakly compatible if they commute at their coincidence points, i.e., if $\mathcal{A}u = \mathcal{B}u$ for some $u \in X$ implies $\mathcal{A}\mathcal{B}u = \mathcal{B}\mathcal{A}u$.*

Lemma 2.1. *([**27**], Lemma 2.5) Let (X, S) be an S-metric space. Then, $S(u, u, v) = S(v, v, u)$ for all $u, v \in X$.*

Lemma 2.2. *([**27**], Lemma 2.12) Let (X, S) be an S-metric space. If $a_n \to a$ and $b_n \to b$ as $n \to +\infty$ then $S(a_n, a_n, b_n) \to S(a, a, b)$ as $n \to +\infty$.*

Lemma 2.3. *([**28**]) The limit of a convergent sequence in an S-metric space (X, S) is unique.*

Lemma 2.4. *([**27**]) In a S-metric space (X, S), any convergent sequence is Cauchy.*

Lemma 2.5. *Let (X, S) be an S-metric space and suppose that the sequence $\{a_n\}$ is S-convergent to a. Then we have*

$$\limsup_{n\to+\infty} S(a_n, a_n, z) \leqslant S(z, z, a).$$

Proof. Let $\lim_{n\to+\infty} a_n = a$. Then for each $\varepsilon > 0$, there exists $n_1 \in \mathbb{N}$ such that for all $n \geqslant n_1$, we have

$$S(a_n, a_n, a) < \frac{\varepsilon}{2}.$$

Then for every $n \geqslant n_1$ by condition $(S2)$, we have

$$\begin{aligned} S(a_n, a_n, z) &\leqslant 2S(a_n, a_n, a) + S(z, z, a) \\ &\leqslant \varepsilon + S(z, z, a). \end{aligned}$$

Taking the upper limit as $n \to +\infty$ in the above inequality, we obtain the desired result. $\square$

In the following lemma we see the relationship between a metric and S-metric.

Lemma 2.6. *([**7**]) Let (X, d) be a metric space. Then the following properties are satisfied:*

(1) *$S_d(u, v, z) = d(u, z) + d(v, z)$ for all $u, v, z \in X$ is an S-metric on X.*
(2) *$a_n \to a$ in (X, d) if and only if $a_n \to a$ in (X, S_d).*
(3) *$\{a_n\}$ is Cauchy in (X, d) if and only if $\{a_n\}$ is Cauchy in (X, S_d).*
(4) *(X, d) is complete if and only if (X, S_d) is complete.*

We call the function S_d defined in Lemma 2.3 (1) as the S-metric generated by the metric d. It can be found an example of an S-metric which is not generated by any metric in [**7**, **18**].

EXAMPLE 2.5. *([**7**]) Let $X = \mathbb{R}$ and the function $S\colon X^3 \to [0, \infty)$ be defined as*

$$S(u, v, z) = |u - z| + |u + z - 2v|,$$

for all $u, v, z \in \mathbb{R}$. Then the function S is an S-metric on X and (X, S) is an S-metric space. Now, we prove that there does not exists any metric d such that $S = S_d$. On the contrary, suppose that there exists a metric d such that

$$S(u, v, z) = d(u, z) + d(v, z),$$

for all $u, v, z \in \mathbb{R}$. Hence, we obtain

$$S(u, u, z) = 2d(u, z) = 2|u - z|,$$

and

$$d(u, z) = |u - z|.$$

Similarly, we get

$$S(v, v, z) = 2d(v, z) = 2|v - z|,$$

and

$$d(v, z) = |v - z|,$$

for all $u, v, z \in \mathbb{R}$. Hence, we have

$$|u - z| + |u + z - 2v| = |u - z| + |v - z|,$$

which is a contradiction. Therefore, $S \neq S_d$ and $(\mathbb{R}, S)$ is a complete S-metric space.

3. Main results

In this section, we shall prove some common fixed point theorems in S-metric spaces using control function. Here we shall use the following concept.

Let Ψ denote the class of all functions $\psi\colon [0, +\infty) \to [0, +\infty)$ such that ψ is nondecreasing, continuous and $\sum_{n=1}^{\infty} \psi^n(t)$ is convergent for each $t > 0$. It is clear that $\psi^n(t) \to 0$ as $n \to +\infty$ for each $t > 0$ and hence, we have $\psi(t) < t$ for each $t > 0$.

THEOREM 3.1. *Let (X, S) be a complete S-metric space and $P, Q\colon X \to X$ be two self-mappings satisfying the following conditions:*

(1)

$$\begin{aligned} S(Px, Py, Pz)) \leqslant \psi\Big(\max\Big\{&S(Qx, Qy, Qz), a_1 S(Py, Py, Pz), \\ &a_2 S(Py, Py, Qz), \frac{a_3}{2}[S(Px, Px, Qz) \\ &+S(Py, Py, Qz)]\Big\}\Big), \end{aligned} \tag{3.1}$$

for all $x, y, z \in X$ and a_1, a_2, a_3 are nonnegative reals such that $0 < a_1, a_2, a_3 < 1$, where $\psi \in \Psi$;

(2) *$P(X) \subseteq Q(X)$ and either $P(X)$ or $Q(X)$ is a closed subset of X;*

(3) *The pair (P, Q) is weakly compatible.*

Then the maps P and Q have a unique common fixed point in X. If Q is continuous at the fixed point v, then P is also continuous at v.

Proof. Let $x_0 \in X$. Define the sequence $r_n = Pu_n = Qu_{n+1}$, $n = 0, 1, 2, \dots$ and let $t_{n+1} = S(r_n, r_n, r_{n+1})$. It follows from (3.1) and using $(S1)$, we have

$$\begin{aligned}
t_{n+1} &= S(r_n, r_n, r_{n+1}) \\
&= S(Pu_n, Pu_n, Pu_{n+1}) \\
&\leqslant \psi\Big(\max\Big\{S(Qu_n, Qu_n, Qu_{n+1}), a_1 S(Pu_n, Pu_n, Pu_{n+1}), \\
&\qquad a_2 S(Pu_n, Pu_n, Qu_{n+1}), \frac{a_3}{2}[S(Pu_n, Pu_n, Qu_{n+1}) \\
&\qquad + S(Pu_n, Pu_n, Qu_{n+1})]\Big\}\Big) \\
&= \psi\Big(\max\Big\{S(r_{n-1}, r_{n-1}, r_n), a_1 S(r_n, r_n, r_{n+1}), \\
&\qquad a_2 S(r_n, r_n, r_n), \frac{a_3}{2}[S(r_n, r_n, r_n) + S(r_n, r_n, r_n)]\Big\}\Big) \\
&= \psi\Big(\max\Big\{S(r_{n-1}, r_{n-1}, r_n), a_1 S(r_n, r_n, r_{n+1}), 0, 0\Big\}\Big) \\
&\leqslant \psi\Big(\max\Big\{t_n, a_1 t_{n+1}, 0, 0\Big\}\Big). \qquad (3.2)
\end{aligned}$$

If $\max\Big\{t_n, a_1 t_{n+1}, a_2 t_n, a_3 t_n\Big\} = a_1 t_{n+1}$, then from equation (3.2), we obtain

$$t_{n+1} \leqslant \psi(a_1 t_{n+1}) < a_1 t_{n+1},$$

which is a contradiction since $a_1 < 1$. Hence, we conclude that

$$t_{n+1} \leqslant \psi(t_n),\ n = 1, 2, \dots. \qquad (3.3)$$

Hence, we have

$$\begin{aligned}
S(r_n, r_n, r_{n+1}) &\leqslant \psi(S(r_{n-1}, r_{n-1}, r_n)) \leqslant \psi^2(S(r_{n-2}, r_{n-2}, r_{n-1})) \leqslant \\
\dots &\leqslant \psi^n(S(r_0, r_0, r_1)).
\end{aligned}$$

Now, we shall show that $\{r_n\}$ is a Cauchy sequence in an S-metric space (X, S). Hence for every $n, m \in \mathbb{N}$ with $m > n$ and using Lemma 2.1, then we have

$$\begin{aligned}
S(r_n, r_n, r_m) &\leqslant 2S(r_n, r_n, r_{n+1}) + S(r_m, r_m, r_{n+1}) \\
&= 2S(r_n, r_n, r_{n+1}) + S(r_{n+1}, r_{n+1}, r_m) \\
&\leqslant 2S(r_n, r_n, r_{n+1}) + 2S(r_{n+1}, r_{n+1}, r_2) \\
&\quad +S(r_{n+2}, r_{n+2}, r_m) \\
&\vdots \\
&\leqslant 2\sum_{j=n}^{m-2} S(r_j, r_j, r_{j+1}) + S(r_{m-1}, r_{m-1}, r_m) \\
&\leqslant 2[\psi^n(S(r_0, r_0, r_1)) + \psi^{n+1}(S(r_0, r_0, r_1)) + \ldots \\
&\quad +\psi^{m-1}(S(r_0, r_0, r_1))] \\
&= 2\sum_{j=n}^{m-1} \psi^j(S(r_0, r_0, r_1)).
\end{aligned}$$

Since $\sum_{n=1}^{\infty} \psi^n(t) < +\infty$ for all $t \geqslant 0$, hence $S(r_n, r_n, r_m) \to 0$ as $n \to +\infty$. Therefore, for each $\varepsilon > 0$, there exists $n_0 \in \mathbb{N}$ such that for each $n, m \geqslant n_0$, we have $S(r_n, r_n, r_m) < \varepsilon$. This shows that the sequence $\{r_n\}$ is a Cauchy sequence in X. By the completeness of the space X, there exists $v \in X$ such that $\lim_{n\to+\infty} r_n = v$ and

$$v = \lim_{n\to+\infty} r_n = \lim_{n\to+\infty} Pu_n = \lim_{n\to+\infty} Qu_{n+1}.$$

Let $Q(X)$ be a closed subset of X. Then there exists $\alpha \in X$ such that $Q\alpha = v$. We claim that $P\alpha = v$. Since

$$\begin{aligned}
S(P\alpha, P\alpha, Pu_n) &\leqslant \psi\Big(\max\Big\{S(Q\alpha, Q\alpha, Qu_n), a_1 S(P\alpha, P\alpha, Pu_n), \\
&\qquad a_2 S(P\alpha, P\alpha, Qu_n), \frac{a_3}{2}[S(P\alpha, P\alpha, Qu_n) \\
&\qquad +S(P\alpha, P\alpha, Qu_n)]\Big\}\Big) \\
&= \psi\Big(\max\Big\{S(Q\alpha, Q\alpha, Qu_n), a_1 S(P\alpha, P\alpha, Pu_n), \\
&\qquad a_2 S(P\alpha, P\alpha, Qu_n), a_3 S(P\alpha, P\alpha, Qu_n)\Big\}\Big) \qquad (3.4) \\
&= \psi\Big(\max\Big\{S(v, v, r_{n-1}), a_1 S(P\alpha, P\alpha, r_n), \\
&\qquad a_2 S(P\alpha, P\alpha, r_{n-1}), a_3 S(P\alpha, P\alpha, r_{n-1})\Big\}\Big).
\end{aligned}$$

Taking the upper limit as $n \to +\infty$ in equation (3.4) and using $(S1)$, Lemma 2.1, Lemma 2.5 and the property of ψ, we obtain

$$\begin{aligned} S(P\alpha, P\alpha, v) &\leqslant \psi\Big(\max\Big\{ \limsup_{n\to\infty} S(v, v, r_{n-1}), a_1 \limsup_{n\to\infty} S(P\alpha, P\alpha, r_n), \\ &\qquad a_2 \limsup_{n\to\infty} S(P\alpha, P\alpha, r_{n-1}), \\ &\qquad a_3 \limsup_{n\to\infty} S(P\alpha, P\alpha, r_{n-1}) \Big\}\Big) \\ &= \psi\Big(\max\Big\{ \limsup_{n\to\infty} S(v, v, r_{n-1}), a_1 \limsup_{n\to\infty} S(r_n, r_n, P\alpha), \\ &\qquad a_2 \limsup_{n\to\infty} S(r_{n-1}, r_{n-1}, P\alpha), \\ &\qquad a_3 \limsup_{n\to\infty} S(r_{n-1}, r_{n-1}, P\alpha) \Big\}\Big) \\ &\leqslant \psi\Big(\max\Big\{ S(v, v, v), a_1 S(P\alpha, P\alpha, v), a_2 S(P\alpha, P\alpha, v), \\ &\qquad\quad a_3 S(P\alpha, P\alpha, v) \Big\}\Big) \\ &= \psi\Big(\max\Big\{ 0, a_1 S(P\alpha, P\alpha, v), a_2 S(P\alpha, P\alpha, v), \\ &\qquad\quad a_3 S(P\alpha, P\alpha, v) \Big\}\Big) \\ &\leqslant \max\{a_1, a_2, a_3\} S(P\alpha, P\alpha, v). \end{aligned} \tag{3.5}$$

This implies that $1 \leqslant \max\{a_1, a_2, a_3\}$, which is a contradiction. Hence, from $\psi(t) < t$ for all $t \geqslant 0$, we have $S(P\alpha, P\alpha, v) = 0$, that is, $P\alpha = v$ and hence $P\alpha = Q\alpha = v$.

Since by hypothesis the pair (P, Q) is weakly compatible, we have $PQ\alpha = QP\alpha$, and hence $Pv = Qv$.

Now, we have to prove that v is a fixed point of P, that is, $Pv = v$. Suppose that $Pv \neq v$. Then from equation (3.1), we have

$$\begin{aligned} S(Pv, Pv, Pu_n)) &\leqslant \psi\Big(\max\Big\{ S(Qv, Qv, Qu_n), a_1 S(Pv, Pv, Pu_n), \\ &\qquad a_2 S(Pv, Pv, Qu_n), \frac{a_3}{2}[S(Pv, Pv, Qu_n) \\ &\qquad + S(Pv, Pv, Qu_n)] \Big\}\Big) \\ &= \psi\Big(\max\Big\{ S(Pv, Pv, r_{n-1}), a_1 S(Pv, Pv, r_n), \\ &\qquad a_2 S(Pv, Pv, r_{n-1}), \frac{a_3}{2}[S(Pv, Pv, r_{n-1}) \\ &\qquad + S(Pv, Pv, r_{n-1})] \Big\}\Big) \\ &= \psi\Big(\max\Big\{ S(Pv, Pv, r_{n-1}), a_1 S(Pv, Pv, r_n), \\ &\qquad a_2 S(Pv, Pv, r_{n-1}), a_3 S(Pv, Pv, r_{n-1}) \Big\}\Big). \end{aligned} \tag{3.6}$$

Taking the upper limit as $n \to +\infty$ in equation (3.6) and using the property of ψ, we obtain

$$\begin{aligned} S(Pv,Pv,v)) &\leqslant \psi\Big(\max\Big\{\limsup_{n\to\infty} S(Pv,Pv,r_{n-1}), a_1 \limsup_{n\to\infty} S(Pv,Pv,r_n), \\ &\qquad a_2 \limsup_{n\to\infty} S(Pv,Pv,r_{n-1}), a_3 \limsup_{n\to\infty} S(Pv,Pv,r_{n-1})\Big\}\Big) \\ &\leqslant \psi\Big(\max\Big\{S(Pv,Pv,v), a_1S(Pv,Pv,v), a_2S(Pv,Pv,v), \\ &\qquad a_3S(Pv,Pv,v)\Big\}\Big) \\ &\leqslant \max\{a_1,a_2,a_3\}S(Pv,Pv,v), \end{aligned} \tag{3.7}$$

which is a contradiction. Since by hypothesis, $\psi(t) < t$ for all $t \geqslant 0$, we have $S(Pv,Pv,v) = 0$, that is, $Pv = v$ and hence $Pv = Qv = v$. This shows that v is a common fixed point of P and Q.

Now, we shall show the uniqueness of the common fixed point. Suppose v' is another common fixed point of P and Q such that $Pv' = Qv' = v'$ with $v' \neq v$. Then, from the given inequality (3.1), we have

$$\begin{aligned} S(v,v,v') &= S(Pv,Pv,Pv')) \\ &\leqslant \psi\Big(\max\Big\{S(Qv,Qv,Qv'), a_1S(Pv,Pv,Pv'), \\ &\qquad a_2S(Pv,Pv,Qv'), \frac{a_3}{2}[S(Pv,Pv,Qv') \\ &\qquad +S(Pv,Pv,Qv')]\Big\}\Big) \\ &= \psi\Big(\max\Big\{S(Qv,Qv,Qv'), a_1S(Pv,Pv,Pv'), \\ &\qquad a_2S(Pv,Pv,Qv'), a_3S(Pv,Pv,Qv')\Big\}\Big) \\ &= \psi\Big(\max\Big\{S(v,v,v'), a_1S(v,v,v'), a_2S(v,v,v'), \\ &\qquad a_3S(v,v,v')\Big\}\Big). \end{aligned}$$

If $S(v,v,v') \leqslant \psi(S(v,v,v'))$, then $S(v,v,v') \leqslant \psi(S(v,v,v')) < S(v,v,v')$, which is a contradiction. Hence, $S(v,v,v') = 0$, that is, $v = v'$. If $S(v,v,v') \leqslant a\psi(S(v,v,v'))$, where $a = \max\{a_1,a_2,a_3\}$, then $S(v,v,v') \leqslant a\psi(S(v,v,v')) < aS(v,v,v')$, which is also a contradiction, since $0 < a < 1$. Hence, we have $S(v,v,v') = 0$, that is, $v = v'$. Thus, v is the unique common fixed point of P and Q.

Next, we shall show the continuity of the mapping in S-metric spaces.

Let $\{s_n\}$ be any sequence in X such that $\{s_n\}$ is convergent to v. Then, we have

$$\begin{aligned} S(Pv,Pv,Ps_n)) &\leqslant \psi\Big(\max\Big\{S(Qv,Qv,Qs_n),a_1S(Pv,Pv,Ps_n), \\ &\qquad a_2S(Pv,Pv,Qs_n),\frac{a_3}{2}[S(Pv,Pv,Qs_n) \\ &\qquad +S(Pv,Pv,Qs_n)]\Big\}\Big) \\ &= \psi\Big(\max\Big\{S(Qv,Qv,Qs_n),a_1S(Pv,Pv,Ps_n), \\ &\qquad a_2S(Pv,Pv,Qs_n),a_3S(Pv,Pv,Qs_n)\Big\}\Big). \end{aligned}$$

Taking the upper limit as $n \to +\infty$ in the above inequality and from the continuity of Q at a point v, using $(S1)$ and the property of ψ, we obtain

$$\begin{aligned} \limsup_{n\to+\infty} S(v,v,Ps_n) &= \limsup_{n\to+\infty} S(Pv,Pv,Ps_n) \\ &\leqslant \psi\Big(\max\Big\{\limsup_{n\to+\infty} S(Qv,Qv,Qs_n),a_1\limsup_{n\to+\infty} S(Pv,Pv,Ps_n), \\ &\qquad a_2\limsup_{n\to+\infty} S(Pv,Pv,Qs_n),a_3\limsup_{n\to+\infty} S(Pv,Pv,Qs_n)\Big\}\Big) \\ &\leqslant \psi\Big(\max\Big\{0,a_1\limsup_{n\to+\infty} S(v,v,Ps_n),a_2S(Pv,Pv,Qv), \\ &\qquad a_3S(Pv,Pv,Qv)\Big\}\Big) \\ &= \psi\Big(\max\Big\{0,a_1\limsup_{n\to+\infty} S(v,v,Ps_n),a_2S(Pv,Pv,Pv), \\ &\qquad a_3S(Pv,Pv,Pv)\Big\}\Big) \\ &= \psi\Big(\max\Big\{0,a_1\limsup_{n\to+\infty} S(v,v,Ps_n),0,0\Big\}\Big) \\ &\leqslant a_1\limsup_{n\to+\infty} S(v,v,Ps_n), \end{aligned}$$

we have

$$\limsup_{n\to+\infty} S(v,v,Ps_n) \leqslant a_1 \limsup_{n\to+\infty} S(v,v,Ps_n).$$

This implies that $\limsup_{n\to+\infty} S(v,v,Ps_n)=0$. Thus, we deduce that P is continuous at v. This completes the proof. □

Setting $Q=I$ (the identity map on X) in Theorem 3.1, then we have the following result.

COROLLARY 3.1. *Let (X,S) be a complete S-metric space and $P\colon X\to X$ be a self-mapping satisfying the following inequality:*

$$\begin{aligned} S(Px,Py,Pz)) \leqslant\ & \psi\Big(\max\Big\{S(x,y,z), a_1 S(Py,Py,Pz), \\ & a_2 S(Py,Py,z), \frac{a_3}{2}[S(Px,Px,z) \\ & +S(Py,Py,z)]\Big\}\Big), \end{aligned} \tag{3.8}$$

for all $x,y,z\in X$ and a_1,a_2,a_3 are nonnegative reals such that $0<a_1,a_2,a_3<1$, where $\psi\in\Psi$. Then the mapping P has a unique fixed point $v\in X$. And the mapping P is continuous at v.

By setting

$$\max\Big\{S(x,y,z), a_1 S(Py,Py,Pz), a_2 S(Py,Py,z),$$

$$\frac{a_3}{2}[S(Px,Px,z)+S(Py,Py,z)]\Big\} = S(x,y,z)$$

and $\psi(t)=kt$ for all $t\geqslant 0$, where $k\in[0,1)$ in Corollary 3.1, then we have the following result.

COROLLARY 3.2. *([**27**]) Let (X,S) be a complete S-metric space and $P\colon X\to X$ be a self-mapping satisfying the following inequality:*

$$S(Px,Py,Pz)) \leqslant k\,S(x,y,z), \tag{3.9}$$

for all $x,y,z\in X$, where $k\in[0,1)$. Then the mapping P has a unique fixed point $v\in X$.

COROLLARY 3.3. *Let (X,S) be a complete S-metric space such that for some positive integer n, P^n satisfies the contraction condition (3.9) for all $x,y,z\in X$, where k is as in Corollary 3.2. Then P has a unique fixed point in X.*

PROOF. From Corollary 3.2, let v_0 be the unique fixed point of P^n, that is, $P^n(v_0)=v_0$. Then

$$P(P^n v_0) = Pv_0 \quad \text{or} \quad P^n(Pv_0) = Pv_0.$$

This gives $Pv_0=v_0$. This shows that v_0 is a unique fixed point of P. This completes the proof. □

REMARK 3.1. Corollary 3.2 extends the well-known Banach fixed point theorem [**3**] from complete metric space to the setting of complete S-metric space.

REMARK 3.2. Corollary 3.2 is a special case of Corollary 3.3 for $n=1$.

THEOREM 3.2. *Let (X,S) be a complete S-metric space and $P,Q\colon X\to X$ be continuous and Q be commutative with P. If for every $n\in\mathbb{N}$, the following conditions are satisfied:*

(1)

$$S(P^n x, P^n y, P^n z)) \leqslant \psi\Big(\max \Big\{ S(Q^n x, Q^n y, Q^n z), a_1 S(P^n y, P^n y, P^n z), a_2 S(P^n y, P^n y, Q^n z), \frac{a_3}{2}[S(P^n x, P^n x, Q^n z) + S(P^n y, P^n y, Q^n z)] \Big\} \Big), \tag{3.10}$$

for all $x, y, z \in X$ and a_1, a_2, a_3 are nonnegative reals such that $0 < a_1, a_2, a_3 < 1$, where $\psi \in \Psi$;

(2) *$P^n(X) \subseteq Q^n(X)$ and either $P^n(X)$ or $Q^n(X)$ is a closed subset of X;*

(3) *the pair (P^n, Q^n) is weakly compatible.*

Then the mappings P and Q have a unique common fixed point $v \in X$. Further, if Q is continuous at the point v, then P is also continuous at v.

PROOF. Follows from Theorem 3.1. □

Now, we furnish some examples to demonstrate the validity of the hypothesis of Theorem 3.1 and Corollary 3.2.

EXAMPLE 3.1. *Let $X = \mathbb{R}$ and (X, S) be a complete S-metric space. For any $x, y, z \in X$, define the function $S\colon X^3 \to [0, \infty)$ by $S(x, y, z) = |x - z| + |y - z|$ and mappings P, $Q\colon X \to X$ by $P(x) = 1$ and*

$$Q(x) = \begin{cases} 1, & \text{if } x \in \mathbb{Q}, \\ 0, & \text{if } x \notin \mathbb{Q}. \end{cases}$$

(i) Now, if $x \in \mathbb{Q}$, then we have

$$S(Px, Py, Pz) = 0,\ S(Qx, Qy, Qz) = 0,\ S(Py, Py, Pz) = 0,$$

$$S(Py, Py, Qz) = 0,\ S(Px, Px, Qz) = 0.$$

(ii) Now, if $x \notin \mathbb{Q}$, then we have

$$S(Px, Py, Pz) = 0,\ S(Qx, Qy, Qz) = 0,\ S(Py, Py, Pz) = 0,$$

$$S(Py, Py, Qz) = 2,\ S(Px, Px, Qz) = 2.$$

For both the cases, it is easy to see that

$$S(Px, Py, Pz)) \leqslant \psi\Big(\max \Big\{ S(Qx, Qy, Qz), a_1 S(Py, Py, Pz), a_2 S(Py, Py, Qz), \frac{a_3}{2}[S(Px, Px, Qz) + S(Py, Py, Qz)] \Big\} \Big),$$

for all $x, y, z \in X$ and a_1, a_2, a_3 are nonnegative reals such that $0 < a_1, a_2, a_3 < 1$. Thus, all the conditions of Theorem 3.1 hold and $P1 = Q1 = 1$, that is, 1 is the unique common fixed point of P and Q.

EXAMPLE 3.2. *Let* $X = \{1, 2, 3, 4\}$ *and* $S\colon X \times X \times X \to \mathbb{R}$ *be defined by*

$$S(x, y, z) = \begin{cases} |x - z| + |y - z|, & \text{if } x \neq y \neq z, \\ x, & \text{if } x = y = z \neq 1, \\ 0, & \text{if } x = y = z = 1, \end{cases}$$

for all $x, y, z \in X$. *Then* (X, S) *is a complete* S*-metric space.*

Define the mapping $P\colon X \to X$ *by*

$$P(1) = 1,\ P(2) = 1,\ P(3) = 2,\ P(4) = 2.$$

Now, we have

$$S(P(1), P(2), P(3)) = S(1, 1, 2) = 2 \leqslant \frac{2}{3}.3 = \frac{2}{3}S(1, 2, 3),$$

$$S(P(1), P(2), P(4)) = S(1, 1, 2) = 2 \leqslant \frac{2}{3}.5 = \frac{2}{3}S(1, 2, 4),$$

$$S(P(1), P(3), P(4)) = S(1, 2, 2) = 1 \leqslant \frac{2}{3}.4 = \frac{2}{3}S(1, 3, 4),$$

$$S(P(2), P(3), P(4)) = S(1, 2, 2) = 1 \leqslant \frac{2}{3}.3 = \frac{2}{3}S(2, 3, 4).$$

Thus, P *satisfies all the conditions of Corollary 3.2 with* $k = \frac{2}{3} < 1$. *Now by applying Corollary 3.2,* P *has a unique fixed point. Indeed* 1 *is the required unique fixed point in this case.*

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