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BINARY $g^{\star}$-PRE REGULAR AND BINARY $g^{\star}$-PRE NORMAL SPACES

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Abstract. The aim of this paper is to introduce and study two new classes of spaces, called binary $g^{\star}$-pre regular and binary $g^{\star}$-pre normal spaces. Some basic properties of these separation axioms are studied by utilizing binary $g^{\star}$-pre closed and binary $g^{\star}$-pre open sets.

1. Introduction

In 2011, S.Nithyanantha Jothi and P.Thangavelu [**10**] introduced topology between two sets and also studied some of their properties. For background material, papers [**1**] to [**16**] may be perused. The aim of this paper is to introduce and study two new classes of spaces, called binary $g^{\star}$-pre regular and binary $g^{\star}$-pre normal spaces. Some basic properties of these separation axioms are studied by utilizing binary $g^{\star}$-pre closed and binary $g^{\star}$-pre open sets.

2. Binary $g^{\star}$-pre regular space

Definition 2.1. *A binary topological space $(X, Y, \mathcal{M})$ is said to be binary $g^{\star}$-pre regular (briefly binary $g^{\star}p$-regular) if for every binary $g^{\star}p$-closed set (E, F) and a point $(x, y) \notin (E, F)$, there exist disjoint pre-open sets (U, V) and (J, K) such that $(E, F) \subseteq (U, V)$ and $(x, y) \in (J, K)$.*

Theorem 2.1. *For a binary topological space $(X, Y, \mathcal{M})$, the following are equivalent:*

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(1) *$(X,Y,\mathcal{M})$ is binary $g^\star p$-regular.*
(2) *Every binary $g^\star p$-open set (U,V) is a union of binary pre-regular sets.*
(3) *Every binary $g^\star p$-closed set (A,B) is an intersection of binary pre-regular sets.*

PROOF. $(1)\Rightarrow(2)$: Let (U,V) be a binary $g^\star p$-open set and let $(x,y)\in(U,V)$. If $(A,B)=(X,Y)-(U,V)$, then (A,B) is binary $g^\star p$-closed. By assumption there exist disjoint binary open subsets $(P,Q)_1$ and $(P,Q)_2$ of (X,Y) such that $(x,y)\in(P,Q)_1$ and $(A,B)\in(P,Q)_2$. If $(J,K)=bp\text{-}cl((P,Q)_1)$, then (J,K) is binary pre-closed and $(J,K)\cap(A,B)\subseteq(J,K)\cap(P,Q)_2=(\phi,\phi)$. It follows that $(x,y)\in(J,K)\subseteq(U,V)$. Thus (U,V) is a union of binary pre-regular sets.

$(2)\Rightarrow(3)$: This is obvious.

$(3)\Rightarrow(1)$: Let (A,B) be binary $g^\star p$-closed and let $(x,y)\notin(A,B)$. By assumption, there exists a binary pre-regular set (J,K) such that $(A,B)\subseteq(J,K)$ and $(x,y)\in(J,K)$. If $(U,V)=(X,Y)\backslash(J,K)$, then (U,V) is a binary pre open set containing (x,y) and $(U,V)\cap(J,K)=(\phi,\phi)$. Thus $((X,Y,\mathcal{M})$ is binary $g^\star p$-regular. □

DEFINITION 2.2. *A binary topological space $(X,Y,\mathcal{M})$ is called a*

(1) *binary $g^\star p$-T_0-space if for each pair of distinct points there exists an binary $g^\star p$-open set containing one point but not the other.*
(2) *binary $(p,g^\star p)$-R_0-space if $bp\text{-}cl(\{x,y\})\subseteq(U,V)$ whenever (U,V) is binary $g^\star p$-open and $x\in(U,V)$.*

DEFINITION 2.3. *A space $(X,Y,\mathcal{M})$ is said to be binary pre-T_2 if for each pair of distinct points (x,y) and (i,j) in (X,Y), there exist disjoint binary pre open sets (U,V) and (J,K) in (X,Y) such that $(x,y)\in(U,V)$ and $(i,j)\in(J,K)$.*

THEOREM 2.2. *Every binary $g^\star p$-regular space $(X,Y,\mathcal{M})$ is both binary pre-T_2 and binary $(p,g^\star p)$-R_0.*

PROOF. Let $(X,Y,\mathcal{M})$ be $g^\star p$-regular space and let $(x,y),(i,j)\in(X,Y)$ such that $(x,y)\notin(i,j)$. By Theorem 2.1, $\{(x,y)\}$ is either binary pre open or binary pre closed since every space is binary pre-T_2. If $\{(x,y)\}$ is binary pre open, hence binary $g^\star p$-open, then binary pre-regular by Theorem 2.1. Thus $\{(x,y)\}$ and $(X,Y)-\{(x,y)\}$ are separating binary pre open sets. If $\{(x,y)\}$ is binary pre closed, then $(X,Y)-\{(x,y)\}$ is binary pre open and so, by Theorem 2.1, the union of binary pre-regular sets. Hence there is a binary pre-regular set $(J,K)\subseteq(X,Y)-\{(x,y)\}$ containing (i,j). This proves that $(X,Y,\mathcal{M})$ is binary pre-T_2. By Theorem 2.1, it follows immediately that $(X,Y,\mathcal{M})$ is also binary (p,g^starp)-R_0. □

DEFINITION 2.4. *The intersection of all binary $g^\star p$-closed sets containing (A,B) is called binary $g^\star$-pre closure (binary $g^\star p$-closure) of (A,B) and is denoted by $bg^\star p$-$cl(A,B)$.*

DEFINITION 2.5. *Let (A,B) be a subset of a space (X,Y) an $(x,y)\in(X,Y)$. The following properties hold for the $bg^\star p$-$cl(A,B)$:*

(1) $(x,y) \in bg^\star p\text{-}cl(A,B)$ *if and only if* $(A,B) \cap (U,V) \neq (\phi,\phi)$*, for every* $(U,V) \in BPO(X,Y)$ *containing* (x,y)*.*
(2) (A,B) *is binary* $g^\star p$*-closed if and only if* $(A,B) = bg^\star p\text{-}cl(A,B)$*.*
(3) $bg^\star p\text{-}cl(A,B)$ *is binary* $g^\star p$*-closed.*
(4) $bg^\star p\text{-}cl(A,B) \subseteq bg^\star p\text{-}cl(C,D)$ *if* $(A,B) \subseteq (C,D)$*.*
(5) $bg^\star p\text{-}cl(bg^\star p\text{-}cl(A,B)) = bg^\star p\text{-}cl(A,B)$*.*

Proof. The proof is obvious.

Definition 2.6. *A subset* (N,O) *of* (X,Y) *is called a binary* $g^\star$*-pre-neighbourhood (binary* $g^\star p$*-neighbourhood) of a point* (x,y) *in* (X,Y) *if there exists a binary* $g^\star p$*-open set* (U,V) *such that* $(x,y) \in (U,V) \subseteq (N,O)$*.*

Theorem 2.3. *Suppose that* $(C,D) \subseteq (A,B) \subseteq (X,Y)$*,* (C,D) *is a binary* $g^\star p$*-closed relative to* (A,B) *and that* (A,B) *is binary open and binary* $g^\star p$*-closed in* $(X,Y,\mathcal{M})$*. Then* (C,D) *is binary* $g^\star p$*-closed in* $(X,Y,\mathcal{M})$*.*

Theorem 2.4. *If* $(X,Y,\mathcal{M})$ *is a binary* $g^\star p$*-regular space and* (S,T) *is an binary open and binary* $g^\star p$*-closed subset of* $(X,Y,\mathcal{M})$*, then the subspace* (S,T) *is binary* $g^\star p$*-regular.*

Proof. Let (E,F) be any binary $g^\star p$-closed subset of (S,T) and $(i,j) \in (E,F)^c$. By Theorem 2.3, (E,F) is binary $g^\star p$-closed in $(X,Y,\mathcal{M})$. Since $((X,Y,\mathcal{M})$ is binary $g^\star p$-regular, there exist disjoint binary pre-open sets (U,V) and (J,K) of $(X,Y,\mathcal{M})$ such that $(i,j) \in (U,V)$ and $(E,F) \subseteq (J,K)$. We know that (S,T) is binary open and hence binary α-open, we get $(U,V) \cap (S,T)$ and $(J,K) \cap (S,T)$ are disjoint binary pre-open sets of the subspace (S,T) such that $(i,j) \in (U,V) \cap (S,T)$ and $(E,F) \subseteq (J,K) \cap (S,T)$. Hence the subspace (S,T) is binary $g^\star p$-regular. □

Theorem 2.5. *Let* $(X,Y,\mathcal{M})$ *be a binary topological space. Then the following statements are equivalent:*

(1) $(X,Y,\mathcal{M})$ *is binary* $g^\star p$*-regular.*
(2) *For each point* $(x,y) \in (X,Y)$ *and for each binary* $g^\star p$*-open neighbourhood* (P,Q) *of* (x,y)*, there exists a binary pre open set* (U,V) *of* (x,y) *such that* $bp\text{-}cl(U,V) \subseteq (P,Q)$*.*
(3) *For each point* $(x,y) \in (X,Y)$ *and for each binary* $g^\star p$*-closed set* (E,F) *not containing* (x,Y)*, there exists a binary pre-open set* (J,K) *of* (x,y) *such that* $bp\text{-}cl(J,K) \cap (E,F) = (\phi,\phi)$*.*

Proof. $(1) \Rightarrow (2)$: Let (P,Q) be any binary $g^\star p$-open neighbourhood of (x,y). Then there exists an binary $g^\star p$-open set (G,H) such that $(x,y) \in (G,H) \subseteq (P,Q)$. Since $(G,H)^c$ is binary $g^\star p$-closed and $(x,y) \notin (G,H)^c$, by hypothesis, there exist binary pre-open sets (U,V) and (J,K) such that $(G,H)^c \subseteq (U,V)$, $(x,y) \in (J,K)$ and $(U,V) \cap (J,K) = (\phi,\phi)$ and so $(J,K) \subseteq (U,V)^c$. Now $bp\text{-}cl(J,K) \subseteq bp\text{-}cl((U,V)^c) = (U,V)^c$ and $(G,H)^c \subseteq (U,V)$ implies $(U,V)^c \subseteq (G,H) \subseteq (P,Q)$. Therefore $bp\text{-}cl(U,V) \subseteq (P,Q)$.

$(2) \Rightarrow (1)$: Let (E,F) be any binary $g^\star p$-closed set and $(x,y) \notin (E,F)$. Then $(x,y) \in (E,F)^c$ and $(E,F)^c$ is binary $g^\star p$-open and so $(E,F)^c$ is an binary $g^\star p$-neighbourhood of (x,y). By hypothesis, there exists a binary pre-open set (J,K) of

(x, y) such that $(x, y) \in (J, K)$ and $bp\text{-}cl(J, K) \subseteq (E, F)^c$, which implies $(E, F) \subseteq (bp\text{-}cl(J, K))^c$. Then $(bp\text{-}cl(J, K))^c$ is binary pre-open set containing (E, F) and $(J, K) \cap (bp\text{-}cl(J, K))^c = (\phi, \phi)$. Therefore (X, Y) is binary $g^{\star}p$-regular.

(2) $\Rightarrow$ (3) : Let $(x, y) \in (X, Y)$ and (E, F) be a binary $g^{\star}p$-closed set such that $(x, y) \notin (E, F)$. Then $(E, F)^c$ is an binary $g^{\star}p$-neighbourhood of (x, y) and by hypothesis, there exists a binary pre-open set (J, K) of (x, y) such that $bp\text{-}cl(J, K) \subseteq (E, F)^c$ and hence $bp\text{-}cl(J, K) \cap (E, F) = (\phi, \phi)$.

(3) $\Rightarrow$ (2) : Let $(x, y) \in (X, Y)$ and (P, Q) be a binary $g^{\star}p$-neighbourhood of (x, y). Then there exists a binary $g^{\star}p$-open set (G, H) such that $(x, y) \in (G, H) \subseteq (P, Q)$. Since $(G, H)^c$ is binary $g^{\star}p$-closed and $(x, y) \notin (G, H)^c$, by hypothesis, there exists a binary pre-open set (U, V) of (x, y) such that $bp\text{-}cl(U, V) \cap (G, H)^c = (\phi, \phi)$. Therefore $bp\text{-}cl(U, V) \subseteq (G, H) \subseteq (P, Q)$. □

THEOREM 2.6. *A binary topological space $((X, Y, \mathcal{M})$ is binary $g^{\star}p$-regular if and only if for each binary $g^{\star}p$-closed set (E, F) of $(X, Y, \mathcal{M})$ and each $(x, y) \in (E, F)^c$, there exist binary pre-open sets (U, V) and (J, K) of $(X, Y, \mathcal{M})$ such that $(x, y) \in (U, V)$ and $(E, F) \subseteq (J, K)$ and $bp\text{-}cl(U, V) \cap bp\text{-}cl(J, K) = (\phi, \phi)$.*

PROOF. Let (E, F) be a binary $g^{\star}p$-closed set of $(X, Y, \mathcal{M})$ and $(x, y) \notin (E, F)$. Then there exist binary pre-open sets $(U, V)_x$ and (J, K) such that $(x, y) \in (U, V)_x$, $(E, F) \subseteq (J, K)$ and $(U, V)_x \cap (J, K) = (\phi, \phi)$. Which implies that $(U, V)_x \cap bp\text{-}cl(J, K) = (\phi, \phi)$. Since $(X, Y, \mathcal{M})$ is binary $g^{\star}p$-regular, there exist binary pre-open sets (G, H) and (W, Z) of $(X, Y, \mathcal{M})$ such that $(x, y) \in (G, H)$, $bp\text{-}cl(J, K) \subseteq (W, Z)$ and $(G, H) \cap (W, Z) = (\phi, \phi)$. This implies $bp\text{-}cl(G, H) \cap (W, Z) = (\phi, \phi)$. Now put $(U, V) = (U, V)_x \cap (G, H)$, then (U, V) and (J, K) are binary pre-open sets of $(X, Y, \mathcal{M})$ such that $(x, y) \in (U, V)$ and $(E, F) \subseteq (J, K)$ and $bp\text{-}cl(U, V) \cap bp\text{-}cl(J, K) = (\phi, \phi)$.

Converse is obvious. □

3. Binary $g^{\star}$-pre normal spaces

DEFINITION 3.1. *A binary topological space $((X, Y, \mathcal{M})$ is said to be binary $g^{\star}$-pre-normal (binary $g^{\star}p$-normal) if for any pair of disjoint binary $g^{\star}p$-closed sets (A, B) and (C, D), there exist disjoint binary pre-open sets (U, V) and (J, K) such that $(A, B) \subseteq (U, V)$ and $(C, D) \subseteq (J, K)$.*

THEOREM 3.1. *If $(X, Y, \mathcal{M})$ is a binary $g^{\star}p$-normal space and (S, T) is an binary open and binary $g^{\star}p$-closed subset of $(X, Y, \mathcal{M})$, then the subspace (S, T) is binary $g^{\star}p$-normal.*

PROOF. Let (A, B) and (C, D) be any two disjoint binary $g^{\star}p$-closed sets of (S, T). By Theorem 2.3, (A, B) and (C, D) are binary $g^{\star}p$-closed in $(X, Y, \mathcal{M})$. Since $(X, Y, \mathcal{M})$ is binary $g^{\star}p$-normal, there exist disjoint binary pre-open sets (U, V) and (J, K) of $(X, Y, \mathcal{M})$ such that $(A, B) \subseteq (U, V)$ and $(C, D) \subseteq (S, T)$. Since (S, T) is binary open and hence α-open. Then $(U, V) \cap (S, T)$ and $(J, K) \subseteq (S, T)$ are disjoint binary pre-open sets of the subspace (S, T). Hence the subspace (S, T) is binary $g^{\star}p$-normal. □

Theorem 3.2. *Let $(X, Y, \mathcal{M})$ be a binary topological space. Then the following statements are equivalent.*

(1) *$(X, Y, \mathcal{M})$ is binary $g^{\star}p$-normal.*
(2) *For each binary $g^{\star}p$-closed (E, F) and for each binary $g^{\star}p$-open set (U, V) containing (E, F), there exists a binary pre-open set (J, K) containing (E, F) such that $bp\text{-}cl(J, K) \subseteq (U, V)$.*
(3) *For each pair of disjoint binary $g^{\star}p$-closed sets (A, B) and (C, D) in $(X, Y, \mathcal{M})$, there exists a binary pre-open set (U, V) containing (A, B) such that $bp\text{-}cl(U, V) \cap (C, D) = (\phi, \phi)$.*
(4) *For each pair of disjoint binary $g^{\star}p$-closed sets (A, B) and (C, D) in $(X, Y, \mathcal{M})$, there exist a binary pre-open sets (U, V) and (J, K) such that $(A, B) \subseteq (U, V)$, $(C, D) \subseteq (J, K)$ and $bp\text{-}cl(A, B) \cap bp\text{-}cl(C, D) = (\phi, \phi)$.*

Proof. (1) $\Rightarrow$ (2) : Let (E, F) be a binary $g^{\star}p$-closed set and (U, V) be a binary $g^{\star}p$-open set such that $(E, F) \subseteq (U, V)$. Then $(E, F) \cap (U, V)^c = (\phi, \phi)$. By assumption, there exist binary pre-open sets (J, K) and (W, Z) such that $(E, F) \subseteq (J, K)$, $(U, V)^c \subseteq (W, Z)$ and $(J, K) \cap (W, Z) = (\phi, \phi)$, which implies $bp\text{-}cl(J, K) \cap (W, Z) = (\phi, \phi)$.

Now, $bp\text{-}cl(J, K) \cap (U, V)^c \subseteq bp\text{-}cl(J, K) \cap (W, Z) = (\phi, \phi)$ and so $bp\text{-}cl(J, K) \subseteq (U, V)$.

(2) $\Rightarrow$ (3) : Let (A, B) and (C, D) be disjoint binary $g^{\star}p$-closed sets of $(X, Y, \mathcal{M})$. Since $(A, B) \cap (C, D) = (\phi, \phi)$, $(A, B) \subseteq (C, D)^c$ and $(C, D)^c$ is binary $g^{\star}p$-open. By assumption, there exists binary pre-open set (U, V) containing (A, B) such that $bp\text{-}cl(U, V) \subseteq (C, D)^c$ and so $bp\text{-}cl(U, V) \cap (C, D) = (\phi, \phi)$.

(3) $\Rightarrow$ (4) : Let (A, B) and (C, D) be any binary $g^{\star}p$-closed sets of $(X, Y, \mathcal{M})$. Then by assumption, there exists a binary pre-open set (U, V) containing (A, B) such that $bp\text{-}cl(U, V) \cap (C, D) = (\phi, \phi)$. Since $bp\text{-}cl(A, B)$ is binary pre-closed, it is binary $g^{\star}p$-closed and so (C, D) and $bp\text{-}cl(A, B)$ are disjoint binary $g^{\star}p$-closed sets in $(X, Y, \mathcal{M})$. Therefore again by assumption, there exists a binary pre open set (J, K) containing (C, D) such that $bp\text{-}cl(A, B) \cap bp\text{-}cl(C, D) = (\phi, \phi)$.

(4) $\Rightarrow$ (1) : Let (A, B) and (C, D) be any disjoint binary $g^{\star}p$-closed sets of $(X, Y, \mathcal{M})$. By assumption, there exist binary pre-open sets (U, V) and (J, K) such that $(A, B) \subseteq (U, V)$, $(C, D) \subseteq (J, K)$ and $bp\text{-}cl(U, V) \cap bp\text{-}cl(J, K) = (\phi, \phi)$, we have $(U, V) \cap (S, T) = (\phi, \phi)$ and thus $(X, Y, \mathcal{M})$ is binary $g^{\star}p$-normal. □

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