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NOTE ON WEAKLY 1-ABSORBING PRIME FILTERS

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Abstract. Let $\pounds$ be a bounded distributive lattice. Following the concept of weakly 1-absorbing prime ideals, we define weakly 1-absorbing prime filters of $\pounds$. A proper filter $\mathbf{p}$ of $\pounds$ is called weakly 1-absorbing prime filter of $\pounds$ if whenever non-zero elements $a, b, c \in \pounds$ and $1 \neq a \vee b \vee c \in \mathbf{p}$, then either $a \vee b \in \mathbf{p}$ or $c \in \mathbf{p}$. We will make an intensive investigation of the basic properties and possible structures of these filters.

1. Introduction

All lattices considered in this paper are assumed to have a least element denoted by 0 and a greatest element denoted by 1, in other words they are bounded. As algebraic structures, lattices are definitely a natural choice of generalizations of rings, and it is appropriate to ask which properties of rings can be extended to lattices. The main aim of this article is that of extending some results obtained for rings theory to the theory of lattices. Nevertheless, growing interest in developing the algebraic theory of lattices can be found in several papers and books (see for example [5, 6, 8, 9]).

Since prime ideals have an important role in the theory of commutative rings, there are several ways to generalize the concept of prime ideals. Badawi generalized the concept of prime ideals in [3]. In 2002, Anderson and Smith in [2] defined weakly prime ideals which is a generalization of prime ideals (also see [10]). A proper ideal $\mathbf{p}$ of a ring R is said to be a weakly prime if $0 \neq xy \in \mathbf{p}$ for each $x, y \in R$ implies either $x \in \mathbf{p}$ or $y \in \mathbf{p}$. Recently, Yassine et. al. defined a new class of ideals, which is an intermediate class of ideals between prime ideals and 2-absorbing ideals. Recall from [13] that a proper ideal $\mathbf{p}$ of R is said to be a 1-absorbing prime ideal if for

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each nonunits $x, y, z \in R$ with $xyz \in \mathbf{p}$, then either $xy \in \mathbf{p}$ or $z \in \mathbf{p}$. Afterwards, Badawi and Celikel in [4] defined and studied weakly 1-absorbing primary ideals, i.e. a proper ideal I of R is called a weakly 1-absorbing primary ideal of R if whenever $a, b, c \in R$ and $0 \neq abc \in I$, then $ab \in I$ or $c \in \sqrt{I}$ (also see [1], [12]). Koc et. al. in [11] investigated weakly 1-absorbing prime ideals. A proper ideal I of R is said to be a weakly 1-absorbing prime ideal if whenever $0 \neq abc \in I$ for some nonunits $a, b, c \in R$, then $ab \in I$ or $c \in I$. Our objective in this paper is to extend the notion of weakly 1-absorbing property in commutative rings to weakly 1-absorbing property in the lattices, and to investigate the relations between weakly 1-absorbing prime filters and weakly prime filters. Among many results in this paper, the first, introduction section contains elementary observations needed later on.

Section 2 is devoted to prove that the Chinese remainder theorem. The Chinese remainder theorem in number theory is essentially solving the congruence equations. In the modern number theory, the Chinese remainder theorem theory is of great importance, and also gains some applications in several different algebras. At first, we give some properties of decomposable lattices which will be used in the sequel. Next, quotient lattices are determined by equivalence relations rather than by ideals as in the ring case. There are many different definition of a quotient lattice appearing in the literature. Here, quotient filters are studied and some possible properties of these filters are investigated. It is shown (Theorem 2.1) that if F and G are two comaximal filters of a lattice $\pounds$, then $\frac{\pounds}{F \cap G} \cong \frac{\pounds}{F} \times \frac{\pounds}{G}$. Also, it is proved (Theorem 2.2) that if $F_1, \cdots, F_k$ are k pairwise comaximal filters of a lattice $\pounds$, then $\frac{\pounds}{\cap_{i=1}^{k} F_i} \cong \frac{\pounds}{F_1} \times \frac{\pounds}{F_2}, \times \cdots \times \frac{\pounds}{F_k}$.

Section 3 is dedicated to the investigate the basic properties of weakly 1-absorbing prime filters. At first, we define the definition of weakly 1-absorbing prime filters (Definition 3.1) and we give an example (Example 3.1) of a weakly 1-absorbing prime filter of $\pounds$ that is not a 1-absorbing prime filter (so it is not a prime filter of $\pounds$). It is proved (Theorem 3.1) that $\mathbf{p}$ is a weakly 1-absorbing prime filter of $\pounds$ if and only if for each proper filters F, G, K of $\pounds$ such that $\{1\} \neq F \vee G \vee K \subseteq \mathbf{p}$, either $F \vee G \subseteq \mathbf{p}$ or $K \subseteq \mathbf{p}$. It is shown that (Theorem 3.2) that if $\mathbf{p}$ is a weakly 1-absorbing prime filter of a local lattice $\pounds$ that is not 1-absorbing prime, then $\mathbf{p} = \{1\}$. In the Theorem 3.4, we give a condition under which a weakly 1-absorbing prime filter of $\pounds$ is not a 1-absorbing prime filter. Theorem 3.5 determines the class of non-local lattices for which their weakly 1-absorbing prime filters and weakly prime filters are the same. In the Theorem 3.7, we provide an example of lattices for which their 1-absorbing prime filters and weakly 1-absorbing prime filters are the same. In the Theorem 3.8, we give a characterization of weakly 1-absorbing prime filters of decomposable lattices. Also, we characterize lattices with the property that all proper filters are weakly 1-absorbing prime (Theorem 3.10). In particular, we prove that if every proper filter of a lattice $\pounds$ is a weakly 1-absorbing prime, then $|\mathrm{Max}(\pounds)| \leqslant 2$ (Theorem 3.11).

Let us recall some notions and notations. A lattice $\pounds$ is called a distributive lattice if $(a \vee b) \wedge c = (a \wedge c) \vee (b \wedge c)$ for all a, b, c in $\pounds$ (equivalently, $\pounds$ is distributive if $(a \wedge b) \vee c = (a \vee c) \wedge (b \vee c)$ for all a, b, c in $\pounds$). A non-empty subset F of a lattice

$\pounds$ is called a filter, if for $a \in F$, $b \in \pounds$, $a \leqslant b$ implies $b \in F$, and $x \wedge y \in F$ for all $x, y \in F$ (so if $\pounds$ is a lattice with 1, then $1 \in F$ and $\{1\}$ is a filter of $\pounds$). A proper filter F of $\pounds$ is called prime if $x \vee y \in F$, then $x \in F$ or $y \in F$. A proper filter F of $\pounds$ is said to be maximal if G is a filter in $\pounds$ with $F \subsetneqq G$, then $G = \pounds$. The set of all maximal filters of $\pounds$ is denoted $\mathrm{Max}(\pounds)$. Let A be subset of a lattice $\pounds$. Then the filter generated by A, denoted by $T(A)$, is the intersection of all filters that is containing A. An element x of $\pounds$ is called an identity join of $\pounds$, if there exists $1 \neq y \in \pounds$ such that $x \vee y = 1$. The set of all identity joins of a lattice $\pounds$ is denoted $\mathrm{Id}(\pounds)$. An element $a \in \pounds$ is called co-regular if a is not an identity join element and $Co - \mathrm{Reg}(\pounds) = \pounds \smallsetminus \mathrm{Id}(\pounds)$. A simple lattice (resp. simple filter) is a lattice (resp. a filter) that has no filters besides the $\{1\}$ and itself. We will use $\pounds^*$ to denote the set of all non-zero elements of $\pounds$. First we need the following lemma proved in [5, 6, 8, 9].

LEMMA 1.1. *Let $\pounds$ be a lattice.*

(1) A non-empty subset F of $\pounds$ is a filter of $\pounds$ if and only if $x \vee z \in F$ and $x \wedge y \in F$ for all $x, y \in F$, $z \in \pounds$. Moreover, since $x = x \vee (x \wedge y)$, $y = y \vee (x \wedge y)$ and F is a filter, $x \wedge y \in F$ gives $x, y \in F$ for all $x, y \in \pounds$.

(2) If $F_1, \cdots, F_n$ are filters of $\pounds$ and $a \in \pounds$, then $\bigvee_{i=1}^n F_i = \{\bigvee_{i=1}^n a_i : a_i \in F_i\}$ and $a \vee F_i = \{a \vee a_i : a_i \in F_i\}$ are filters of $\pounds$ and $\bigvee_{i=1}^n F_i = \bigcap_{i=1}^n F_i$.

(3) If A is a non-empty subset of $\pounds$, then $T(A) = \{x \in L : a_1 \wedge a_2 \wedge \cdots \wedge a_n \leqslant x$ for some $a_i \in A$ $(1 \leqslant i \leqslant n)\}$.

(4) If $\pounds$ is distributive, F is a filter of $\pounds$, and $y \in \pounds$, then $(F :_L y) = \{a \in L : a \vee y \in F\}$ and $(1 :_\pounds F) = \{x \in \pounds : x \vee F = \{1\}\}$ are filters of L.

(5) If $\pounds$ is distributive and $F_1, \cdots, F_n$ are filters of $\pounds$, then $\bigwedge_{i=1}^n F_i = \{\wedge_{i=1}^n a_i : a_i \in F_i\}$ is a filter of $\pounds$ and $F_i \subseteq \bigwedge_{i=1}^n F_i$ for each i.

2. The Chinese remainder theorem

Throughout this paper, we shall assume unless otherwise stated, that $\pounds$ is a bounded distributive lattice. In this section, we extend the Chinese remainder theorem in the distributive lattices.

Assume that $(\pounds_1, \leqslant_1), (\pounds_2, \leqslant_2), \cdots, (\pounds_n, \leqslant_n)$ are lattices $(n \geqslant 2)$ and let $\pounds = \pounds_1 \times \pounds_2 \times \cdots \times \pounds_n$. We set up a partial order $\leqslant_c$ on $\pounds$ as follows: for each $x = (x_1, x_2, \cdots, x_n), y = (y_1, y_2, \cdots, y_n) \in \pounds$, we write $x \leqslant_c y$ if and only if $x_i \leqslant_i y_i$ for each $i \in \{1, 2, \cdots, n\}$. The following notation below will be kept in this paper: It is straightforward to check that $(\pounds, \leqslant_c)$ is a lattice with $x \vee_c y = (x_1 \vee y_1, x_2 \vee y_2, \cdots, x_n \vee y_n)$ and $x \wedge_c y = (x_1 \wedge y_1, \cdots, x_n \wedge y_n)$. In this case, we say that $\pounds$ is a decomposable lattice. We start with the following trivial result, and hence we omit its proof.

LEMMA 2.1. *If $\pounds_1, \pounds_2, \pounds_3$ are three lattices and $f : \pounds_1 \to \pounds_2$ is a lattice isomorphism, then there is a lattice isomorphism $g : \pounds_1 \times \pounds_3 \to \pounds_2 \times \pounds_3$ that sends each element (a, b) of $\pounds_1 \times \pounds_3$ to the pair $(f(a), b)$.*

DEFINITION 2.1. *Let $F, G, F_1, F_2, \cdots, F_n$, where $n \geqslant 2$, be filters of a lattice $\pounds$. We say that F and G are comaximal precisely when $F \wedge G = \pounds$; also, we say* *that the family* $\{F_i\}_{i=1}^n$ *is pairwise comaximal if and only if* $F_i \wedge F_j = \pounds$ *whenever* $1 \leqslant i, j \leqslant n$ *and* $i \neq j$.

LEMMA 2.2. *If* $\{F_i\}_{i=1}^n$ *(where* $n \geqslant 2$*) is a pairwise comaximal family of filters of the lattice* $\pounds$*, then* $\bigcap_{i=1}^{n-1} F_i$ *and* F_n *are comaximal.*

PROOF. Set $F = \bigcap_{i=1}^{n-1} F_i$. On the contrary, assume that $F \wedge F_n \subsetneqq \pounds$. Suppose that $\mathbf{m}$ is a maximal filter of $\pounds$ such that $F \wedge F_n \subseteq \mathbf{m}$. Then $F_n \subseteq \mathbf{m}$ and $F \subseteq \mathbf{m}$; hence, by [7, Lemma 2.1], there is a $j \in \{1, 2, \cdots, n-1\}$ such that $F_j \subseteq \mathbf{m}$ and so $F_j \wedge F_n \subseteq \mathbf{m}$ which is impossible. Thus F and F_n are comaximal. □

Quotient lattices are determined by equivalence relations rather than by ideals as in the ring case. If F is a filter of a lattice $(L, \leqslant)$, we define a relation on L, given by $x \sim y$ if and only if there exist $a, b \in F$ satisfying $x \wedge a = y \wedge b$. Then $\sim$ is an equivalence relation on L, and we denote the equivalence class of a by $a \wedge F$ and these collection of all equivalence classes by $\frac{L}{F}$. We set up a partial order $\leqslant_Q$ on $\frac{L}{F}$ as follows: for each $a \wedge F, b \wedge F \in \frac{L}{F}$, we write $a \wedge F \leqslant_Q b \wedge F$ if and only if $a \leqslant b$. The following notation below will be kept in this paper: It is straightforward to check that $(\frac{L}{F}, \leqslant_Q)$ is a lattice with $(a \wedge F) \vee_Q (b \wedge F) = (a \vee b) \wedge F$ and $(a \wedge F) \wedge_Q (b \wedge F) = (a \wedge b) \wedge F$ for all elements $a \wedge F, b \wedge F \in \frac{L}{F}$. We need the following Lemma proved in [8, Remark 4.2 and Lemma 4.3].

LEMMA 2.3. *Let* G *be a a sub filter of a filter* F *of* $\pounds$.

(1) If $a \in F$*, then* $a \wedge F = F$*. By the definition of* $\leqslant_Q$*, it is easy to see that* $1 \wedge F = F$ *is the greatest element of* $\frac{\pounds}{F}$ *and* $0 \wedge F$ *is a least of element* $\frac{\pounds}{F}$.

(2) If $a \in F$*, then* $a \wedge F = b \wedge F$ *(for every* $b \in \pounds$*) if and only if* $b \in F$*. In particular,* $c \wedge F = F$ *if and only if* $c \in F$*. Moreover, if* $a \in F$*, then* $a \wedge F = F = 1 \wedge F$.

(3) $a \wedge F = 0 \wedge F$ *if and only if* $a \wedge f = 0$ *for some* $f \in F$.

(4) By the definition $\leqslant_Q$*, we can easily show that if* $\pounds$ *is distributive, then* $\frac{\pounds}{F}$ *is distributive.*

(5) $\frac{F}{G} = \{a \wedge G : a \in F\}$ *is a filter of* $\frac{\pounds}{G}$.

(6) If K *is a filter of* $\frac{\pounds}{G}$*, then* $K = \frac{F}{G}$ *for some filter* F *of* $\pounds$.

Next, we state the following immediate lemma.

LEMMA 2.4. *If* $\pounds$ *is a lattice, then there is a lattice isomorphism* $h : \pounds \to \frac{\pounds}{\{1\}}$ *that sends each element* x *of* $\pounds$ *to* $x \wedge \{1\}$.

THEOREM 2.1. *(The Chinese remainder theorem for two filters) Let* F *and* G *be two comaximal filters of a lattice* $\pounds$*. Then there is a lattice isomorphism* $\phi : \frac{\pounds}{F \cap G} \to \frac{\pounds}{F} \times \frac{\pounds}{G}$ *that sends each residue class* $x \wedge (F \cap G)$ *to the pair* $(x \wedge F, x \wedge G)$.

PROOF. If $x \wedge (F \cap G) = y \wedge (F \cap G)$, then $x \wedge a = y \wedge b$ for some $a, b \in F \cap G$; so $x \wedge F = y \wedge F$ and $x \wedge G = y \wedge G$. This shows that ϕ is well defined.

If $A = x \wedge (F \cap G)$, and $B = y \wedge (F \cap G)$ are elements of $\frac{\pounds}{F \cap G}$, then $\phi(A \vee_Q B) = \phi((x \vee y) \wedge F \cap G) = ((x \vee y) \wedge F, (x \vee y) \wedge G) = (x \wedge F, x \wedge G) \vee_c (y \wedge F, y \wedge G) = \phi(A) \vee_c \phi(B)$. Similarly, $\phi(A \wedge_Q B) = \phi(A) \wedge_c \phi(B)$. Now, we claim that ϕ is injective. If $\phi(x \wedge (F \cap G)) = \phi(y \wedge (F \cap G))$, then $x \wedge F = y \wedge F$ and $x \wedge G = y \wedge G$ which implies that $x \wedge s = y \wedge s'$ and $x \wedge t = y \wedge t'$ for some $s, s' \in F$ and $t, t' \in G$ and so $x \wedge (s \vee t) = (x \wedge s) \vee (x \wedge t) = (y \wedge s') \vee (y \wedge t') = y \wedge (s' \vee t')$. Since $s \vee t, s' \vee t' \in F \cap G$, we conclude that $x \wedge (F \cap G) = y \wedge (F \cap G)$. This shows that ϕ is injective. It remains to show that ϕ is subjective By assumption, $0 = a \wedge b$ for some $a \in F$ and $b \in G$ which gives $0 \wedge F = b \wedge G$ and $0 \wedge G = a \wedge G$ by Lemma 2.3 (3). It then follows from Lemma 2.3 (2) that $\phi(a \wedge (F \cap G)) = (a \wedge F, a \wedge G) = (1 \wedge F, 0 \wedge G)$. Similarly, $\phi(b \wedge (F \cap G)) = (0 \wedge F, 1 \wedge G)$. Now, for every $x, y \in \pounds$, set $D = (y \vee a) \wedge (x \vee b)$. Then we have $\phi(D \wedge (F \cap G)) =$

$$\phi((y \vee a) \wedge F \cap G) \wedge_c \phi((x \vee b) \wedge F \cap G) =$$

$(1 \wedge F, y \wedge G) \wedge_c (x \wedge F, 1 \wedge G) = (x \wedge F, y \wedge G)$. Thus, every element of the form $(x \wedge F, y \wedge G)$ for some $x \in \pounds$ and $y \in \pounds$ lies in the image of ϕ. Hence ϕ is subjective, as needed. □

THEOREM 2.2. *(The Chinese remainder theorem for k filters) Let* $F_1, \cdots, F_k$ *be k pairwise comaximal filters of a lattice* $\pounds$*. Then there is a lattice isomorphism* $\phi : \frac{\pounds}{\cap_{i=1}^{k} F_i} \to \frac{\pounds}{F_1} \times \frac{\pounds}{F_2}, \times \cdots \times \frac{\pounds}{F_k}$ *that sends each residue class* $x \wedge \cap_{i=1}^{k} F_i$ *to the k-tuple* $(x \wedge F_1, x \wedge F_2, \cdots, x \wedge F_k)$.

PROOF. We use induction on k. We can take $k = 1$ as a base case. Let n be a positive integer. We now turn to the inductive step. Assume, inductively, that $k = n - 1$, where $n \geqslant 3$, and that the result has been proved in the case where $k = n - 1$. Thus we have $\frac{\pounds}{\cap_{i=1}^{n-1} F_i} \cong \frac{\pounds}{F_1} \times \cdots \times \frac{\pounds}{F_{n-1}}$. Since $F = \cap_{i=1}^{n-1} F_i$ and F_n are comaximal by Lemma 2.2, we conclude that $\frac{\pounds}{F \cap F_n} \cong \frac{\pounds}{F} \times \frac{\pounds}{F_n}$ by Theorem 2.1. Now the assertion follows from the induction hypothesis and Lemma 2.1. □

COROLLARY 2.1. *Let* $\mathbf{m_1}$*,* $\mathbf{m_2}$ *and* $\mathbf{m_3}$ *be three distinct maximal filters of a lattice* $\pounds$*. The following hold:*

(1) If $\mathbf{m_1} \cap \mathbf{m_2} = \{1\}$*, then* $\pounds \cong \frac{\pounds}{\mathbf{m_1}} \times \frac{\pounds}{\mathbf{m_2}}$.

(2) If $\mathbf{m_1} \cap \mathbf{m_2} \cap \mathbf{m_3} = \{1\}$*, then* $\pounds \cong \frac{\pounds}{\mathbf{m_1}} \times \frac{\pounds}{\mathbf{m_2}} \times \frac{\pounds}{\mathbf{m_3}}$.

PROOF. (1) Since $\mathbf{m_1} \subsetneqq \mathbf{m_1} \wedge \mathbf{m_2} \subseteq \pounds$, we conclude that $\mathbf{m_1} \wedge \mathbf{m_2} = \pounds$; so $\mathbf{m_1}$ and $\mathbf{m_2}$ are comaximal filters of $\pounds$. Now the assertion is a direct consequence of Theorem 2.1 and Lemma 2.4.

(2) It is easy to see that $\mathbf{m_1}$, $\mathbf{m_2}$ and $\mathbf{m_3}$ are pairwise comaximal filters of $\pounds$. Now the claim is a direct consequence of Theorem 2.2 and Lemma 2.4. □

3. Characterization of weakly 1-absorbing prime filters

In this section, the concept of weakly 1-absorbing prime filters is introduced and investigated. We remind the reader with the following definition.

DEFINITION 3.1. *A proper filter* $\mathbf{p}$ *of* $\pounds$ *is called weakly* 1*-absorbing prime if for all* $x, y, z \in \pounds^*$ *such that* $1 \neq x \vee y \vee z \in \mathbf{p}$*, then* $x \vee y \in \mathbf{p}$ *or* $z \in \mathbf{p}$.

EXAMPLE 3.1. *(1) It is easy to see that every* 1*-absorbing prime filter is a weakly* 1*-absorbing prime filter.*

(2) Let $D = \{a, b, c\}$. Then $£(D) = \{X : X \subseteq D\}$ forms a distributive lattice under set inclusion with greatest element D and least element $\emptyset$ (note that if $x, y \in £(D)$, then $x \vee y = x \cup y$ and $x \wedge y = x \cap y$). Set $\mathbf{p} = \{1\}$. Then $\mathbf{p}$ is clearly a weakly 1-absorbing prime filter of $£(D)$. Since $\{a\} \vee \{b\} \vee \{c\} \in \mathbf{p}$, $\{a\} \vee \{b\} \notin \mathbf{p}$ and $\{c\} \notin \mathbf{p}$, it follows that $\mathbf{p}$ is not a 1-absorbing prime filter of $£(D)$. Thus a weakly 1-absorbing prime filter need not be a 1-absorbing prime filter.

Proposition 3.1. *Let $\mathbf{p}$ be a weakly 1-absorbing prime filter of $£$. The following assertions hold:*

(1) $(\mathbf{p} :_£ a)$ is a weakly prime for each $a \in Co - \text{Reg}(£) \smallsetminus \mathbf{p}$;

(2) If $\{1\}$ is a 1-absorbing prime filter, then $\mathbf{p}$ is a 1-absorbing prime filter.

Proof. (1) Assume that $a \in Co - \text{Reg}(£) \smallsetminus \mathbf{p}$ and let $1 \neq b \vee c \in \mathbf{p}$ for some $b, c \in £$. We may assume that $b, c \in £^*$. Since $a \in Co - \text{Reg}(£)$ and $1 \neq b \vee c \in (\mathbf{p} :_£ a)$, we have $1 \neq a \vee b \vee c \in \mathbf{p}$. Then either $a \vee b \in \mathbf{p}$ or $c \in \mathbf{p}$. This shows that $b \in (\mathbf{p} :_£ a)$ or $c \in \mathbf{p} \subseteq (\mathbf{p} :_£ a)$, i.e. (1) holds.

(2) Let $a \vee b \vee c \in \mathbf{p}$ for some $a, b, c \in £^*$. If $a \vee b \vee c \neq 1$, then we have either $a \vee b \in \mathbf{p}$ or $c \in \mathbf{p}$. So assume that $a \vee b \vee c \in \{1\}$. Since $\{1\}$ is a 1-absorbing prime filter, we conclude that either $a \vee b = 1 \in \mathbf{p}$ or $c = 1 \in \mathbf{p}$, as needed. □

In the following theorem we give five other characterizations of weakly 1-absorbing prime filters.

Theorem 3.1. *Let $\mathbf{p}$ be a proper filter of a lattice $£$. The following statements are equivalent:*

(1) $\mathbf{p}$ is a weakly 1-absorbing prime filter of $£$;

(2) For each $x, y \in £^*$ with $x \vee y \notin \mathbf{p}$, $(\mathbf{p} :_£ x \vee y) = \mathbf{p} \cup (1 :_£ x \vee y)$;

(3) For each $x, y \in £^*$ with $x \vee y \notin \mathbf{p}$, either $(\mathbf{p} :_£ x \vee y) = \mathbf{p}$ or $(\mathbf{p} :_£ x \vee y) = (1 :_£ x \vee y)$;

(4) For each $x, y \in £^*$ and proper filter F of $£$ such that $\{1\} \neq (x \vee y) \vee F \subseteq \mathbf{p}$, either $x \vee y \in \mathbf{p}$ or $F \subseteq \mathbf{p}$;

(5) For each $x \in £^*$ and proper filters F, G of $£$ such that $\{1\} \neq x \vee F \vee G \subseteq \mathbf{p}$, either $x \vee F \subseteq \mathbf{p}$ or $G \subseteq \mathbf{p}$;

(6) For each proper filters F, G, K of $£$ such that $\{1\} \neq F \vee G \vee K \subseteq \mathbf{p}$, either $F \vee G \subseteq \mathbf{p}$ or $K \subseteq \mathbf{p}$.

Proof. (1) $\Rightarrow$ (2) As the inclusion $\mathbf{p} \cup (1 :_£ x \vee y) \subseteq (\mathbf{p} :_£ x \vee y)$ is clear, we will prove the reverse inclusion. Let $z \in (P :_£ x \vee y)$. Since $x \vee y \notin \mathbf{p}$ and $x \vee y \vee z \in \mathbf{p}$, we conclude that $z \neq 0$. If $x \vee y \vee z = 1$, then $z \in (1 :_£ x \vee y)$. So suppose that $x \vee y \vee z \neq 1$. Since $x \vee y \notin \mathbf{p}$, $x \vee y \vee z \in \mathbf{p}$ and $\mathbf{p}$ is a weakly 1-absorbing prime filter, we have $z \in \mathbf{p}$, and so we have equality.

(2) $\Rightarrow$ (3) By [9, Remark 2.3 (i)], $(\mathbf{p} :_£ x \vee y) \subseteq \mathbf{p} \cup (1 :_£ x \vee y)$ gives either $(\mathbf{p} :_£ x \vee y) \subseteq \mathbf{p}$ or $(\mathbf{p} :_£ x \vee y) \subseteq (1 :_£ x \vee y)$; hence either $(\mathbf{p} :_£ x \vee y) = \mathbf{p}$ or $(\mathbf{p} :_£ x \vee y) = (1 :_£ x \vee y)$.

(3) $\Rightarrow$ (4) Let $\{1\} \neq x \vee y \vee F \subseteq \mathbf{p}$ for some $x, y \in £^*$ and a proper filter F of $£$. If $x \vee y \in \mathbf{p}$, then we are done. So suppose that $x \vee y \notin \mathbf{p}$. Then by (3), we have either $(\mathbf{p} :_£ x \vee y) = \mathbf{p}$ or $(\mathbf{p} :_£ x \vee y) = (1 :_£ x \vee y)$. If $(\mathbf{p} :_£ x \vee y) = (1 :_£ x \vee y)$,

then $x \vee y \vee F \subseteq \mathbf{p}$ gives $F \subseteq (\mathbf{p} :_{\pounds} x \vee y) = (1 :_{\pounds} x \vee y)$ and so $x \vee y \vee F = \{1\}$ which is impossible. Thus $(\mathbf{p} :_{\pounds} x \vee y) = \mathbf{p}$ and this shows that $F \subseteq (\mathbf{p} :_{\pounds} x \vee y) = \mathbf{p}$, i.e. (4) holds.

(4) $\Rightarrow$ (5) Let $\{1\} \neq x \vee F \vee G \subseteq \mathbf{p}$ for some $x \in \pounds^*$ and proper filters F, G of $\pounds$. On the contrary, assume that $x \vee F \nsubseteq \mathbf{p}$ and $G \nsubseteq \mathbf{p}$. Then there exists $z \in F$ (so $z \neq 0$, as F is a proper filter) such that $z \vee x \notin \mathbf{p}$. Since $x \vee F \vee G \neq \{1\}$, we conclude that $x \vee u \vee G \neq \{1\}$ for some $0 \neq u \in F$. We claim that $x \vee z \vee G = \{1\}$. Otherwise, by (4), we would have $x \vee z \in \mathbf{p}$ or $G \subseteq \mathbf{p}$ which is a contradiction. Thus $x \vee z \vee G = \{1\}$. Since $z, u \in F$, we have $u \wedge z \neq 0$ and so $\{1\} \neq x \vee u \vee G \subseteq x \vee (z \wedge u) \vee G \subseteq x \vee F \vee G \subseteq \mathbf{p}$. Then by (4), we have $x \vee (u \wedge z) = (x \vee u) \wedge (x \vee z) \in \mathbf{p}$, and so $x \vee z \in \mathbf{p}$ by Lemma 1.1 (1), a contradiction. Therefore, $x \vee F \subseteq \mathbf{p}$ or $G \subseteq \mathbf{p}$.

(5) $\Rightarrow$ (6) Let $\{1\} \neq F \vee G \vee K \subseteq \mathbf{p}$ for some proper filters F, G, K of $\pounds$. On the contrary, assume that $F \vee G \nsubseteq \mathbf{p}$ and $K \nsubseteq \mathbf{p}$. Then there exists $y \in F$ such that $y \vee G \nsubseteq \mathbf{p}$. If $y \vee G \vee K \neq \{1\}$, then we have either $y \vee G \subseteq \mathbf{p}$ or $K \subseteq \mathbf{p}$ which is impossible. Thus $y \vee G \vee K = \{1\}$. Since $F \vee G \vee K \neq \{1\}$, $x \vee G \vee K \neq \{1\}$ for some $x \in F$. As $x \vee G \vee K \subseteq \mathbf{p}$, we conclude that $x \vee G \subseteq \mathbf{p}$. Clearly, $0 \neq x \wedge y \in F$. Since $\{1\} \neq x \vee G \vee K \subseteq (x \wedge y) \vee G \vee K \subseteq \mathbf{p}$, we have $(x \wedge y) \vee G \subseteq \mathbf{p}$. If $g \in G$, then $g \vee (x \wedge y) = (g \vee x) \wedge (g \vee y) \in \mathbf{p}$ gives $g \vee y \in \mathbf{p}$ by Lemma 1.1 (1); so $y \vee G \subseteq \mathbf{p}$, a contradiction, i.e. (6) holds.

(6) $\Rightarrow$ (1) Let $1 \neq x \vee y \vee z \in \mathbf{p}$ for some $x, y, z \in \pounds^*$. Now, put $F = T(\{x\})$, $G = T(\{y\})$ and $K = T(\{z\})$. Then by (6), $\{1\} \neq F \vee G \vee K \subseteq \mathbf{p}$ gives either $x \vee y \in F \vee G \subseteq \mathbf{p}$ or $z \in K \subseteq \mathbf{p}$ which completes the proof. □

PROPOSITION 3.2. *Assume that* $\mathbf{p}$ *is a weakly* 1*-absorbing prime filter of a local lattice* $\pounds$ *and let there exist non-zero elements* $a, b, c \in \pounds$ *such that* $a \vee b \vee c = 1$*,* $a \vee b \notin \mathbf{p}$ *and* $c \notin \mathbf{p}$*. The following assertions hold:*

(1) $(a \vee b) \vee \mathbf{p} = (a \vee c) \vee \mathbf{p} = (b \vee c) \vee \mathbf{p} = \{1\}$.

(2) $a \vee \mathbf{p} = c \vee \mathbf{p} = b \vee \mathbf{p} = \{1\}$.

PROOF. (1) On the contrary, assume that $(a \vee b) \vee \mathbf{p} \neq \{1\}$. Then there is an element $p \in \mathbf{p}$ such that $a \vee b \vee p \neq 1$ which implies that $(a \vee b) \vee (c \wedge p) = a \vee b \vee p \neq 1$. Since $a, b, c \wedge p \in \pounds^*$, $a \vee b \notin \mathbf{p}$ and $\mathbf{p}$ is a weakly 1-absorbing prime filter, $p \wedge c \in \mathbf{p}$; hence $c \in \mathbf{p}$ by Lemma 1.1 (1), a contradiction. Thus $(a \vee b) \vee \mathbf{p} = \{1\}$. Now suppose that $(a \vee c) \vee \mathbf{p} \neq \{1\}$. So $a \vee c \vee p' \neq 1$ for some $p' \in \mathbf{p}$. Therefore $a \vee (b \wedge p') \vee c = a \vee c \vee p' \neq 1$ with $a, c, b \wedge p' \in \pounds^*$. Since $c \notin \mathbf{p}$ and $\mathbf{p}$ is a weakly 1-absorbing prime filter, $a \vee (b \wedge p') = (a \vee b) \wedge (a \vee p') \in \mathbf{p}$, and so $a \vee b \in \mathbf{p}$ by Lemma 1.1 (1), a contradiction. Hence $(a \vee c) \vee \mathbf{p} = \{1\}$. Similarly, $(b \vee c) \vee \mathbf{p} = \{1\}$.

(2) Suppose that $a \vee \mathbf{p} \neq \{1\}$. Then $a \vee p \neq 1$ for some $p \in \mathbf{p}$. Since $a \vee (b \wedge p) \vee (c \wedge p) = (a \vee p) \wedge (a \vee b \vee c) = a \vee p \neq 1$, $a, b \wedge p, c \wedge p \in \pounds^*$ and $\mathbf{p}$ is a weakly 1-absorbing prime filter, we conclude that either $a \vee (b \wedge p) = (a \vee b) \wedge (a \vee p) \in \mathbf{p}$ or $c \wedge p \in \mathbf{p}$. Hence either $a \vee b \in \mathbf{p}$ or $c \in \mathbf{p}$ by Lemma 1.1 (1) which is impossible. Thus $a \vee \mathbf{p} = \{1\}$. Similarly, $b \vee \mathbf{p} = c \vee \mathbf{p} = \{1\}$. □

THEOREM 3.2. *Let* $\mathbf{p}$ *be a weakly* 1*-absorbing prime filter of a local lattice* $\pounds$ *with unique maximal filter* $\mathbf{m}$ *that is not* 1*-absorbing prime. Then* $\mathbf{p} = \{1\}$*.*

Proof. Since $\mathbf{p}$ is a weakly 1-absorbing prime filter of $\pounds$ that is not 1-absorbing prime, there exist $a, b, c \in \pounds^*$ such that $a \vee b \vee c = 1$, $a \vee b \notin \mathbf{p}$ and $c \notin \mathbf{p}$. On the contrary, assume that $\mathbf{p} \neq \{1\}$ and consider $1 \neq p \in \mathbf{p}$. Since $(a \wedge p) \vee (b \wedge p) \vee (c \wedge p) = p \wedge (a \vee b \vee c) = p \neq 1$, $p \wedge a, p \wedge b, p \wedge c \in \mathbf{m}$ and $\mathbf{p}$ is a weakly 1-absorbing prime filter, we conclude that either $(a \wedge p) \vee (b \wedge p) = p \wedge (a \vee b) \in \mathbf{p}$ or $c \wedge p \in \mathbf{p}$, and so either $a \vee b \in \mathbf{p}$ or $c \in \mathbf{p}$ by Lemma 1.1 (1), a contradiction. Therefore $\mathbf{p} = \{1\}$. □

A lattice $\pounds$ with 1 is called a $\pounds$-domain if $a \vee b = 1$ $(a, b \in \pounds)$, then either $a = 1$ or $b = 1$. Clearly, a lattice $\pounds$ is a $\pounds$-domain if and only if $\{1\}$ is a prime filter.

Proposition 3.3. *If $\mathbf{p}$ is a proper filter of a $\pounds$-domain $\pounds$, then $\mathbf{p}$ is a weakly 1-absorbing prime filter of $\pounds$ if and only if $\mathbf{p}$ is a 1-absorbing prime filter.*

Proof. One side is clear. To see the other side, assume that $\mathbf{p}$ is a weakly 1-absorbing prime filter of $\pounds$ and $a \vee b \vee c \in \mathbf{p}$ for some $a, b, c \in \pounds^*$. If $a \vee b \vee c \neq 1$, then either $a \vee b \in \mathbf{p}$ or $c \in \mathbf{p}$. So we may assume that $a \vee b \vee c = 1$ and $c \notin \mathbf{p}$. Since $a \vee b \vee c \in \{1\}$ which is a prime filter, we conclude that $a \vee b \in \{1\} \subseteq \mathbf{p}$. □

Theorem 3.3. *Let $\mathbf{q} \subseteq \mathbf{p}$ be two filter of a lattice $\pounds$. The following hold:*

(1) If $\mathbf{p}$ is a weakly 1-absorbing prime filter of $\pounds$, then $\frac{\mathbf{p}}{\mathbf{q}}$ is a weakly 1-absorbing prime filter of $\frac{\pounds}{\mathbf{q}}$;

(2) If $(\frac{\pounds}{\mathbf{q}})^* = \{a \wedge \mathbf{q} : a \in \pounds^*\}$, $\mathbf{q}$ is a weakly 1-absorbing prime filter and $\frac{\mathbf{p}}{\mathbf{q}}$ is a weakly 1-absorbing prime filter of $\frac{\pounds}{\mathbf{q}}$, then $\mathbf{p}$ is a weakly 1-absorbing prime filter.

Proof. (1) Let $1_{\frac{\pounds}{\mathbf{q}}} = 1 \wedge \mathbf{q} \neq (a \wedge \mathbf{q}) \vee_Q (b \wedge \mathbf{q}) \vee_Q (c \wedge \mathbf{q}) = (a \vee b \vee c) \wedge \mathbf{q} \in \frac{\mathbf{p}}{\mathbf{q}}$ for some $(a \wedge \mathbf{q}), (b \wedge \mathbf{q}), (c \wedge \mathbf{q}) \in (\frac{\pounds}{\mathbf{q}})^*$. This shows that $1 \neq a \vee b \vee c \in \mathbf{p}$ by Lemma 2.3. Since $a, b, c \in \pounds^*$ and $\mathbf{p}$ is a weakly 1-absorbing prime filter of $\pounds$, we conclude that either $a \vee b \in \mathbf{p}$ or $c \in \mathbf{p}$ which implies that $(a \wedge \mathbf{q}) \vee_Q (b \wedge \mathbf{q}) \in \frac{\mathbf{p}}{\mathbf{q}}$ or $c \wedge \mathbf{q} \in \frac{\mathbf{p}}{\mathbf{q}}$ by Lemma 2.3. Thus, $\frac{\mathbf{p}}{\mathbf{q}}$ is a weakly 1-absorbing prime filter of $\frac{\pounds}{\mathbf{q}}$.

(2) Let $1 \neq a \vee b \vee c \in \mathbf{p}$ for some a, b, c of $\pounds^*$. If $a \vee b \vee c \in \mathbf{q}$, then by assumption, either $a \vee b \in \mathbf{q} \subseteq \mathbf{p}$ or $c \in \mathbf{q} \subseteq \mathbf{p}$. So assume that $a \vee b \vee c \notin \mathbf{q}$. This implies that $1 \wedge \mathbf{q} \neq (a \vee b \vee c) \wedge \mathbf{q} = (a \wedge \mathbf{q}) \vee_Q (b \wedge \mathbf{q}) \vee_Q (c \wedge \mathbf{q}) \in \frac{\mathbf{p}}{\mathbf{q}}$ by Lemma 2.3. By the hypothesis, $a \wedge \mathbf{q}, b \wedge \mathbf{q}, c \wedge \mathbf{q} \in (\frac{\pounds}{\mathbf{q}})^*$. Since $\frac{\mathbf{p}}{\mathbf{q}}$ is a weakly 1-absorbing prime filter, we obtain either $(a \wedge \mathbf{q}) \vee_Q (b \wedge \mathbf{q}) = (a \vee b) \wedge \mathbf{q} \in \frac{\mathbf{p}}{\mathbf{q}}$ or $c \wedge \mathbf{q} \in \frac{\mathbf{p}}{\mathbf{q}}$ and this gives $a \vee b \in \mathbf{p}$ or $c \in \mathbf{p}$, as required. □

In the following theorem, we give a condition under which a weakly 1-absorbing prime filter of $\pounds$ is not a 1-absorbing prime filter.

Theorem 3.4. *Let $\mathbf{p}$ be a weakly 1-absorbing prime filter of a local lattice $\pounds$ and there exist $a, b, c \in \pounds^*$ such that $a \vee b \vee c = 1$, $a \vee b \notin \{1\}$ and $c \notin \{1\}$. Then $\mathbf{p}$ is not a 1-absorbing prime filter if and only if $\mathbf{p} = \{1\}$.*

Proof. If $\mathbf{p}$ is not a 1-absorbing prime filter, then $\mathbf{p} = \{1\}$ by Theorem 3.2. Conversely, assume that $\mathbf{p} = \{1\}$. By the hypothesis, $a \vee b \vee c \in \mathbf{p}$ and $a \vee b, c \notin \mathbf{p}$ gives $\mathbf{p}$ is not a 1-absorbing prime filter. □

The next result determines the class of non-local lattices for which their weakly 1-absorbing prime filters and weakly prime filters are the same.

THEOREM 3.5. *Suppose that $\pounds$ is a non-local lattice and* $\mathbf{p}$ *a proper filter of $\pounds$ having the property that* $(1 :_{\pounds} x)$ *is not a maximal filter of $\pounds$ for every $x \in \mathbf{p}$. Then* $\mathbf{p}$ *is a weakly* 1*-absorbing prime filter if and only if* $\mathbf{p}$ *is a weakly prime filter.*

PROOF. One side is clear. To see the other side, assume that $\mathbf{p}$ is a weakly 1-absorbing prime filter of $\pounds$ and let $1 \neq a \vee b \in \mathbf{p}$ for some $a, b \in \pounds$. We may assume that $a, b \in \pounds^*$. Since $a \vee b \neq 1$ and $0 \notin (1 :_{\pounds} a \vee b)$, we conclude that $(1 :_{\pounds} a \vee b)$ is a nontrivial filter of $\pounds$, and so $(1 :_{\pounds} a \vee b) \subseteq \mathbf{m}$ for some maximal filter $\mathbf{m}$ of $\pounds$. But $\pounds$ is a non-local lattice, so there exists a maximal filter $\mathbf{m}'$ of $\pounds$ such that $\mathbf{m} \neq \mathbf{m}'$. Consider $m' \in \mathbf{m}' \setminus \mathbf{m}$. Then $m' \notin (1 :_{\pounds} a \vee b)$ which implies that $1 \neq a \vee b \vee m' \in \mathbf{p}$. Since $\mathbf{p}$ is a weakly 1-absorbing prime filer of $\pounds$, either $a \vee m' \in \mathbf{p}$ or $b \in \mathbf{p}$. If $b \in \mathbf{p}$, then we are done. So suppose that $b \notin \mathbf{p}$. Therefore $a \vee m' \in \mathbf{p}$. Since $m' \notin \mathbf{m}$, $T(\{m'\}) \wedge \mathbf{m} = \pounds$ by maximality of $\mathbf{m}$; so $0 = m \wedge (m' \vee t)$ for some $t \in \pounds$ and $m \in \mathbf{m}$. We split the proof into two cases.

Case 1. $m \notin (1 :_{\pounds} a \vee b)$. Then $1 \neq a \vee b \vee m \in \mathbf{p}$. Since $\mathbf{p}$ is a weakly 1-absorbing prime filter of $\pounds$ and $b \notin \mathbf{p}$, we conclude that $a \vee m \in \mathbf{p}$. But $a = a \vee 0 = (a \vee m) \wedge (a \vee m' \vee t) \in \mathbf{p}$ since $a \vee m, a \vee m' \in \mathbf{p}$ and $\mathbf{p}$ is a filter, and so $\mathbf{p}$ is a weakly prime filter.

Case 2. $m \in (1 :_{\pounds} a \vee b)$. Then $a \vee b \vee m = 1$. Since $(1 :_{\pounds} a \vee b)$ is not a maximal filter of $\pounds$ and $(1 :_{\pounds} a \vee b) \subsetneqq \mathbf{m}$, there exists an element $s \in \mathbf{m} \setminus (1 :_{\pounds} a \vee b)$. Therefor $1 \neq a \vee b \vee s \in \mathbf{p}$ and $b \notin \mathbf{p}$ gives $a \vee s \in \mathbf{p}$, as $\mathbf{p}$ is a weakly 1-absorbing prime filter. Since $m, s \in \mathbf{m}$, we have $s \wedge m \neq 0$. It follows that $(m \wedge s) \vee (a \vee b) = (a \vee b \vee m) \wedge (a \vee b \vee s) = a \vee b \vee s \neq 1$ and $(a \vee b) \vee (m \wedge s) \in \underline{p}$. Now, since $\mathbf{p}$ is a weakly 1-absorbing prime filter of $\pounds$ and $b \notin \mathbf{p}$, we have $a \vee (m \wedge s) = (a \vee m) \wedge (a \vee s) \in \mathbf{p}$; so $a \vee m, a \vee s \in \mathbf{p}$ by Lemma 1.1 (1). But $a = a \vee 0 = (a \vee m) \wedge (a \vee m' \vee t) \in \mathbf{p}$, as $\mathbf{p}$ is a filter and therefore $\mathbf{p}$ is a weakly prime filter of $\pounds$. □

Let $\pounds_1$ and $\pounds_2$ be two lattices and $f : \pounds_1 \to \pounds_2$ be a lattice homomorphism such that $f(1) = 1$. Then it is easy to see that $\mathrm{Ker}(f) = \{x \in \pounds_1 : f(x) = 1\}$ is a filter of $\pounds_1$.

THEOREM 3.6. *Let $\pounds_1$ and $\pounds_2$ be two lattices and $f : \pounds_1 \to \pounds_2$ be a lattice homomorphism such that $f(1) = 1$ and $f(a)$ is non-zero in $\pounds_2$ for every non-zero element a of $\pounds_1$. The following statements hold:*

(1) If f is monomorphism and $\mathbf{p}$ *is a weakly* 1*-absorbing prime filter of $\pounds_2$, then* $f^{-1}(\mathbf{p}))$ *is a weakly* 1*-absorbing prime filter of $\pounds_1$.*

(2) If $\pounds_1$ is a complemented lattice, f is onto and $\mathbf{p}'$ *is a weakly* 1*-absorbing prime filter of $\pounds_1$ with* $\mathrm{Ker}(f) \subseteq \mathbf{p}'$*, then* $f(\mathbf{p}')$ *is a weakly* 1*-absorbing prime filter.*

PROOF. (1) Let $x, y \in f^{-1}(\mathbf{p})$ and $t \in \pounds_1$. Then $f(x \wedge y) = f(x) \wedge f(y)$, $f(x \vee t) = f(x) \vee f(t) \in \mathbf{p}$ gives $x \wedge y, x \vee t \in f^{-1}(\mathbf{p})$, as $\mathbf{p}$ is a filter. Thus $f^{-1}(\mathbf{p})$ is a filter of $\pounds_1$. Suppose that $1 \neq a \vee b \vee c \in f^{-1}(\mathbf{p})$ for some $a, b, c \in \pounds_1^*$. Then by assumption, $f(a \vee b \vee c) = f(a) \vee f(b) \vee f(c) \in \mathbf{p}$ for some $f(a), f(b), f(c) \in \pounds_2^*$. As f is monomorphism, we have $f(a \vee b \vee c) \neq 1$. Since $\mathbf{p}$ is a weakly 1-absorbing prime filter of $\pounds_2$, we conclude that $f(a) \vee f(b) = f(a \vee b) \in \mathbf{p}$ or $f(c) \in \mathbf{p}$ and this implies that $a \vee b \in f^{-1}(\mathbf{p})$ or $c \in f^{-1}(\mathbf{p})$.

(2) Clearly, $f(\mathbf{p}')$ is a filter of $\pounds_2$. Suppose that $1 \neq x \vee y \vee z \in f(\mathbf{p}')$ for some $x, y, z \in \pounds_2^*$. Then there are $a, b, c \in \pounds_1^*$ such that $x = f(a)$, $y = f(b)$ and $z = f(c)$. Therefore $1 \neq f(a \vee b \vee c) = f(a) \vee f(b) \vee f(c) = x \vee y \vee z \in f(\mathbf{p}')$; so $f(a \vee b \vee c) = f(d)$ for some $d \in \mathbf{p}'$. By the hypothesis, there exists $e \in \pounds_1$ such that $e \vee d = 1$ and $d \wedge e = 0$. Set $u = a \vee b \vee c$ (so $u \vee d \in \mathbf{p}'$). Then $f(u \vee e) = f(u) \vee f(e) = f(d) \vee f(e) = f(1) = 1$; hence $u \vee e \in \text{Ker}(f) \subseteq \mathbf{p}'$. Now $\mathbf{p}'$ is a filter gives $(u \vee d) \wedge (u \vee e) = u \in \mathbf{p}'$. Therefore $a \vee b \in \mathbf{p}'$ or $c \in \mathbf{p}'$, and so $x \vee y \in f(\mathbf{p}')$ or $z \in f(\mathbf{p}')$. Hence $f(\mathbf{p}')$ is a 1-absorbing prime filter of $\pounds_2$. $\square$

LEMMA 3.1. *If $\pounds = \pounds_1 \times \pounds_2 \times \pounds_3$ is a decomposable lattice, then every filter of $\pounds$ is of the form $F_1 \times F_2 \times F_3$, where for each $i \in \{1, 2, 3\}$, F_i is a filter of $\pounds_i$.*

PROOF. Let F be a filter of $\pounds$, $F_1 = \{x_1 \in \pounds_1 : (x_1, x_2, x_3) \in F$ for some $x_2 \in \pounds_2, x_3 \in \pounds_3\}$, $F_2 = \{x_2 \in \pounds_2 : (x_1, x_2, x_3) \in F$ for some $x_1 \in \pounds_1, x_3 \in \pounds_3\}$ and $F_3 = \{x_3 \in \pounds_3 : (x_1, x_2, x_3) \in F$ for some $x_1 \in \pounds_1, x_2 \in \pounds_2\}$. It is easy to see that for each $i \in \{1, 2, 3\}$, F_i is a filter of $\pounds_i$. We claim that $F = F_1 \times F_2 \times F_3$. Since the inclusion $F \subseteq F_1 \times F_2 \times F_3$ is clear, we will prove the reverse inclusion. Let $(x_1, x_2, x_3) \in F_1 \times F_2 \times F_3$. Then there exist $u_2, y_2 \in \pounds_2$, $u_1, z_1 \in \pounds_1$ and $z_3, y_3 \in \pounds_3$ such that $(x_1, y_2, y_3)), (z_1, x_2, z_3), (u_1, u_2, x_3) \in F$. Since F is a filter, $(0, 1, 1) \vee_c (x_1, y_2, y_3) = (x_1, 1, 1) \in F$. Similarly, $(1, x_2, 1), (1, 1, x_3) \in F$ which gives $(x_1, 1, 1) \wedge_c (1, x_2, 1) \wedge_c (1, 1, x_3) = (x_1, x_2, x_3) \in F$, and so we have equality. $\square$

In the following results we show that weakly 1-absorbing prime filters are really of interest in indecomposable lattices.

PROPOSITION 3.4. *Suppose that $\pounds = \pounds_1 \times \pounds_2$ is a decomposable lattice and $\mathbf{p}$ is a proper filter of $\pounds_1$. Then the following statements are equivalent.*

(1) $\mathbf{p} \times \pounds_2$ is a weakly 1-absorbing prime filter of $\pounds$;

(2) $\mathbf{p}$ is a 1-absorbing prime filter of $\pounds_1$;

(3) $\mathbf{p} \times \pounds_2$ is a 1-absorbing prime filter of $\pounds$.

PROOF. (1) $\Rightarrow$ (2) Suppose that $x \vee y \vee z \in \mathbf{p}$ for some $x, y, z \in \pounds_1^*$. If $1 \neq s \in \pounds_2$, then $(1, 1) \neq (x, 0) \vee_c (y, 0) \vee_c (z, s) = (x \vee y \vee z, s) \in \mathbf{p} \times \pounds_2$, and so either $(x, 0) \vee_c (y, 0) = (x \vee y, 0) \in \mathbf{p} \times \pounds_2$ or $(z, s) \in \mathbf{p} \times \pounds_2$. Hence, either $x \vee y \in \mathbf{p}$ or $z \in \mathbf{p}$. The implications (2) $\Rightarrow$ (3) and (3) $\Rightarrow$ (1) are clear. $\square$

In the next theorem, we provide an example of lattices for which their 1-absorbing prime filters and weakly 1-absorbing prime filters are the same.

THEOREM 3.7. *Suppose that $\pounds = \pounds_1 \times \pounds_2 \times \pounds_3$ is a decomposable lattice and $\mathbf{p}$ is a nontrivial filter of $\pounds$. Then $\mathbf{p}$ is a weakly 1-absorbing prime filter if and only if $\mathbf{p}$ is a 1-absorbing prime filter.*

PROOF. One side is clear. To see the other side, assume that $\mathbf{p} \neq \{1\}$ is a weakly 1-absorbing prime filter of $\pounds$. By Lemma 3.1, $\mathbf{p} = \mathbf{p_1} \times \mathbf{p_2} \times \mathbf{p_3}$, where $\mathbf{p_i}$ is a filter of $\pounds_i$ for each $i \in \{1, 2, 3\}$. Then there exists an element $(1, 1, 1) \neq (x, y, z) \in \mathbf{p}$. Since $(x, 0, 0) \vee_c (0, y, 0) \vee_c (0, 0, z) = (x, y, z) \in \mathbf{p}$ and $\mathbf{p}$ is a weakly

1-absorbing prime filter, we conclude that either $(x,0,0)\vee_c(0,y,0)=(x,y,0)\in\mathbf{p}$ or $(0,0,z)\in\mathbf{p}$. Therefore either $\mathbf{p_3}=\pounds_3$ or $\mathbf{p_1}=\pounds_1$ and $\mathbf{p_2}=\pounds_2$, and so $\mathbf{p}=\mathbf{p_1}\times\mathbf{p_2}\times\pounds_3$ or $\mathbf{p}=\pounds_1\times\pounds_2\times\mathbf{p_3}$. Hence, by Proposition 3.4, $\mathbf{p}$ is a 1-absorbing prime filter of $\pounds$. □

THEOREM 3.8. *Assume that $\pounds=\pounds_1\times\pounds_2$ is a decomposable lattice such that $\pounds_1,\pounds_2$ are not simples and let $\mathbf{p}$ be a nontrivial filter of $\pounds$. The following statements are equivalent.*

(1) $\mathbf{p}$ is a weakly 1-absorbing prime filter of $\pounds$;

(2) $\mathbf{p}=\mathbf{p_1}\times\pounds_2$ for some prime filter $\mathbf{p_1}$ of $\pounds_1$ or $\mathbf{p}=\pounds_1\times\mathbf{p_2}$ for some prime filter $\mathbf{p_2}$ of $\pounds_2$;

(3) $\mathbf{p}$ is a prime filter of $\pounds$;

(4) $\mathbf{p}$ is a weakly prime filter of $\pounds$;

(5) $\mathbf{p}$ is a 1-absorbing prime filter of $\pounds$.

PROOF. (1) ⇒ (2) Let $\mathbf{p}$ be a nontrivial filter of $\pounds$. Set $\mathbf{p_1}=\{x_1\in\pounds_1:(x_1,y_2)\in\mathbf{p}$ for some $y_2\in\pounds_2\}$ and $\mathbf{p_2}=\{y_2\in\pounds_2:(x_1,y_2)\in\mathbf{p}$ for some $x_1\in\pounds_1\}$. Then $\mathbf{p_1}$ and $\mathbf{p_2}$ are filters of $\pounds_1$ and $\pounds_2$, respectively with $\mathbf{p}=\mathbf{p_1}\times\mathbf{p_2}$ by Lemma 3.1. Since $\mathbf{p}\neq\{1\}$, either $\mathbf{p_1}\neq\{1\}$ or $\mathbf{p_2}\neq\{1\}$. Without loss of generality, we may assume that $\mathbf{p_1}\neq\{1\}$. So there is an element $x\in\mathbf{p_1}$ such that $x\neq 1$. Since $\mathbf{p}$ is a weakly 1-absorbing prime filter and $(1,1)\neq(0,1)\vee_c(0,1)\vee_c(x,0)=(x,1)\in\mathbf{p}$, we conclude that either $(0,1)\in\mathbf{p}$ or $(x,0)\in\mathbf{p}$ which implies that $\mathbf{p_1}=\pounds_1$ or $\mathbf{p_2}=\pounds_2$. Suppose that $\mathbf{p_1}=\pounds_1$. Now we will show that $\mathbf{p_2}$ is a prime filter of $\pounds_2$. Let $a\vee b\in\mathbf{p_2}$ for some $a,b\in\pounds_2$. If $a=0$ or $b=0$, then we are done. So suppose that $a,b\in\pounds_2^*$. Since $\pounds_1$ is not a simple lattice, there exists a non-zero element $s\in\pounds_1$ such that $s\neq 1$. This implies that $(1,1)\neq(s,0)\vee_c(0,a)\vee_c(0,b)=(s,a\vee b)\in\mathbf{p}$. Since $\mathbf{p}$ is a weakly 1-absorbing prime filter, we conclude that either $(s,0)\vee_c(0,a)=(s,a)\in\mathbf{p}$ or $(0,b)\in\mathbf{p}$. Therefore we obtain that $a\in\mathbf{p_2}$ or $b\in\mathbf{p_2}$ and so $\mathbf{p_2}$ is a prime filter of $\pounds_2$. Similarly, we can show that $\mathbf{p}=\mathbf{p_1}\times\pounds_2$ and $\mathbf{p_1}$ is a prime filter of $\pounds_1$.

(2) ⇒ (3) Without loss of generality, we may assume that $\mathbf{p}=\mathbf{p_1}\times\pounds_2$. Let $(x,y)\vee_c(x',y')=(x\vee x',y\vee y')\in\mathbf{p}$ for some $(x,y),(x',y')\in\pounds$. Then $x\vee x'\in\mathbf{p_1}$ gives either $x\in\mathbf{p_1}$ or $x'\in\mathbf{p_1}$ which implies that either $(x,y)\in\mathbf{p}$ or $(x',y')\in\mathbf{p}$.

(3) ⇔ (4) Clearly, every prime filter is a weakly prime filter. Conversely, suppose that $\{1\}\neq\mathbf{p}=\mathbf{p_1}\times\mathbf{p_2}$ is a weakly prime filter of $\pounds$. Consider $(1,1)\neq(a,b)\in\mathbf{p}$. Then $(a,0)\vee_c(0,b)=(a,b)\in\mathbf{p}$ gives $(a,0)\in\mathbf{p}$ or $(0,b)\in\mathbf{P}$. Suppose that $(a,0)\in\mathbf{p}$. Since $0\in\mathbf{p_2}$, we conclude that $\mathbf{p_2}=\pounds_2$ and so $\mathbf{p}=\mathbf{p_1}\times\pounds_2$. We show that $\mathbf{p_1}$ is a prime filter of $\pounds_1$; hence $\mathbf{p}$ is a prime filter of $\pounds$. Let $x\vee y\in\mathbf{p_1}$, where $x,y\in\pounds_1$. If $x=1$ or $y=1$, then we are done. So assume that $x\neq 1$ and $y\neq 1$. Then $(1,1)\neq(x,0)\vee_c(y,0)=(x\vee y,0)\in\mathbf{p}$, so $(x,0)\in\mathbf{p}$ or $(y,0)\in\mathbf{p}$ and hence $x\in\mathbf{p_1}$ or $y\in\mathbf{p_1}$. The case where $(0,b)\in\mathbf{p}$ is similar. The implication (3) ⇒ (5) is clear by definition of a 1-absorbing prime filter. The implication (5) ⇒ (1) is clear by definition of a weakly 1-absorbing prime filter. □

THEOREM 3.9. *Let $\pounds=\pounds_1\times\pounds_2$ be a decomposable lattice, $\mathbf{p_1}$ is a nontrivial filter of $\pounds_1$ and $\mathbf{p_2}$ is a proper filter of $\pounds_2$. If $\mathbf{p_1}\times\mathbf{p_2}$ is a weakly 1-absorbing prime* *filter of $\pounds$ that is not a* 1*-absorbing prime filter, then* $\mathbf{p_1}$ *is a weakly prime filer of* $\pounds_1$ *that is not a prime filter and* $\mathbf{p_2} = \{1\}$ *is a prime filter of* $\pounds_2$.

Proof. Suppose that $\mathbf{p_1} \times \mathbf{p_2}$ has the stated property and $\mathbf{p_2} \neq \{1\}$. Therefore, by Theorem 3.8, $\mathbf{p_1} \times \mathbf{p_2}$ is a 1-absorbing prime filter of $\pounds$ which is impossible, and so $\mathbf{p_2} = \{1\}$. Suppose that $a \vee b \in \mathbf{p_2}$ for some $a, b \in \pounds_2$. Consider $1 \neq c \in \mathbf{p_1}$. Then $(1,1) \neq (c,0) \vee_c (0,a) \vee_c (0,b) = (c, a \vee b) \in \mathbf{p_1} \times \mathbf{p_2}$. But $\mathbf{p_1}$ is a proper filter gives $(0,a),(0,b) \notin \mathbf{p_1} \times \mathbf{p_2}$. We may assume that $a, b \in \pounds_2^*$ and $c \in \pounds_1^*$. Since $\mathbf{p_1} \times \mathbf{p_2}$ is a weakly 1-absorbing prime filter, we conclude that $(c,0) \vee_c (0,a) = (c,a) \in \mathbf{p_1} \times \mathbf{p_2}$; hence $\mathbf{p_2} = \{1\}$ is a prime filter of $\pounds_2$. Now, we show that $\mathbf{p_1}$ is a weakly prime filter of $\pounds_1$. Let $1 \neq a \vee b \in \mathbf{p_1}$ for some $a, b \in \pounds_1$. We can assume that $a, b \in \pounds_1^*$. Since $(1,1) \neq (b,0) \vee_c (0,1) \vee_c (a \vee b, 0) = (a \vee b, 1) \in \mathbf{p_1} \times \mathbf{p_2}$, $(a \vee b, 0) \notin \mathbf{p_1} \times \mathbf{p_2} = \mathbf{p_1} \times \{1\}$ and $\mathbf{p_1} \times \mathbf{p_2}$ is a weakly 1-absorbing prime filter, we conclude that $(b,1) \in \mathbf{p_1} \times \{1\}$, and so $b \in \mathbf{p_1}$. Hence $\mathbf{p_1}$ is a weakly prime filter of $\pounds_1$. It remains to show that $\mathbf{p_1}$ is not a prime filter. On the contrary, assume that $\mathbf{p_1}$ is a prime filter. Since $\mathbf{p_1}$ is a nontrivial filter, there is an element $x \in \mathbf{p_1}$ such that $x \neq 1$. Then $(1,1) \neq (x,0) \vee_c (x,0) \vee_c (0,1) = (x,1) \in \mathbf{p_1} \times \mathbf{p_2}$ gives either $(x,0) \vee_c (x,0) = (x,0) \in \mathbf{p_1} \times \mathbf{p_2}$ or $(0,1) \in \mathbf{p_1} \times \mathbf{p_2}$ which is impossible. □

The following remark shows that the converse of Theorem 3.9 need not be true.

Remark 3.1. Let $\pounds, \pounds_1, \pounds_2$ and $\mathbf{p_1}, \mathbf{p_2}$ be as in Theorem 3.9. Suppose that $\mathbf{p_1}$ is a weakly prime filter of $\pounds_1$ that is not a prime filter and $\mathbf{p_2} = \{1\}$ is a prime filter of $\pounds_2$. We claim that $\mathbf{p_1} \times \mathbf{p_2}$ need not be a weakly 1-absorbing prime filter of $\pounds$. Since $\mathbf{p_1}$ is a nontrivial filter, there is an element $x \in \mathbf{p_1}$ such that $a \neq 1$, and so $(1,1) \neq (a,0) \vee_c (a,0) \vee_c (0,1) = (a,1) \in \mathbf{p_1} \times \mathbf{p_2}$. Since $(a,0) \vee_c (a,0) = (a,0) \notin \mathbf{p_1} \times \{1\}$ and $(0,1) \notin \mathbf{p_1} \times \{1\}$ (as $\mathbf{p_1}$ is nontrivial), we conclude that $\mathbf{p_1} \times \mathbf{p_2}$ is not a weakly 1-absorbing prime filter of $\pounds$.

Theorem 3.10. *Let $\pounds = \pounds_1 \times \pounds_2 \times \cdots \times \pounds_n$ be a decomposable filter ($n \geqslant 2$). The following statements are equivalent.*

(1) Every proper filter of $\pounds$ is a weakly 1-absorbing prime filter;

(2) $n = 2$ and for each $i \in \{1,2\}$, $\pounds_i$ is a simple $\pounds_i$-domain.

Proof. (1) ⇒ (2) On the contrary, assume that $n \geqslant 3$. Set

$$\mathbf{p} = \{1\} \times \{1\} \times \times \pounds_3 \times \cdots \times \pounds_n.$$

Consider $1 \neq a \in \pounds_3$. Since $(1,1,\cdots,1) \neq (0,1,0,0,\cdots,0) \vee_c (0,1,0,0,\cdots,0) \vee_c (1,0,a,0,0,\cdots 0) = (1,1,a,0,0,\cdots,0) \in \mathbf{p}$ and $\mathbf{p}$ is a weakly 1-absorbing prime filter by (1), we conclude either $(0,1,0,0,\cdots,0) \in \mathbf{p}$ or $(1,0,a,0,0,\cdots,0) \in \mathbf{p}$ which both of them are contradictions. Hence $n = 2$ and $\pounds = \pounds_1 \times \pounds_2$. Now, we will show that for each $i \in \{1,2\}$, $\pounds_i$ is a simple $\pounds_i$-domain. Now, put $\mathbf{q}' = \{1\} \times \pounds_2$. Since $\mathbf{q}'$ is a weakly 1-absorbing prime filter of $\pounds$, by Theorem 3.8, $\{1\}$ is a prime filter of $\pounds_1$, i.e. $\pounds_1$ is a $\pounds_1$-domain. Similarly, $\pounds_2$ is a $\pounds_2$-domain. Let $\{1\} \subsetneqq S \subsetneqq \pounds_1$ for some filter S of $\pounds_1$. Choose an element s of S with $s \neq 1$. Suppose that $\mathbf{q} = S \times \{1\}$. Since $\mathbf{q}$ is a weakly 1-absorbing prime filter by (1), $(1,1) \neq (s,0) \vee_c (s,0) \vee_c (0,1) = (s,1) \in \mathbf{q}$ and both $(s,0) \vee_c (s,0) = (s,0) \notin \mathbf{q}$ and

$(0,1) \notin \mathbf{q}$, we have a contradiction. Hence $\pounds_1$ is a simple lattice. Likewise, $\pounds_2$ is a simple lattice.

(2) $\Rightarrow$ (1) Suppose that $n = 2$ and for each $i \in \{1,2\}$, $\pounds_i$ is a simple $\pounds_i$-domain. Let $\mathbf{p}$ be a proper filter of $\pounds$. Then $\pounds$ has exactly three proper filters, i.e., $\{1\} \times \{1\}$, $\{1\} \times \pounds_2$ and $\pounds_1 \times \{1\}$. If $\mathbf{p} = \{1\} \times \pounds_2$ or $\mathbf{p} = \pounds_1 \times \{1\}$, then $\mathbf{p}$ is a weakly 1-absorbing prime filter by Theorem 3.8. If $\mathbf{p} = \{1\} \times \{1\}$, then $\mathbf{p}$ is trivially a weakly 1-absorbing prime filter of $\pounds$. □

Lemma 3.2. *If $\mathbf{m}$ is a maximal filter of $\pounds$, then $\frac{\pounds}{\mathbf{m}}$ is a simple $\pounds$-domain.*

Proof. Let $H \neq 1 \wedge \mathbf{m} = \{\bar{1}\}$ be a filter of $\frac{\pounds}{\mathbf{m}}$. Then $H = \frac{K}{\mathbf{m}}$ for some filter $\mathbf{m} \subsetneqq K$ of $\pounds$ by Lemma 2.3 (6); hence $K = \pounds$ and so $\frac{\pounds}{\mathbf{m}}$ is a simple lattice. Let $(a \wedge \mathbf{m}) \vee_Q (b \wedge \mathbf{m}) = (a \vee b) \wedge \mathbf{m} = 1 \wedge \mathbf{m}$ for some $a, b \in \pounds$. Since $\mathbf{m}$ is a prime filter, we conclude that $a \in \mathbf{m}$ or $b \in \mathbf{m}$ by Lemma 2.3 (2); so either $a \wedge \mathbf{m} = 1 \wedge \mathbf{m}$ or $b \wedge \mathbf{m} = 1 \wedge \mathbf{m}$. Thus $\frac{\pounds}{\mathbf{m}}$ is a $\pounds$-domain. □

We close this section with the following theorem:

Theorem 3.11. *If every proper filter of a lattice $\pounds$ is a weakly 1-absorbing prime, then $|\mathrm{Max}(\pounds)| \leqslant 2$.*

Proof. Let $\pounds$ be a lattice such that every proper filter is weakly 1-absorbing prime. On the contrary, assume that $|\mathrm{Max}(\pounds)| \geqslant 3$. We suppose that $\mathbf{m_1}, \mathbf{m_2}$ and $\mathbf{m_3}$ are distinct maximal filters of $\pounds$, and look for a contradiction. We split the proof into two cases.

Case 1: Suppose that $\mathbf{m_1} \vee \mathbf{m_2} \vee \mathbf{m_3} \neq \{1\}$. Since $\mathbf{m_1} \vee \mathbf{m_2} \vee \mathbf{m_3} \subseteq \mathbf{m_1} \vee \mathbf{m_2} \vee \mathbf{m_3}$ and $\mathbf{m_1} \vee \mathbf{m_2} \vee \mathbf{m_3}$ is weakly 1-absorbing prime filter, we conclude that either $\mathbf{m_1} \vee \mathbf{m_2} \subseteq \mathbf{m_1} \vee \mathbf{m_2} \vee \mathbf{m_3} \subseteq \mathbf{m_3}$ or $\mathbf{m_3} \subseteq \mathbf{m_1} \vee \mathbf{m_2} \vee \mathbf{m_3} \subseteq \mathbf{m_1}$ by Theorem 3.1 (6). This shows that either $\mathbf{m_1} = \mathbf{m_3}$ or $\mathbf{m_2} = \mathbf{m_3}$ which is impossible.

Case 2: Suppose that $\mathbf{m_1} \vee \mathbf{m_2} \vee \mathbf{m_3} = \mathbf{m_1} \cap \mathbf{m_2} \cap \mathbf{m_3} = \{1\}$. Then by Corollary 2.1, $\pounds$ is isomorphic to $\frac{\pounds}{\mathbf{m_1}} \times \frac{\pounds}{\mathbf{m_2}} \times \frac{\pounds}{\mathbf{m_3}}$. Take $\pounds = \pounds_1 \times \pounds_2 \times \pounds_3$, where for each $i \in \{1,2,3\}$, $\pounds_i$ is a simple $\pounds_i$-domain by Lemma 3.2. Then by Theorem 3.10, we conclude that $\pounds_1 = \{1\}$ or $\pounds_2 = \{1\}$ or $\pounds_3 = \{1\}$ which is impossible. Hence, $|\mathrm{Max}(\pounds)| \leqslant 2$. □

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