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PSEUDO-PARALLEL HYPERSURFACES IN SASAKIAN SPACE FORMS

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Abstract. The main purpose of this work is to investigate the existence of pseudo-parallel hypersurfaces in $(2n+1)$-dimensional Sasakian space form $\widetilde{M}^{2n+1}$ of constant φ-sectional curvature c.

1. Introduction

Given an isometric immersion $f : M \longrightarrow \widetilde{M}$, let σ be the second fundamental form and $\widetilde{\nabla}$ the van der Waerden-Bortolloti connection of $\widetilde{M}$. Then Deprez [**4, 5**] defined the immersion to be *semi-parallel* if $\widetilde{R}(X,Y).\sigma = (\widetilde{\nabla}_X\widetilde{\nabla}_Y - \widetilde{\nabla}_Y\widetilde{\nabla}_X - \widetilde{\nabla}_{[X,Y]})\sigma = 0$, holds for any vectors X, Y tangent to M.
In [**6**], authors obtained some results on hypersurfaces in 4-dimension space $N^4(c)$ satisfying the curvature condition

$$\widetilde{R}\,.\,\sigma = L\,Q(g,\sigma)\,, \tag{1.1}$$

where L is some smooth function on M and $Q(g,\sigma)$ is a $(0,4)$-tensor on M determined by $Q(g,\sigma)(Z,W;X,Y) = ((X\wedge Y)\,.\,\sigma)(Z,W)$, where $X\wedge Y$ is an endomorphism given by $(X\wedge Y)Z = g(Y,Z)X - g(X,Z)Y$, for all $X,Y,Z \in TM$.
The submanifolds satisfying (1.1) are called *pseudo-parallel* [**1, 2**].
In [**2**], authors showed that a pseudo-parallel hypersurface of a space form is either quasi-umbilical or cyclic of Dupin.
In the present study, we consider pseudo-parallel hypersurfaces in a Sasakian space form $\widetilde{M}^{2n+1}(c)$ of constant φ-sectional curvature.

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1.1. Preliminaries. Let M be a submanifold of a Riemannian manifold $\widetilde{M}$ with a Riemannian metric g. Then, the Gauss-Weingarten formulas are given by

$$\widetilde{\nabla}_X Y = \nabla_X Y + \sigma(X,Y)\,,\;\; \widetilde{\nabla}_X V = -A_V X + \nabla^{\perp}_X V$$

$X, Y \in TM$, $V \in T^{\perp}M$, where $\widetilde{\nabla}$ (resp. ∇) is the covariant differentiation in $\widetilde{M}$ (resp. in M) and $\nabla^{\perp}$ is the connection in the normal bundle, σ is the second fundamental form of M and A_V is the Weingarten endomorphism associated with V. A_V and σ are related by

$$g(A_V X, Y) = g(\sigma(X,Y), V) = g(X, A_V Y)\,. \tag{1.2}$$

The mean curvature vector H of M is defined to be $H = \frac{1}{n}Tr(\sigma)$. A submanifold M in a Riemannian manifold is called *minimal* if its mean curvature vector vanishes identically.
The covariant derivative $\widetilde{\nabla}\sigma$ of σ is defined by

$$\widetilde{\nabla}_X \sigma(Y,Z) = \nabla^{\perp}_X(\sigma(Y,Z)) - \sigma(\nabla_X Y, Z) - \sigma(Y, \nabla_X Z),$$

where, $\widetilde{\nabla}\sigma$ is a normal bundle valued tensor of type $(0,3)$ and is called the *third fundamental form* of M. If $\widetilde{\nabla}\sigma = 0$, then M is called *parallel* [**7**].
As a parallel submanifold (in particular, totally geodesic submanifold, i.e. $\sigma = 0$) is semi-parallel it is obvious that also is pseudo-parallel.
We denote by $\widetilde{R}$ and R the curvature tensors associated with $\widetilde{\nabla}$ and ∇ respectively. Then, the Gauss equation is given by

$$\begin{aligned} g(\widetilde{R}(X,Y)Z,W) &= g(R(X,Y)Z,W) + g(\sigma(X,Z),\sigma(Y,W)) \\ &\quad - g(\sigma(X,W),\sigma(Y,Z)), \end{aligned}$$

For any vector fields Z, W on M, the curvature operator $\widetilde{R}(X,Y)$ with respect to $\widetilde{\nabla}$ and $X \wedge Y$ can be extended as derivations of tensor fields in the usual way, so

$$\begin{aligned} (\widetilde{R}(X,Y)\,.\,\sigma)(Z,W) &= R^{\perp}(X,Y)(\sigma(Z,W)) - \sigma(R(X,Y)Z,W) \\ &\quad - \sigma(Z, R(X,Y)W), \end{aligned} \tag{1.3}$$

and

$$\begin{aligned} Q(g,\sigma)(Z,W;X,Y) &= ((X \wedge Y)\,.\,\sigma)(Z,W) \\ &= -\sigma((X \wedge Y)Z,W) - \sigma(Z,(X \wedge Y)W) \\ &= -g(Y,Z)\sigma(X,W) + g(X,Z)\sigma(Y,W) \\ &\quad - g(Y,W)\sigma(Z,X) + g(X,W)\sigma(Z,Y), \end{aligned} \tag{1.4}$$

2. Submanifolds of Sasakian space forms

Let $\widetilde{M}$ be a $(2n+1)$-dimensional Sasakian manifold with structure tensors (φ, ξ, η, g), where φ is a tensor field of type $(1,1)$, ξ a unit vector field, η a one-form dual to ξ. Then they satisfy

$$\varphi^2 = -I + \eta \otimes \xi, \quad \eta \circ \varphi = 0\,, \quad \varphi\xi = 0 \tag{2.1}$$

$$g(\varphi X, \varphi Y) = g(X,Y) - \eta(X)\eta(Y), \quad d\eta(X,Y) = g(\varphi X, Y)\,, \tag{2.2}$$

$$g(\varphi X, Y) = -g(X, \varphi Y), \quad g(X, \xi) = \eta(X)\,, \tag{2.3}$$

for all $X, Y \in T\widetilde{M}$. We denote by $\widetilde{\nabla}$ the Levi-Civita connection on $\widetilde{M}$, then we have [**3**]

$$(\widetilde{\nabla}_X \varphi)Y = g(X, Y)\xi - \eta(Y)X\,, \tag{2.4}$$

$$\widetilde{\nabla}_X \xi = -\varphi X\,, \tag{2.5}$$

for any vector fields X, Y tangent to $\widetilde{M}$.
If moreover the structure is normal, that is if $[\varphi X, \varphi Y] + \varphi^2[X, Y] - \varphi[X, \varphi Y] - \varphi[\varphi X, Y] = -2d\eta(X, Y)\xi$, then the contact metric structure is called a *Sasakian structure* and $\widetilde{M}^{2n+1}$ is called a *Sasakian manifold*.
A plane section in the tangent space $T_p\widetilde{M}$ at $p \in \widetilde{M}$ is called a *φ-section* if it is spanned by a vector X orthogonal to ξ and φX. The curvature of φ-section is called *φ-sectional curvature.*
A Sasakian space form is defined as a Sasakian manifold with constant φ-sectional curvature c and is denoted by $\widetilde{M}^{2n+1}(c)$, for more details see [**3**,**12**]. The curvature tensor of a Sasakian space form $\widetilde{M}^{2n+1}(c)$ is given by [**3**]

$$\begin{aligned} \widetilde{R}(X,Y)Z &= \frac{c+3}{4}\{g(Y,Z)X - g(X,Z)Y\} + \frac{c-1}{4}\{\eta(X)\eta(Z)Y \\ &\quad -\eta(Y)\eta(Z)X + g(X,Z)\eta(Y)\xi - g(Y,Z)\eta(X)\xi \\ &\quad +g(\varphi Y,Z)\varphi X - g(\varphi X,Z)\varphi Y - 2g(\varphi X,Y)\varphi Z\}\,, \end{aligned} \tag{2.6}$$

for all $X, Y, Z \in T\widetilde{M}$.

EXAMPLE 2.1. *[3] We consider $\mathbb{R}^{2n+1}$ with the coordinates (x^i, y^i, z), $i = 1, ..., n$ and its usual contact form $\eta = \frac{1}{2}(dz - \sum_{i=1}^n y^i dx^i)$. The characteristic field ξ is given by $\xi = 2\frac{\partial}{\partial z}$, the tensor field φ is given by the matrix*

$$\begin{pmatrix} 0 & \delta_{ij} & 0 \\ -\delta_{ij} & 0 & 0 \\ 0 & y^j & 0 \end{pmatrix}$$

and the Riemannian metric $g = \eta \otimes \eta + \frac{1}{4}\sum_{i=1}^n (dx^i)^2 + (dy^i)^2$ is an associated metric for η. In this case $\mathbb{R}^{2n+1}$ is a Sasakian space form with φ-sectional curvature $c = -3$ denoted by $\mathbb{R}^{2n+1}(-3)$.

Let M^m be an m-dimensional submanifold immersed in $\widetilde{M}^{2n+1}$. For any vector field X tangent to M, we put

$$\varphi X = TX + NX\,, \tag{2.7}$$

where TX is the tangential part and NX the normal part of φX. Then T is an endomorphism on the tangent bundle TM and N is a normal bundle valued 1-form on the tangent bundle.
The submanifold M is said to be *invariant* if ξ is tangent to M and $\varphi X \in TM$, for any $X \in TM$. It is easy to show that an invariant submanifold of a Sasakian manifold is a Sasakian manifold too. If M is invariant, then NX in (2.7) vanishes

identically for all $X \in TM$.

Now, let M^{2n} be an immersed hypersurface of a Sasakian manifold $\widetilde{M}^{2n+1}$, then we have the formulas of Gauss and Weingarten:

$$\widetilde{\nabla}_X Y = \nabla_X Y + g(A_V X, Y)V,$$
$$\widetilde{\nabla}_X V = -A_V X,$$

where X and Y are tangent vector fields, V a unit normal vector field to M.

THEOREM 2.1. *[**9**] There does not exist an invariant hypersurface of a contact manifold.*

It is known that a Sasakian manifold is a normal contact metric manifold.

Let M be an hypersurface of a Sasakian space form $\widetilde{M}^{2n+1}(c)$, tangent to the structure vector field ξ. We denote by V a unit normal vector field to M and we put [**8**]

$$\zeta = -\varphi V.$$

Since V is orthogonal to M, then by using (2.2) and (2.3), we have

$$g(\zeta, \zeta) = 1, \qquad g(\zeta, V) = 0.$$

Hence ζ is a unit vector field tangent to M.
For any vector field X tangent to M, we set [**11**]

$$\varphi X = TX + u(X)V, \tag{2.8}$$

where u and T are tensor fields on M of type $(0, 1)$ and $(1, 1)$ respectively, also TX represents the tangent part of φX, and it is easily shown that T is a skew-symmetric operator.
Thus, from (2.7) we get

$$NX = u(X)V,$$

for any vector field X tangent to M.
Moreover, it is easy to verify that

$$u(X) = g(X, \zeta), \qquad \varphi\zeta = V. \tag{2.9}$$

Since the structure vector field ξ is tangent to M, (2.8) implies that

$$T\xi = 0, \qquad T\zeta = 0. \tag{2.10}$$

$$u(\xi) = 0 = \eta(\zeta), \qquad u(\zeta) = 1, \qquad u(TX) = 0. \tag{2.11}$$

$$T^2 X = -X + \eta(X)\xi + u(X)\zeta. \tag{2.12}$$

LEMMA 2.1. *Let M be an hypersurface of a Sasakian space form $\widetilde{M}^{2n+1}(c)$ tangent to the structure vector field ξ. We have*

$$A\xi = -\zeta, \tag{2.13}$$

where A is the shape operator of M.

PROOF. By using the Gauss formula, (2.5) and (2.8) we obtain

$$\begin{aligned}\overline{\nabla}_X\xi &= \nabla_X\xi + g(AX,\xi)V\\ &= -\varphi X = -TX - u(X)V.\end{aligned}$$

Considering the tangential and normal parts of the above relation and since $u(X) = g(X,\zeta)$, we get

$$\nabla_X\xi = -TX, \qquad g(AX,\xi)V = -u(X)V = g(X,A\xi)V = -g(X,\zeta)V.$$

Which implies $A\xi = -\zeta$. □

On the other hand, the Gauss formula implies that

$$-\varphi\xi = \overline{\nabla}_\xi\xi = \nabla_\xi\xi + g(A\xi,\xi)V.$$

Since $\varphi\xi = 0$, we have

$$\nabla_\xi\xi = 0, \qquad g(A\xi,\xi) = 0 \tag{2.14}$$

3. Main results

THEOREM 3.1. *There are no pseudo-parallel hypersurfaces in a Sasakian space form $\widetilde{M}^{2n+1}(c)$ tangent to the structure vector field ξ, with $c \neq 1$.*

PROOF. Assume that M^{2n} is a pseudo-parallel hypersurface in a Sasakian space form $\widetilde{M}^{2n+1}(c)$, tangent to the structure vector field ξ.
Since M is an hypersurface, and from (1.3) the $(0,4)$-tensor field $\widetilde{R}\,.\,\sigma$ is defined by

$$(\widetilde{R}\,.\,\sigma)(X,Y,Z,W) = -\sigma(R(X,Y)Z,W) - \sigma(Z,R(X,Y)W),$$

for any vector fields X, Y, Z and W tangent to M. Then (1.1) becomes

$$\sigma(R(X,Y)Z,W) + \sigma(Z,R(X,Y)W) + L\,Q(g,\sigma)(Z,W;X,Y) = 0\,. \tag{3.1}$$

The curvature tensor R of M is given by the Gauss equation

$$R(X,Y)Z = \widetilde{R}(X,Y)Z + g(AY,Z)AX - g(AX,Z)AY\,. \tag{3.2}$$

From (3.2), (2.6) and (2.8) , we obtain

$$\begin{aligned}R(X,Y)Z &= \frac{c+3}{4}\{g(Y,Z)X - g(X,Z)Y\}\\ &\quad+\frac{c-1}{4}\{\eta(X)\eta(Z)Y - \eta(Y)\eta(Z)X + g(X,Z)\eta(Y)\xi\\ &\quad-g(Y,Z)\eta(X)\xi + g(TY,Z)TX - g(TX,Z)TY - 2g(TX,Y)TZ\}\\ &\quad+g(AY,Z)AX - g(AX,Z)AY\,.\end{aligned} \tag{3.3}$$

Using (3.3) and (1.4), the condition (3.1) may be written in the form:

$$\begin{aligned}0 \quad = \quad & \frac{c+3}{4}\{g(Y,Z)g(AX,W)V - g(X,Z)g(AY,W)V + g(Y,W)g(AX,Z)V \\ & -g(X,W)g(AY,Z)V\} + \frac{c-1}{4}\{\eta(X)\eta(Z)g(AY,W)V - \eta(Y)\eta(Z)g(AX,W)V \\ & +g(X,Z)\eta(Y)g(A\xi,W)V - g(Y,Z)\eta(X)g(A\xi,W)V + g(TY,Z)g(ATX,W)V \\ & -g(TX,Z)g(ATY,W)V - 2g(TX,Y)g(ATZ,W)V + \eta(X)\eta(W)g(AY,Z)V \\ & -\eta(Y)\eta(W)g(AX,Z)V + g(X,W)\eta(Y)g(A\xi,Z)V - g(Y,W)\eta(X)g(A\xi,Z)V \\ & +g(TY,W)g(ATX,Z)V - g(TX,W)g(ATY,Z)V - 2g(TX,Y)g(ATW,Z)V\} \\ & +g(AY,Z)g(A^2X,W)V - g(AX,Z)g(A^2Y,W)V \\ & +g(AY,W)g(A^2X,Z)V - g(AX,W)g(A^2Y,Z)V + L\{-g(Y,Z)g(AX,W)V \\ & +g(X,Z)g(AY,W)V - g(Y,W)g(AX,Z)V + g(X,W)g(AY,Z)V\}\,,\end{aligned} \tag{3.4}$$

as $\sigma(X,Y) = g(AY,X)V = g(AX,Y)V$, for all X,Y,Z,W in TM and $V \in T^{\perp}M$. Since T is a skew-symmetric operator, then $g(ATW,Z) = -g(W,TAZ)$ and the equation (3.4) becomes

$$\begin{aligned}0 \quad = \quad & (\frac{c+3}{4} - L)\{g(Y,Z)AX - g(X,Z)AY + g(AX,Z)Y - g(AY,Z)X\} \\ & +\frac{c-1}{4}\{\eta(X)\eta(Z)AY - \eta(Y)\eta(Z)AX + g(X,Z)\eta(Y)A\xi \\ & -g(Y,Z)\eta(X)A\xi + g(TY,Z)ATX - g(TX,Z)ATY \\ & -2g(TX,Y)ATZ + \eta(X)g(AY,Z)\xi - \eta(Y)g(AX,Z)\xi \\ & +\eta(Y)g(A\xi,Z)X - \eta(X)g(A\xi,Z)Y + g(ATX,Z)TY \\ & -g(ATY,Z)TX + 2g(TX,Y)TAZ\} \\ & +g(AY,Z)A^2X - g(AX,Z)A^2Y + g(A^2X,Z)AY - g(A^2Y,Z)AX\,.\end{aligned} \tag{3.5}$$

If we set $Y = Z$ in (3.5), we obtain

$$\begin{aligned}0 \quad = \quad & (\frac{c+3}{4} - L)\{g(Z,Z)AX - g(X,Z)AZ + g(AX,Z)Z - g(AZ,Z)X\} \\ & +\frac{c-1}{4}\{\eta(X)\eta(Z)AZ - \eta(Z)\eta(Z)AX + g(X,Z)\eta(Z)A\xi - g(Z,Z)\eta(X)A\xi \\ & -3g(TX,Z)ATZ + \eta(X)g(AZ,Z)\xi - \eta(Z)g(AX,Z)\xi + \eta(Z)g(A\xi,Z)X \\ & -\eta(X)g(A\xi,Z)Z + g(ATX,Z)TZ - g(ATZ,Z)TX + 2g(TX,Z)TAZ\} \\ & +g(AZ,Z)A^2X - g(AX,Z)A^2Z + g(A^2X,Z)AZ - g(A^2Z,Z)AX\,,\end{aligned} \tag{3.6}$$

since T is skew-symmetric we have $g(TZ, Z) = 0$.
If Z and ξ are orthogonal in (3.6), we have

$$\begin{aligned} 0 = {} & (\frac{c+3}{4} - L)\{g(Z,Z)AX - g(X,Z)AZ + g(AX,Z)Z - g(AZ,Z)X\} \\ & + \frac{c-1}{4}\{-g(Z,Z)\eta(X)A\xi - 3g(TX,Z)ATZ + \eta(X)g(AZ,Z)\xi \\ & -\eta(X)g(A\xi,Z)Z + g(ATX,Z)TZ - g(ATZ,Z)TX + 2g(TX,Z)TAZ\} \\ & + g(AZ,Z)A^2X - g(AX,Z)A^2Z + g(A^2X,Z)AZ - g(A^2Z,Z)AX \, . \end{aligned} \tag{3.7}$$

We consider $\{e_1, ..., e_{2n}\}$ a local orthonormal frame of TM. We insert $Z = e_k$ in (3.7) and taking summation over $k = 1, ..., 2n$, we get

$$\begin{aligned} 0 = {} & (\frac{c+3}{4} - L)\{2n\, AX - tr(A)X\} + \frac{c-1}{4}\{-(2n+1)\eta(X)A\xi - 3AT^2X \\ & + tr(A)\eta(X)\xi + 3TATX - tr(AT)TX\} + tr(A)A^2X - tr(A^2)AX \, , \end{aligned}$$

for any vector field X tangent to M.
From (2.12), it follows that $AT^2X = -AX + \eta(X)A\xi + u(X)A\zeta$ and the above equation becomes

$$\begin{aligned} 0 = {} & (\frac{c+3}{4} - L)\{2n\, AX - tr(A)X\} + \frac{c-1}{4}\{-2(n+2)\eta(X)A\xi \\ & + 3AX - 3u(X)A\zeta + tr(A)\eta(X)\xi + 3TATX - tr(AT)TX\} \\ & + tr(A)A^2X - tr(A^2)AX \, . \end{aligned} \tag{3.8}$$

For any $W \in TM$, Equation (3.8) turns into

$$\begin{aligned} 0 = {} & (\frac{c+3}{4} - L)\{2n\, g(AX,W) - tr(A)g(X,W)\} \\ & + \frac{c-1}{4}\{-2(n+2)\eta(X)g(A\xi,W) + 3g(AX,W) - 3u(X)g(A\zeta,W) \\ & + tr(A)\eta(X)\eta(W) + 3g(TATX,W) - tr(AT)g(TX,W)\} \\ & + tr(A)g(A^2X,W) - tr(A^2)g(AX,W) \, . \end{aligned} \tag{3.9}$$

Exchanging X and W in (3.9), we get

$$\begin{aligned} 0 = {} & (\frac{c+3}{4} - L)\{2n\, g(AW,X) - tr(A)g(W,X)\} \\ & + \frac{c-1}{4}\{-2(n+2)\eta(W)g(A\xi,X) + 3g(AW,X) - 3u(W)g(A\zeta,X) \\ & + tr(A)\eta(W)\eta(X) + 3g(TATW,X) - tr(AT)g(TW,X)\} \\ & + tr(A)g(A^2W,X) - tr(A^2)g(AW,X) \, . \end{aligned} \tag{3.10}$$

Subtracting (3.9) to (3.10) and using (1.2), (2.13) and (2.9) , we have

$$\begin{aligned} 0 = {} & \frac{c-1}{4}\{2(n+2)\eta(W)u(X) - 2(n+2)\eta(X)u(W) - 3u(W)g(A\zeta,X) \\ & + 3u(X)g(A\zeta,W) - tr(AT)g(TW,X) + tr(AT)g(TX,W)\} \, . \end{aligned} \tag{3.11}$$

If we insert $W = \xi$ in (3.11) and using (2.10), (2.11), (2.13) and (2.14) , we get

$$(2n+1)\frac{c-1}{4}u(X) = 0,$$

for any $X \in TM$. Since $c \neq 1$, then $u(X) = 0$. Hence, from (2.8) we deduce that φX is tangent, and as ξ is tangent to M, this shows that M is an invariant hypersurface in a Sasakian space form $\widetilde{M}^{2n+1}(c)$. According to Theorem 2.1 this leads to a contradiction. □

Corollary 3.1. *There are no pseudo-parallel hypersurfaces in* $\mathbb{R}^{2n+1}(-3)$.

Remark 3.1. Let S^{2n+1} be a $(2n+1)$-dimensional unit sphere, i.e., $S^{2n+1} = \{z \in \mathbb{C}^{n+1} : \mid z \mid = 1\}$. For any point $z \in S^{2n+1}$, put $\xi = Jz$, where J is the almost complex structure of $\mathbb{C}^{n+1}$. We consider the orthogonal projection

$$\pi : T_x(\mathbb{C}^{n+1}) \longrightarrow T_x(S^{2n+1}).$$

Putting $\varphi = \pi \circ J$, we have a Sasakian structure (φ, ξ, η, g) on S^{2n+1}, where η is a 1-form dual to ξ and g the standard metric tensor field on S^{2n+1}. We see that S^{2n+1} is of constant φ-sectional curvature 1, that is, of constant curvature 1 (cf. [**10**]).
Now we consider the Clifford hypersurface $M_{p,q}$ defined by

$$M_{p,q} = S^{2p+1}(\sqrt{\frac{p}{2n}}) \times S^{2q+1}(\sqrt{\frac{q}{2n}}),\ p+q = n-1$$

Then $M_{p,q}$ is a minimal hypersurface of S^{2n+1} tangent to the structure vector field ξ of S^{2n+1} and $M_{p,q}$ has the parallel second fundamental form, so it is pseudo-parallel. Therefore the assumption in Theorem 3.1 on the φ-sectional curvature $c \neq 1$ of the ambient space $\widetilde{M}^{2n+1}(c)$ is essential.

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