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PAIRED DOMINATION IN TRANSFORMATION GRAPHS: G^{xy+} and G^{xy-}

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Abstract. In this paper, we investigate the results of paired domination in transformation graphs, which are based on the vertex set of the total graph. Haynes and Slater introduce paired dominating set, in which the dominating set induced-subset has a perfect matching. The paired domination number, denoted as $\gamma_{pr}(G)$, represents the minimum cardinality of such a set. We obtained results for the paired domination number values for eight permutations of the transformation graph. Additionally, we derived upper bounds associated with the edge domination number and the covering number from these results.

1. Introduction

Domination parameters and their alterations are considered deeply in graph theory literature. There are two important books about this concept published by Haynes et al. [**13**,**14**]. A subset $S \subseteq V$ is called a *dominating set* of G, if every vertex in $V - S$ is adjacent to any vertex in S and *the domination number* of G; denoted by $\gamma(G)$ is the minimum cardinality of a dominating set. A subset $S' \subseteq E$ is called an *edge-dominating set* of G, abbreviated by $ED-set$, if every edge in $E - S'$ is adjacent to at least one edge in S'. Edge domination number that has minimum cardinality is denoted by $\gamma'(G)$. A *total dominating* set, denoted by $TD-set$, of G with no isolated vertex is a set S of vertices of G and total domination number that is the minimum cardinality of a total dominating set denoted by $\gamma_t(G)$. *Total domination* was introduced by Cockayne et al. [**7**]. A set $S \subseteq V$ is a *paired-dominating set* is S, denoted by $PD-$set, dominates and induced subgraph $\langle S \rangle$ has a perfect matching

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that covers every vertex of the graph. A paired dominating set with the smallest cardinality is represented by $\gamma_{pr}-set$. *Paired domination* was introduced by Haynes and Slater [**11**]. If we need to consider these parameters in terms of application, we can interpret the domination, total domination, and paired domination parameters as follows; if we consider each $s \in S$ as a guard that can protect each vertex where s dominates then every guard guards itself. As for total domination, every guard must be protected by another guard. For paired domination, the positions of the guardians must be chosen as adjacent vertex pairs so that each guardian is assigned to the each other and exists as a backup to the each other. There are many domination parameters being studied and we can group these parameters under several categories. The paired domination parameter that we consider in this paper has a restricted form that is based on $\langle S \rangle$. The main studies in the literature on paired domination, we can refer to the readers as: [**9**, **11**, **12**, **15**].

For notation and terminology, we refer to the book [**8**]. In this work, we consider the graph as a simple graph. Let $G = (V, E)$ be a graph with vertex set $V(G)$ and edge set $E(G)$. A subgraph G' of a graph G is a graph whose vertex set and edge set are subsets of G. *Vertex disjoint subgraph* is a subgraph that has no common vertex. *The open neighborhood* of any vertex in $V(G)$, denoted by $N(v) = \{u \in V(G) \mid (uv) \in E(G)\}$. *The degree of a vertex v* denoted by $deg(v)$, is the cardinality of its neighborhood and the vertex that has degree one is called as *pendant vertex*. If each vertex of a graph has the same vertex degree then the graph is called as *regular graph*. Also, *the maximum degree* of G denoted by $\Delta(G)$, is the degree of the vertex with the largest number of edges incident to it and analogously *the minimum degree* of G denoted by $\delta(G)$, is the degree of a vertex with the smallest number of edges incident to it. *The distance* between two vertices u and v is the shortest path between them denoted by $d(u, v)$ and *the graph diameter* of a graph is the shortest path with the maximum length between any two graph vertices (u, v), it can be symbolized as $\max\limits_{(u,v)\in V\times V}\{d(u, v)\}$. If a vertex and an edge are incident in a graph then it is said that it covers each other. A *vertex cover* of G, is a set of vertices that every vertex of a graph that incident to at least one endpoint of all edges and an *edge cover* of G, is a set of edges that every vertex of G is incident to at least one edge in the edge covering set. Therefore, The vertex covering number and the edge covering number is the minimum cardinality of them and it is denoted by $\alpha(G)$ and $\alpha'(G)$, respectively.

In this paper, transformation graphs G^{xy-} and G^{xy+} are studied and values of their paired-domination number are obtained. There are many studies and approaches related to "transformation graphs" in the literature. Transformation graphs definitions are based on total graphs [**5**]. The total graph has $V(G) \cup E(G)$ as the vertex set in which two vertices are adjacent if and only if vertices and edges are adjacent or incident in G. We will consider 8 cases of the structure which is the generalization of the total graph concept [**18**]. In order to explain the definition of transformation graphs, we need to give some fundamental graph concepts such as complement graph and line graph. *The complement* graph of G, denoted by $\overline{G}$, has the same vertex set $V(G)$ as its vertex sets, but a different edge set that completes the graph G to complement graph using vertices of G [**8**]. The *line graph* $L(G)$ of G is the graph that has the edge set $E(G)$ as its vertex set and two vertices are adjacent in $L(G)$ if and only if associated edges are adjacent in G [**17**]. Let define *transformation graphs* which is introduced by Wu and Meng [**18**] as abbreviated by G^{xyz}. Let x, y, z be three variables taking value + or −. The transformation graph of G, G^{xyz} is a simple graph having as the vertex set $V(G) \cup E(G)$, and for $\alpha, \beta \in V(G) \cup E(G)$ and their relationships in G^{xyz} can be explained as follows:

(i) Let $\alpha, \beta \in V(G)$. α and β are adjacent in G^{xyz} if $x = +$; otherwise $x = -$.

(ii) Let $\alpha, \beta \in E(G)$. α and β are adjacent in G^{xyz} if $y = +$; otherwise $y = -$.

(iii) Let $\alpha \in V(G)$, $\beta \in E(G)$. α and β are incident in G^{xyz} if $z = +$; otherwise $z = -$.

Since there are eight distinct 3-permutations of $\{+, -\}$, we may obtain eight kinds of transformation graphs, in which G^{+++} is the total graph of G, and G^{---} is its complement. Also, $G^{--+}, G^{-+-}$and G^{-++} are the complements of G^{++-}, G^{+-+} and G^{+--}, respectively. In addition, the transformation graph can be expressed as relationship between G (or $\overline{G}$) and its line graph $L(G)$ (or $\overline{L(G)}$) according to be sign of x, y, z. If $x = +$ then transformation graph includes G as a subgraph, otherwise includes $\overline{G}$ as a subgraph. If $y = +$ then transformation graph includes $L(G)$ as a subgraph, otherwise includes $\overline{L(G)}$ as a subgraph. Therefore, according to the sign of z, the relationship between G (or $\overline{G}$) and $L(G)$ (or $\overline{L(G)}$) has been constructed based on the case (iii) mentioned above. Let $V(G) \cup V(L(G))$ (or their complements in terms of the sign of x, y) be vertex set of transformation graph where $V(G) = \{1, 2, ..., n\}$ and $V(L(G)) = \{(ij) : (ij) \in E(G)\}$. For detailed information and studies on some domination parameters on transformation graphs, we refer to readers the papers [**1–3**, **10**, **16**, **18**, **20**].

2. Known results

In this section, we will provide the information available in the literature that will be used in the paper.

Observation 2.1. *[**11**] If u is adjacent to an end vertex of G then u is in every paired dominating set.*

Also, for some special graph families $\gamma_{pr}(K_n) = \gamma_{pr}(K_{r,s}) = \gamma_{pr}(W_n) = 2$ and $\gamma_{pr}(P_n) = \gamma_{pr}(C_n) = 2\left\lceil \frac{n}{4} \right\rceil$ [**11**].

Observation 2.2. *[**11**]If $diam(G) \geqslant 3$, then $\gamma_{pr}(G) = 2$.*

Theorem 2.1. *[**11**] For any graph $G, \gamma_{pr}(\overline{G}) \geqslant 4$ if and only if $diam(G) = 2$.*

Theorem 2.2. *[**11**]If a graph G has no isolated vertices, then $2 \leqslant \gamma_{pr}(G) \leqslant n$.*

Observation 2.3. *[**11**] If a graph G has no isolated vertices, then $\gamma(G) \leqslant \gamma_t(G) \leqslant \gamma_{pr}(G)$ and $\gamma_{pr}(G)$ is even.*

Theorem 2.3. *[**11**] If a graph G has no isolated vertices, then $\gamma_{pr}(G) \leqslant 2\gamma(G)$.*

Observation 2.4. *[**11**] If $diam(G) \geqslant 3$, then $\gamma_{pr}(\overline{G}) = 2$.*

THEOREM 2.4. *[11] For any graph* G, $\gamma_{pr}(\overline{G}) \geqslant 4$ *if and only if* $diam(G) = 2$.

THEOREM 2.5. *[1] Let* G *be connected graph with* n *vertices then* $\gamma(G^{++-}) = 2$.

THEOREM 2.6. *[3]Let* G *be a connected graph. If the minimum vertex degree* $\delta(G) \geqslant 2$, *then*

$$\gamma(G^{---}) = \begin{cases} 2 & , \text{ if } diam(G) \geqslant 3 \\ 3 & , \text{ otherwise} \end{cases} .$$

OBSERVATION 2.5. *[11]Both total domination and paired domination require that there be no isolated vertices.*

3. G^{xy-} graphs

In this section, some exact values for the paired domination numbers of the transformation graph G^{xy-} are obtained.

THEOREM 3.1. *G be graph with n vertices and $diam(G) \geqslant 4$ then*

$$\gamma_{pr}(G^{+--}) = 2.$$

PROOF. Since $diam(G) \geqslant 4$, there exist a path $uxyvz$ in G where $u, x, y, z, v \in V(G)$. Since vertices ux, $vz \in V(\overline{L(G)})$ do not have a common end point,

$$N_{\overline{L(G)}}(ux) \cap N_{\overline{L(G)}}(vz) = \emptyset.$$

Let the set S be any $PD-$set of the graph G^{+--}. If ux, $vz \in S$ then all vertices in the graph G^{+--} are paired dominated. Therefore, $\gamma_{pr}(G^{+--}) = |S| \leqslant 2$. From Theorem 2.2, it is known that $\gamma_{pr}(G) \geqslant 2$. Hence, we get $\gamma_{pr}(G^{+--}) = 2$. □

OBSERVATION 3.1. *If the graph has vertex disjoint subgraphs and graph is r-regular then $n \geqslant 2r + 2$. This is because it is necessary to place $2(r-1) = 2r - 2$ vertices between these two edges in order to obtain at least one pair of disjoint edges. Thus, the graph contains at least $2r - 2 + 4 = 2r + 2$ vertices. Therefore, based on this observation, we can obtain the following result.*

THEOREM 3.2. *Let G be $r-$ regular graph with n vertices and m edges.*
(a) If $n > 2r + 1$ then $\gamma_{pr}(G^{+--}) = 2$
(b) If $n \leqslant 2r + 1$ then $\gamma_{pr}(G^{+--}) = 4$.

PROOF. (a) $\forall u \in V(G)$ we have $d_{G^{+--}}(u) = \frac{nr}{2} = m$ and $\forall (xy) \in V(\overline{L(G)})$ we have $d_{G^{+--}}(xy) = \frac{nr}{2} + -2r + 1$. Therefore, it can be stated that $d_{G^{+--}}(u) < d_{G^{+--}}(xy)$. Let S be a $PD-$set of G^{+--}. In this case, it is reasonable to choose vertices in S from $V(\overline{L(G)})$. According to Observation 3.1, when $n \geqslant 2r + 2$, there exist two edges (uv) and (w_1w_2) such that $w_1, w_2 \notin N_G(u)$ and $N_G(v)$. When both (uv) and (w_1w_2) are in S, then all vertices in G^{+--} are paired dominated. Hence, we obtain $\gamma_{pr}(G^{+--}) = |S| \leqslant 2$. Furthermore, as established by Theorem 2.2, we know that $\gamma_{pr}(G^{+--}) \geqslant 2$. Consequently, we conclude that $\gamma_{pr}(G^{+--}) = 2$.

(b) If $n < 2r + 1$, then $d_{G^{+--}}(u) > d_{G^{+--}}(xy)$. In addition, from Observation 3.1, it is not possible to find any two pairs of edges (u, v) and (w_1, w_2) in $E(G)$ that satisfy $w_1, w_2 \notin N_G(u) \cup N_G(v)$. Let S be a $PD-$set of G^{+--}.

Case1: Similar as case (a), if we choose vertices (uv) and (w_1w_2) from $V(\overline{L(G)})$, there remains at least one vertex of type $(uw_1), (uw_2), (vw_1)$, and (vw_2) that is not dominated in the graph G^{+--}. Therefore, vertices z and t must be added to S provided that $u, v, w_1, w_2 \neq z \in V(G)$ and $t \in N_G(z)$. Thus, we have $S = \{uv, w_1w_2, z, t\}$, and $|S| \leqslant 4$. Hence, we get $\gamma_{pr}(G^{+--}) \leqslant 4$.

Case 2: If we make the selection of vertices from the part G where the vertex degrees are larger, then S includes u and v that are adjacent vertices in G. Therefore, there is at least one non-dominated vertex (uv) in G^{+--} . In this case, two more adjacent vertices will need to be added to S to dominate (uv). Therefore, it can be said that $|S| \leqslant 4$. Hence, we get $\gamma_{pr}(G^{+--}) \leqslant 4$.

Case 3: Assume that we add two adjacent vertices to the S, one vertex from G and one vertex from $\overline{L(G)}$. Let u and v be two adjacent vertices in G. When u and vt are included by S then there is at least one vertex such as uv or ut which are not dominated in G^{+--}. Analogously, we get $\gamma_{pr}(G^{+--}) \leqslant 4$.

In all three cases (Case 1, Case 2, Case 3), it can be concluded that $2 < |S| \leqslant 4$. Additionally, according to Observation 2.3, the γ_{pr} value must be even. In this case, there is no other possibility for $|S|$ than the value 4. Thus, $\gamma_{pr}(G^{+--}) = 4$. □

THEOREM 3.3. *Let G be a graph with n vertices and m edges, then*

$$\gamma_{pr}(G^{++-}) = 4.$$

PROOF. We will prove this theorem in two cases.

Case1: It is known from Theorem 2.3 $\gamma_{pr}(G) \leqslant 2\gamma(G)$, and from Theorem 2.5 $\gamma(G^{++-}) = 2$. Therefore, we get $\gamma_{pr}(G^{++-}) \leqslant 4$.

Case 2: Let S be any $PD-$ set of G^{++-}. Let assume that $|S| = 2$.

- For the vertices in the form that $u, v \in V(G)$, let u and v be the elements of S. According to the form of the graph G^{++-}, we have $uv \in E(G)$. Furthermore, the vertex $uv \in V(L(G))$ cannot be dominated by S. In this case, we find that $|S| > 2$.
- For the vertices in the form that $u = xy$, $v = xt \in V(L(G))$, let u and v be elements of S. According to the form of the graph G^{++-}, we have $x \in V(G)$. Furthermore, the vertex $x \in V(G)$ cannot be dominated by S. In this case, again we have that $|S| > 2$.
- For the vertices in the form that $u \in V(G)$, $v = xy \in V(L(G))$, let u and v be elements of S. According to the form of the graph G^{++-}, the vertices $x, y \notin N_G(u)$. Therefore, the vertices x and y can not be dominated by S. In this case as well, $|S| > 2$.

Hence, in each situation in Case 2, it is clear that $|S| > 2$. Moreover, as per Observation 2.3, the γ_{pr} value must be even. Therefore, considering the results of Case 1 and Case 2, we conclude that $|S| = 4$. Thus, it can be stated that $\gamma_{pr}(G^{++-}) = 4$.

□

THEOREM 3.4. *Let G be a connected graph with more than one pendant vertices. Then, we have*

$$\gamma_{pr}(G^{---}) = 2.$$

PROOF. Let the vertices $u, v \in V(\overline{G})$ with $\deg_G(u) = \deg_G(v) = 1$. Also, let S be any $PD-$set of G^{---}. It is logical to choose the set $\{u, v\}$ as a subset of S, due to the structure of G^{---} and all the vertices will be paired dominated. Therefore, we obtain the result that $\gamma_{pr}(G^{---}) = |S| \leqslant 2$. From Theorem 2.2, it is known that $\gamma_{pr}(G) \geqslant 2$. Thus, $\gamma_{pr}(G^{---}) = 2$. □

THEOREM 3.5. *Let G be a connected graph with $\delta(G) \geqslant 2$. Then, we obtain that*

$$\gamma_{pr}(G^{---}) = \begin{cases} 2, & diam(G) \geqslant 3 \\ 4, & otherwise \end{cases}.$$

PROOF. Let's examine the results under two cases, depending on the diameter:

Case 1: If $diam(G) \geqslant 3$, then a path of length 3 can be found between two vertices u and v in G, denoted by $uxyv$. Let S be any $PD-$set of G^{---}. When $u \in S$, then $yv \in V(\overline{L(G)})$ must be chosen as an element of S. Therefore, all vertices in G^{---} can be dominated by $\{u, yv\} \subseteq S$. Then, we get $\gamma_{pr}(G^{---}) \leqslant 2$. From Theorem 2.2, it is known that $\gamma_{pr}(G^{---}) \geqslant 2$. Hence, we conclude that $\gamma_{pr}(G^{---}) = 2$ for $diam(G) \geqslant 3$.

Case 2: If $diam(G) < 3$ then, there can be at most a path of length 2 between two vertices u and v in G, denoted by uxv. Let S be any $PD-$set of G^{---}. When $u, xv \in S$, there is at least one non-dominated vertex such as $x \in V(G)$. Therefore, $\gamma_{pr}(G^{---}) > 2$. According to Theorem 2.6, to dominate the remaining non-dominated vertex, it is enough to add only one more vertex to the set u, xv. However, it is known from Observation 2.3 that $\gamma_{pr}(G)$ must be even. Therefore, $\gamma_{pr}(G^{---}) = 4$ for $diam(G) < 3$. □

OBSERVATION 3.2. *Let G be a graph with n vertices. If $\Delta(G) = n - 1$, then there is no paired domination set in the graph G^{-+-}. This is due to the presence of isolated vertices in G^{-+-}, as indicated by Observation 2.5.*

THEOREM 3.6. *Let G be a graph and G includes a star graph as a spanning subgraph, denoted by H. Let $\delta(H) = 1$ and $\Delta(G) < n - 1$. Then, we have*

$$\gamma_{pr}(G^{-+-}) = 4.$$

PROOF. Let u is a vertex which satisfy $\delta(H) = 1$. Assume that $|V(H)| = t$, then $\overline{G}$ can be expressed as a bipartite graph such as $\overline{G} = G_1 \cup K_{t-1}$. Subgraph G_1includes center vertex, v, of star graph H. The vertex v is not adjacent to any vertex of K_{t-1}. In order to dominates v, there are two cases as follows. Let S be $PD-$set of G^{-+-}.

Case 1: Let $v_1 \in N_{\overline{G}}(u)$ and $v, v_1 \in V(G_1)$. If $v, v_1 \in S$ then all vertices in $L(G)$ is dominated but it remains some non-dominated vertices in $\overline{G}$. Therefore, $|S| > 2$ and also it is known from Observation 2.3 that $\gamma_{pr}(G)$ must be even, then we have $|S| \geqslant 4$. If the vertex u is added to S then all vertices in $\overline{G}$ is dominated by u except v which is dominated by v_1. Therefore, $\{u, v, v_1\}$ is a $TD-$set of G^{-+-}. In order to get perfect matching in $< S >$, at least one vertex which is adjacent to any vertex in $\{u, v, v_1\}$ must be added. Assume that this vertex is denoted by $w \in V(G^{-+-})$. Thus, $\{u, v, v_1, w\}$ is a $PD-$set. We have $|S| \leqslant 4$.

Case 2: Let $v_1, v_2 \in G_1$, $(v_1, v_2 \neq v)$ and $v \in N_{G^{-+-}}(v_1v_2)$. If the vertices $v \in V(\overline{G})$ and $v_1v_2 \in V(L(G))$ are added to S, some vertices remain as non-dominated in $L(G)$. Therefore, $|S| > 2$ and also it is known from Observation 2.3 that $\gamma_{pr}(G)$ must be even, then we have $|S| \geqslant 4$. As in Case 1, if the vertex u and any of its neighbors, such as $v_1 \in N_{\overline{G}}(u)$, are added to S, then we have $S = \{v, v_1v_2, u, v_1\}$ as a $PD-$set. We have $|S| \leqslant 4$.

Hence, from all cases above, we get $\gamma_{pr}(G^{-+-}) = 4$. □

4. G^{xy+} graphs

In this part, we obtain some exact values and upper bounds, in terms of edge domination and covering number, for the paired domination numbers of the transformation graph G^{xy+}. It can be observed from the definitions of the line graph, domination and edge domination that for a connected graph G, $\gamma'(G) = \gamma(L(G))$ [**4**]. Using this information, we can provide the following theorem.

Theorem 4.1. *Let G be a connected graph and $\gamma'(G) = 1$. Then, the result can be obtained as:*

$$\gamma_{pr}(G^{-++}) = \begin{cases} 2 & , \textit{ if } G \simeq S_n \\ 4 & , \textit{otherwise} \end{cases} .$$

where S_n is a star graph.

Proof. Let S be a PD-set of G^{-++}, and the edge domination number of G, denoted as $\gamma'(G)$, is equal to 1. Therefore, there exists an edge in G, abbreviated as (uv), which is adjacent to all edges in the graph G. Additionally, all vertices in $L(G)$ and the vertices u and v in $V(G)$ can be dominated by the vertex $uv \in V(L(G))$. Due to the form of G^{-++}, vertices u and v are not adjacent. Hence, two situations are observed:

- If $N_G(u) \cap N_G(v) \neq \emptyset$, since $\gamma'(G) = 1$, the vertices other than u and v form a complete graph. Hence, there exists a PD-set such that $\{u, uv, t, z\} \subseteq S$ where $t, z \in V(G)$. Thus, from Theorem 2.2, we conclude that $|S| = 4$.
- If $N_G(u) \cap N_G(v) = \emptyset$, there exist two cases. G can be in the form of a star graph, or the vertices u and v are both cut vertices. It is easy to see that $\{uv, v\}$ is a $PD-$ set of S_n^{-++} provided that $\deg_G\{v\} = 1$. In the other case, since $N_G(u) \cap N_G(v) = \emptyset$ and G is edge-dominated by (uv), then the vertices u and v dominate all vertices in G. Therefore $\{u, v, uv\}$ is a dominating set for G^{-++}. In order to dominate G^{-++} in a paired manner, it is needed to add one more vertex which is adjacent to either u

or v. Thus, $\{u, v, uv, t\}$ is a $PD-$set of G^{-++} where t$\in N_G(u)$ or $N_G(v)$. We conclude that $|S| = 4$. Therefore, considering all possibilities, we have

$$\gamma_{pr}(G^{-++}) = \begin{cases} 2 & , \text{ if } G = S_n \\ 4 & , \text{otherwise} \end{cases} .$$

□

THEOREM 4.2. *Let G be a connected graph and $\gamma'(G)$ be the edge domination number of G and $\gamma'(G) > 1$. Then, we have*

$$\gamma_{pr}(G^{-++}) \leqslant 2\gamma'(G).$$

PROOF. Let $S \subseteq V(G^{-++})$ be a $PD-$set and S_1 be a dominating set of $L(G)$. Additionally, define $S_1 = \{uv \in V(L(G))$: where $(uv) \in E(G)$ is in $ED - set$ of G $\}$. Therefore, S_1 dominates all vertices in $L(G)$ and it also dominates the $2\gamma'(G)$ vertex in graph $\overline{G}$ that are neighbors of S_1. Remaining vertices which are not dominated by S_1, form a complete graph in $\overline{G}$. Furthermore, let (uv) be any vertex in S_1 and let $w \in V(\overline{G}) - N_{\overline{G}}[S_1]$ be a remaining non-dominated vertex in $V(\overline{G})$, it can be concluded that $w \in N_{\overline{G}}(u)$ or $w \in N_{\overline{G}}(v)$ due to the form of G^{-++}.

If any vertices in S_1 are not adjacent to each other, $|\gamma'(G)|$ vertices from $N_{\overline{G}}[S_1]$ should be added to S_1 at most. Therefore, we get an $PD-$set S_1 such as $|S_1| = 2\gamma'(G)$. Thus, we have:

$$\gamma_{pr}(G^{-++}) \leqslant 2\gamma'(G).$$

□

THEOREM 4.3. *Let G be a graph with n vertices and $\alpha(G)$ be the vertex covering number of G. Then, we obtain that:*

$$\gamma_{pr}(G^{+x+}) \leqslant 2\alpha(G).$$

PROOF. Let S be minimum vertex cover set of G. According to the definition of the vertex cover set of a graph, all vertices in the G^{+x+} can be dominated by S. However, if the elements of S cannot form discrete pairs, then these elements should be matched with the vertices up to $|S|$. Therefore, we have $\gamma_{pr}(G^{+x+}) \leqslant 2\,|S| = 2\alpha(G)$. □

THEOREM 4.4. *Let G be a connected graph with n vertices. If the G graph has pendant vertex and $\Delta(G) = n - 1$ then we get*

$$\gamma_{pr}(G^{+-+}) = 2.$$

PROOF. Let $\deg_G(u) = n-1$ and v be a pendant vertex of G. According to the form of G^{+-+}, $\{u, uv\}$ be a $PD-$set of G^{+-+}. Therefore, we have $\gamma_{pr}(G^{+-+}) \leqslant 2$. From Theorem 2.2, it is known that $\gamma_{pr}(G^{+-+}) \geqslant 2$. Hence, we conclude that $\gamma_{pr}(G^{+-+}) = 2$. □

THEOREM 4.5. *Let G be n order, r-regular ($r \neq n-1$) graph and $diam(G) \geqslant 3$. Then,*

$$\gamma_{pr}(G^{--+}) = 4.$$

Proof. Let S be any $PD-$set of G^{--+}. In order to construct S, there are three options:

Case 1: If S includes $x \in V(\overline{G})$ and $(xy) \in V(\overline{L(G)})$., then there are $n+\frac{nr}{2}-2r$ vertices are dominated. Therefore, the remaining non-dominated vertices are $2r$.

Case 2: If S includes (uv), $(xy) \in V(\overline{L(G)})$ such as $x, y \notin N_G(u) \cup N_G(v)$, then all vertices in $\overline{L(G)}$ and $u, v, x, y \in \overline{G}$ are dominated. In addition, there are $(n-4)$ remaining non-dominated vertices left.

Case 3: If S includes $x, y \in V(\overline{G})$ such as $d_G(x, y) \geqslant 3$, then there are $n+2r$ vertices are dominated. Therefore, the remaining non-dominated vertices are $\frac{nr}{2} - 2r$.

According to the three cases above, it is clear that $|S| > 2$. Since it is known from Observation 2.3 that $\gamma_{pr}(G)$ must be even, we have $|S| \geqslant 4$. Without loss of generality, let $\{(uv), (xy)\} \subseteq S$. Since $diam(G) \geqslant 3$, there exist two vertices m, p such that $p \in N_{G^{--+}}(m)$ and due to the form of $\overline{G}$, all non-dominated vertices can be dominated by m, p. Thus, $S = \{uv, xy, m, p\}$ is a $PD-$set of G^{--+}, and we conclude that $\gamma_{pr}(G^{--+}) = |S| \leqslant 4$. Because of the upper and lower bounds of $|S|$, we obtain $\gamma_{pr}(G^{--+}) = 4$. □

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