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COMPARISON OF SMALLEST EIGENVALUES FOR NABLA FRACTIONAL BOUNDARY VALUE PROBLEMS

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Abstract. In this article, we establish the existence of and then compare smallest eigenvalues for nabla fractional boundary value problems involving a fractional difference boundary condition, using the theory of u_0- positive operators with respect to a cone in a Banach space.

1. Introduction

The theories of Krein–Rutman [**30**] and Krasnosel'skii [**29**] have been used by many authors to establish the existence of and then compare smallest eigenvalues of boundary value problems for differential equations [**5, 10, 11, 12, 13, 32**] difference equations [**7, 16**] dynamic equations on time scales [**6, 19**], fractional differential equations [**8, 9, 18, 28**], and delta fractional difference equations [**17, 33, 34**].

Motivated by these works, in this paper, we obtain the existence of and then compare smallest eigenvalues for the eigenvalue problems

$$\left(\nabla^{\alpha}_{\rho(a)}u\right)(t) + \lambda_1 p(t)u(t) = 0, \quad t \in \mathbb{N}^{b}_{a+2}, \tag{1.1}$$

$$\left(\nabla^{\alpha}_{\rho(a)}u\right)(t) + \lambda_2 q(t)u(t) = 0, \quad t \in \mathbb{N}^{b}_{a+2}, \tag{1.2}$$

satisfying the boundary condition

$$u(a) = \left(\nabla^{\beta}_{\rho(a)}u\right)(b) = 0. \tag{1.3}$$

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Here $1 < \alpha < 2$, $0 \leqslant \beta \leqslant 1$, a, $b \in \mathbb{R}$ with $b - a \in \mathbb{N}_2$, p, $q : \mathbb{N}_{a+2}^b \to (0, \infty)$, $\nabla_{\rho(a)}^{\alpha}$ and $\nabla_{\rho(a)}^{\beta}$ are α^{th} and β^{th}-order nabla fractional difference operators, respectively. Observe that the pair of boundary conditions in (1.3) reduces to conjugate [**14, 24**], right-focal [**22**] and right-focal type [**23**] boundary condition as $\beta \to 0^+$, $\beta \to 1^-$ and $\beta \to (\alpha - 1)$, respectively.

This article is organized as follows: In Section 2, we state the preliminary definitions and results from nabla fractional calculus and the theory of u_0-positive operators with respect to a cone in a Banach space. In Section 3, we define the appropriate Banach space and establish the existence of and then compare smallest eigenvalues of (1.1) - (1.3) and (1.2) - (1.3).

2. Preliminaries

Denote the set of all real numbers by $\mathbb{R}$. For any a, $b \in \mathbb{R}$ such that $b - a \in \mathbb{N}_1$, define $\mathbb{N}_a = \{a, a+1, a+2, \ldots\}$ and $\mathbb{N}_a^b = \{a, a+1, a+2, \ldots, b\}$. Assume that empty sums and products are taken to be 0 and 1, respectively.

Definition 2.1. [**4**] *The backward jump operator* $\rho : \mathbb{N}_{a+1} \to \mathbb{N}_a$ *is defined by*

$$\rho(t) = t - 1, \quad t \in \mathbb{N}_{a+1}.$$

Definition 2.2. [**27, 31**] *The Euler gamma function is defined by*

$$\Gamma(z) = \int_0^{\infty} t^{z-1} e^{-t} dt, \quad \Re(z) > 0.$$

Using the reduction formula

$$\Gamma(z+1) = z\Gamma(z), \quad \Re(z) > 0,$$

the Euler gamma function can also be extended to the half-plane $\Re(z) \leqslant 0$ *except for* $z \in \{\cdots, -2, -1, 0\}$.

Definition 2.3. [**15**] *For* $t \in \mathbb{R} \setminus \{\ldots, -2, -1, 0\}$ *and* $r \in \mathbb{R}$ *such that* $(t+r) \in \mathbb{R} \setminus \{\ldots, -2, -1, 0\}$, *the generalized rising function is defined by*

$$t^{\overline{r}} = \frac{\Gamma(t+r)}{\Gamma(t)}.$$

We use the convention that if $t \in \{\ldots, -2, -1, 0\}$ *and* $r \in \mathbb{R}$ *such that* $(t+r) \in \mathbb{R} \setminus \{\ldots, -2, -1, 0\}$, *then*

$$t^{\overline{r}} = 0.$$

Definition 2.4. [**4**] *Let* $u : \mathbb{N}_a \to \mathbb{R}$ *and* $N \in \mathbb{N}_1$. *The first order backward (nabla) difference of* u *is defined by*

$$\big(\nabla u\big)(t) = u(t) - u(t-1), \quad t \in \mathbb{N}_{a+1},$$

and the N^{th}*-order nabla difference of* u *is defined recursively by*

$$\big(\nabla^N u\big)(t) = \Big(\nabla\big(\nabla^{N-1} u\big)\Big)(t), \quad t \in \mathbb{N}_{a+N}.$$

DEFINITION 2.5. [**15**] *Let $u : \mathbb{N}_{a+1} \to \mathbb{R}$ and $N \in \mathbb{N}_1$. The N^{th}-order nabla sum of u based at a is given by*

$$\left(\nabla_a^{-N} u\right)(t) = \frac{1}{(N-1)!} \sum_{s=a+1}^{t} (t-\rho(s))^{\overline{N-1}} u(s), \quad t \in \mathbb{N}_a,$$

where by convention $\left(\nabla_a^{-N} u\right)(a) = 0$. We define

$$\left(\nabla_a^{-0} u\right)(t) = u(t), \quad t \in \mathbb{N}_{a+1}.$$

DEFINITION 2.6. [**15**] *Let $u : \mathbb{N}_{a+1} \to \mathbb{R}$ and $\nu > 0$. The ν^{th}-order nabla fractional sum of u based at a is given by*

$$\left(\nabla_a^{-\nu} u\right)(t) = \frac{1}{\Gamma(\nu)} \sum_{s=a+1}^{t} (t-\rho(s))^{\overline{\nu-1}} u(s), \quad t \in \mathbb{N}_a,$$

where by convention $\left(\nabla_a^{-\nu} u\right)(a) = 0$.

DEFINITION 2.7. [**15**] *Let $u : \mathbb{N}_{a+1} \to \mathbb{R}$, $\nu > 0$ and choose $N \in \mathbb{N}_1$ such that $N-1 < \nu \leqslant N$. The ν^{th}-order Riemann–Liouville nabla fractional difference of u based at a is given by*

$$\left(\nabla_a^{\nu} u\right)(t) = \left(\nabla^N \left(\nabla_a^{-(N-\nu)} u\right)\right)(t), \quad t \in \mathbb{N}_{a+N}.$$

DEFINITION 2.8. [**29**] *Let $\mathcal{B}$ be a real Banach space. A set $\mathcal{P} \subset \mathcal{B}$ is called a cone if the following conditions are satisfied:*

(1) *$\mathcal{P}$ is closed;*
(2) *if u, $v \in \mathcal{P}$ then $\alpha u + \beta v \in \mathcal{P}$ for all α, $\beta \geqslant 0$;*
(3) *if $u \in \mathcal{P}$ and $-u \in \mathcal{P}$ then $u = 0$.*

A cone $\mathcal{P}$ is solid if the interior, $\mathcal{P}^0$, of $\mathcal{P}$, is nonempty. A cone is called reproducing if every element $w \in \mathcal{B}$ can be represented in the form

$$w = u - v, \quad u, v \in \mathcal{P}.$$

REMARK 2.1. [**29**] Every solid cone is reproducing.

By means of a cone, we define a partial ordering relation in a Banach space as follows.

DEFINITION 2.9. [**29**] *Let $\mathcal{P}$ be a cone in a real Banach space $\mathcal{B}$. For all u, $v \in \mathcal{B}$, we write $u \leqslant v$ with respect to $\mathcal{P}$ if $v - u \in \mathcal{P}$.*

Further, we also introduce a partial ordering relation on bounded linear operators defined on a Banach space.

DEFINITION 2.10. [**29**] *Let $\mathcal{P}$ be a cone in a real Banach space $\mathcal{B}$ and T, $S : \mathcal{B} \to \mathcal{B}$ are bounded linear operators. We write $T \leqslant S$ with respect to $\mathcal{P}$ if $Tu \leqslant Su$ for all $u \in \mathcal{P}$.*

DEFINITION 2.11. [**29**] *A bounded linear operator $T : \mathcal{B} \to \mathcal{B}$ is u_0-positive with respect to $\mathcal{P}$ if there exists $u_0 \in \mathcal{P} \setminus \{0\}$ such that for each $u_0 \in \mathcal{P} \setminus \{0\}$, there exist positive constants $k_1(u)$ and $k_2(u)$ such that $k_1(u_0) \leqslant Tu \leqslant k_2(u_0)$ with respect to $\mathcal{P}$.*

We use the following three theorems to establish our main results.

THEOREM 2.1. [**29**] *Let $\mathcal{P} \subset \mathcal{B}$ be a solid cone. If $T : \mathcal{B} \to \mathcal{B}$ is a linear operator such that $T : \mathcal{P} \setminus \{0\} \to \mathcal{P}^0$, then T is u_0-positive.*

THEOREM 2.2. [**29**] *Let $\mathcal{B}$ be a real Banach space, $\mathcal{P} \subset \mathcal{B}$ be a reproducing cone and $T : \mathcal{B} \to \mathcal{B}$ be a compact, u_0-positive linear operator. Then, T has an essentially unique eigenvector in $\mathcal{P}$, and the corresponding eigenvalue is simple, positive, and larger than the absolute value of any other eigenvalue.*

THEOREM 2.3. [**29**] *Let $\mathcal{B}$ be a real Banach space, $\mathcal{P} \subset \mathcal{B}$ be a cone and T, $S : \mathcal{B} \to \mathcal{B}$ be bounded, linear operators. Assume at least one of the operators T and S is u_0-positive. If $T \leqslant S$, $Tu_1 \geqslant \lambda_1 u_1$ for some $u_1 \in \mathcal{P}$ and $\lambda_1 > 0$, and $Su_2 \leqslant \lambda_2 u_2$ for some $u_2 \in \mathcal{P}$ and $\lambda_2 > 0$, then $\lambda_1 \leqslant \lambda_2$. Further, $\lambda_1 = \lambda_2$ implies u_1 is a scalar multiple of u_2.*

3. Main results

The author [**21**] has derived the Green's function $G(\beta; t, s)$ for

$$\left(\nabla^{\alpha}_{\rho(a)} u\right)(t) = 0, \quad t \in \mathbb{N}^{b}_{a+2}, \tag{3.1}$$

satisfying (1.3) and also obtained a few of its properties.

$$G(\beta; t, s) = \begin{cases} \frac{1}{\Gamma(\alpha)}\left[\frac{(b-s+1)^{\overline{\alpha-\beta-1}}}{(b-a)^{\overline{\alpha-\beta-1}}}(t-a)^{\overline{\alpha-1}}\right], & t \in \mathbb{N}^{\rho(s)}_{a}, \\ \frac{1}{\Gamma(\alpha)}\left[\frac{(b-s+1)^{\overline{\alpha-\beta-1}}}{(b-a)^{\overline{\alpha-\beta-1}}}(t-a)^{\overline{\alpha-1}} - (t-s+1)^{\overline{\alpha-1}}\right], & t \in \mathbb{N}^{b}_{s}. \end{cases} \tag{3.2}$$

THEOREM 3.1. [**21**] *The Green's function $G(\beta; t, s)$ defined in* (3.2) *satisfies the following properties:*

(1) $G(\beta; a, s) = 0$ *for all* $0 \leqslant \beta \leqslant 1$ *and* $s \in \mathbb{N}^{b}_{a+1}$.

(2) $G(0; b, s) = 0$ *for all* $s \in \mathbb{N}^{b}_{a+1}$.

(3) $G(\beta; t, a+1) = 0$ *for all* $0 \leqslant \beta \leqslant 1$ *and* $t \in \mathbb{N}^{b}_{a}$.

(4) $G(0; t, s) > 0$ *for all* $(t, s) \in \mathbb{N}^{b-1}_{a+1} \times \mathbb{N}^{b}_{a+2}$.

(5) $G(\beta; t, s) > 0$ *for all* $0 < \beta \leqslant 1$ *and* $(t, s) \in \mathbb{N}^{b}_{a+1} \times \mathbb{N}^{b}_{a+2}$.

Observe that u is a solution of (1.1) - (1.3) if and only if u is a solution of the summation equation

$$u(t) = \lambda_1 \sum_{s=a+2}^{b} G(\beta; t, s) p(s) u(s), \quad t \in \mathbb{N}^{b}_{a}. \tag{3.3}$$

Similarly, u is a solution of (1.2) - (1.3) if and only if u is a solution of the summation equation

$$u(t) = \lambda_2 \sum_{s=a+2}^{b} G(\beta; t, s) q(s) u(s), \quad t \in \mathbb{N}_a^b. \tag{3.4}$$

Denote by

$$\mathcal{B} = \left\{ u : \mathbb{N}_a^b \to \mathbb{R} \mid u(a) = \left(\nabla_{\rho(a)}^{\beta} u \right)(b) = 0 \right\} \subseteq \mathbb{R}^{b-a+1}.$$

Clearly, $\mathcal{B}$ is a Banach space equipped with the maximum norm

$$\|u\| = \max_{t \in \mathbb{N}_a^b} |u(t)|.$$

Define the cone

$$\mathcal{P} = \left\{ u \in \mathcal{B} \mid u(t) \geqslant 0 \text{ for all } t \in \mathbb{N}_a^b \right\}.$$

Since

$$\Omega = \left\{ u \in \mathcal{B} \mid u(t) > 0 \text{ for all } t \in \mathbb{N}_{a+1}^{b-1} \right\} \subset \mathcal{P}^0,$$

$\mathcal{P}$ is solid and hence it is reproducing. Define the operators

$$(Tu)(t) = \sum_{s=a+2}^{b} G(\beta; t, s) p(s) u(s), \quad t \in \mathbb{N}_a^b, \tag{3.5}$$

$$(Su)(t) = \sum_{s=a+2}^{b} G(\beta; t, s) q(s) u(s), \quad t \in \mathbb{N}_a^b. \tag{3.6}$$

Clearly, T, $S : \mathcal{B} \to \mathcal{B}$ are linear. Note that T and S are summation operations on a discrete finite set. Hence, T and S are compact.

Lemma 3.1. *The operators T and S are u_0-positive with respect to $\mathcal{P}$.*

Proof. We prove this statement for the operator T. For this purpose, we apply Theorem 2.1. Clearly, $\mathcal{P} \subset \mathcal{B}$ is a solid cone and $T : \mathcal{B} \to \mathcal{B}$ is a linear operator. It is enough to show that $T : \mathcal{P} \setminus \{0\} \to \mathcal{P}^0$. To see this, let $u \epsilon \mathcal{P} \setminus \{0\}$. Then, there exists a $t_0 \in \mathbb{N}_{a+2}^{b-1}$ such that $u(t_0) > 0$. Since $G(\beta; t, s) > 0$ for all $0 \leqslant \beta \leqslant 1$ and $(t, s) \in \mathbb{N}_{a+1}^{b-1} \times \mathbb{N}_{a+2}^{b}$ and $p(s) > 0$ for all $s \in \mathbb{N}_{a+2}^{b}$, we have

$$\begin{aligned} (Tu)(t) &= \sum_{s=a+2}^{b} G(\beta; t, s) p(s) u(s) \\ &\geqslant G(\beta; t, t_0) p(t_0) u(t_0) > 0, \end{aligned}$$

for all $t \in \mathbb{N}_{a+1}^{b-1}$. So, $Tu \in \Omega \subset \mathcal{P}^0$. The proof is complete. □

Remark 3.1. Let λ_1 be a nonzero eigenvalue of (1.1) - (1.3). If u is an eigenvector corresponding to λ_1 of (1.1) - (1.3), then

$$\frac{1}{\lambda_1} u = Tu.$$

So, the eigenvalues of (1.1) - (1.3) are reciprocals of the eigenvalues of (3.5), and conversely.

THEOREM 3.2. *T has an essentially unique eigenvector $u \in \mathcal{P} \setminus \{0\}$, and the corresponding eigenvalue Λ is positive, simple, and larger than the absolute value of any other eigenvalue.*

PROOF. We know that $\mathcal{P}$ is a reproducing cone and T is a compact, u_0-positive linear operator. Then, by Theorem 2.2, T has an essentially unique eigenvector $u \in \mathcal{P} \setminus \{0\}$, and the corresponding eigenvalue Λ is positive, simple, and larger than the absolute value of any other eigenvalue. □

REMARK 3.2. From the proof of Lemma 3.1, we observe that $(Tu)(t) > 0$ for all $t \in \mathbb{N}_{a+1}^{b-1}$ and hence $Tu \in \mathcal{P}^0$. It follows from Theorem 3.2 that $\Lambda u = Tu$. Thus, we obtain

$$u(t) = \frac{1}{\Lambda}(Tu)(t) > 0$$

for all $t \in \mathbb{N}_{a+1}^{b-1}$. Therefore, $u \in \Omega \subset \mathcal{P}^0$.

THEOREM 3.3. *Let $p(s) \leqslant q(s)$ for all $s \in \mathbb{N}_{a+2}^{b}$. Let Λ_1 and Λ_2 be the eigenvalues defined in Theorem 3.2 associated with T and S, respectively, with the essentially unique eigenvectors u_1 and u_2 in $\mathcal{P} \setminus \{0\}$. Then, $\Lambda_1 \leqslant \Lambda_2$. Furthermore, $\Lambda_1 = \Lambda_2$ if and only if $p(s) = q(s)$ for all $s \in \mathbb{N}_{a+2}^{b}$.*

PROOF. Let $p(s) \leqslant q(s)$ for all $s \in \mathbb{N}_{a+2}^{b}$. Then, for any $u \in \mathcal{P}$ and $t \in \mathbb{N}_{a}^{b}$,

$$(Su - Tu)(t) = \sum_{s=a+2}^{b} G(\beta; t, s)\,(q(s) - p(s))\,u(s) \geqslant 0.$$

So, $(Su - Tu) \in \mathcal{P}$ for all $u \in \mathcal{P}$. That is, $T \leqslant S$ with respect to $\mathcal{P}$. Then, by Theorem 2.3, we obtain $\Lambda_1 \leqslant \Lambda_2$.

Now, we prove the second statement of the theorem. If possible, suppose $p(t_0) < q(t_0)$, for some $t_0 \in \mathbb{N}_{a+2}^{b-1}$. Since $u_1 \in \mathcal{P}^0$, we have $u_1(t_0) > 0$. Then, for all $t \in \mathbb{N}_{a+1}^{b-1}$,

$$\begin{aligned}(Su_1 - Tu_1)(t) &= \sum_{s=a+2}^{b} G(\beta; t, s)\,(q(s) - p(s))\,u_1(s)\\ &\geqslant G(\beta; t, t_0)\,(q(t_0) - p(t_0))\,u_1(t_0) > 0,\end{aligned}$$

implying that $(Su_1 - Tu_1) \in \Omega \subset \mathcal{P}^0$. So, there exists $\epsilon > 0$ such that $(S - T)\,u_1 - \epsilon u_1 \in \mathcal{P}$. Hence,

$$\Lambda_1 u_1 + \epsilon u_1 = Tu_1 + \epsilon u_1 \leqslant Su_1,$$

which implies

$$(\Lambda_1 + \epsilon)\,u_1 \leqslant Su_1.$$

Since $S \leqslant S$ and $Su_2 = \Lambda_2 u_2$, by Theorem 2.3, we obtain

$$\Lambda_1 + \epsilon \leqslant \Lambda_2 \text{ and } \Lambda_1 < \Lambda_2.$$

Thus, by contrapositive, if $\Lambda_1 = \Lambda_2$, then $p(s) = q(s)$ for all $t \in \mathbb{N}_{a+2}^{b}$. □

By Remark 3.1, the following theorem is an immediate consequence of Theorems 3.2 and 3.3.

THEOREM 3.4. *Assume the hypotheses of Theorem 3.3. Then, there exist smallest positive eigenvalues* λ_1 *and* λ_2 *of* (1.1) - (1.3) *and* (1.2) - (1.3), *respectively, each of which is simple, positive, and less than the absolute value of any other eigenvalue of the corresponding problems. Also, eigenvectors corresponding to* λ_1 *and* λ_2 *may be chosen to belong to* $\mathcal{P} \setminus \{0\}$. *Then,* $\lambda_1 \geqslant \lambda_2$. *Furthermore,* $\lambda_1 = \lambda_2$ *if and only if* $p(s) = q(s)$ *for all* $s \in \mathbb{N}_{a+2}^{b}$.

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