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ON A MONOTONE CONVERGENCE RESULT FOR HENSTOCK-KURZWEIL-STIELTJES-◇-DOUBLE INTEGRAL OF INTERVAL-VALUED FUNCTIONS ON TIME SCALES WITH AN APPLICATION

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Abstract. We establish a monotone convergence result for Henstock-Kurweil-Stieltjes-◇-double integral of interval-valued functions on time scales. An interesting application is given as well to support our result.

1. Introduction

The calculus of time scale was introduced by Hilger [**6**] in the year 1988 as a unification of several definitions and results arising from the studies of discrete and continuous dynamical systems. The theory of time scale occurs due to extension of many problems arising in the theories of differential and integral calculus, see [**3**]. Peterson and Thompson [**13**], studied some properties of Henstock delta and nabla integrals on single time scale. Also in [**15**], Thomson investigated other basic properties of Henstock-Kurzweil integrals on single time scale when the integrand is a non-negative function. For a general concept to the time scale of one variable, we refer the reader to references ([**2, 3**], [**10**], [**12, 13**], [**15**], [**19**]).

The introduction of interval analysis can be traced to the celebrated book of Moore [**9**]. Along this line of research, some Henstock-Kurzweil integrals have been extended to the context of interval-valued functions by Yoon [**18**], Hamid et al.[**8**], Hamid and Elmuiz [**7**], Wu and Gong [**17**], Park [**11**], Gong and Shao [**4**]. For the general concept of the Henstock-Stieltjes integrals of interval-valued functions on time scales of single variable, see [**17**]. Motivated by [**7**], [**17**], Afariogun et al.

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[**1**] introduced the Henstock-Kurweil-Stieltjes-$\diamond$-double integral for interval-valued functions on time scales and investigated some of its properties.

It is well known that, the convergence results for integrals are important concept in measure and integration theory [**14**]. Various convergence theorems have been established by several authors in this area of research for Henstock integrable functions involving single time scale variable, see [**12**], [**19**]. Nonetheless, to the best of our knowledge, there is no convergence result of Henstock-Kurweil-Stieltjes-$\diamond$-double integral for interval-valued functions on two time scales. Therefore, it is the purpose of this paper to establish a monotone convergence result for Henstock-Kurweil-Stieltjes-$\diamond$-double integral of interval-valued functions on two variables time scales. An interesting application of an error estimate of a cubature formula is given to support our result.

2. Preliminaries

A time scale $\mathbb{T}$ is any closed non-empty subset of $\mathbb{R}$, with the topology inherited from the standard topology on the real numbers $\mathbb{R}$. Let $\mathbb{T}$ be a time scale. A function $F : \mathbb{T} \to \mathbb{R}$ is continuous at a point s if there exists $\varepsilon > 0$ such that $|F(t) - F(s)| < \varepsilon$, and for any $\delta > 0$, there is $|t - s| < \delta$. Right-scattered at $t \in \mathbb{T}$ if $\sigma(t) > t$ and left-scattered at $t \in \mathbb{T}$ if $\rho(t) < t$. It is right-dense at $t \in \mathbb{T}$ if $t < \sup \mathbb{T}$ and $\sigma(t) = t$ and left-dense at $t \in \mathbb{T}$ if $t > \inf \mathbb{T}$ and $\rho(t) = t$.

Definition 2.1. *A mapping $F : \mathbb{T} \to \mathbb{R}$ is said to be (right dense) rd-continuous if:*
(i) F is continuous at each right-dense point of $\mathbb{T}$
(ii) at each left-dense point $t \in \mathbb{T}$, $\lim_{s \to t^-} g(s) = g(t^-)$ exists.
An interval-valued function $F : \mathbb{T} \to I_{\mathbb{R}}$ is called regulated if its right-sided $\mathbb{T}$-limits in $I_{\mathbb{R}}$ exist at all right-dense points in $\mathbb{T}$, and its left-sided $\mathbb{T}$-limits in $I_{\mathbb{R}}$ exist at all left-dense points in $\mathbb{T}$.

Here, we give a fair introduction into the time scales of two variables.
Let $\mathbb{T}_1$ and $\mathbb{T}_2$ be two given time scales with $a, b \in \mathbb{T}_1, c, d \in \mathbb{T}_2$, where $a < b, c < d$, and $\mathcal{R} = [a, b)_{\mathbb{T}_1} \times [c, d)_{\mathbb{T}_2} = \{(t, s) : t \in [a, b), s \in [c, d), t \in \mathbb{T}_1, s \in \mathbb{T}_2\}$. Suppose $g_1, g_2 : \mathbb{T}_1 \times \mathbb{T}_2 \to \mathbb{R}$ be two non-decreasing functions on $[a, b]_{\mathbb{T}_1}$ and $[c, d]_{\mathbb{T}_2}$, respectively. Let P_1 and P_2 be two partitions of $[a, b]_{\mathbb{T}_1}$ and $[c, d]_{\mathbb{T}_2}$ such that $P_1 = \{t_0, t_1, ..., t_n\} \subset [a, b]_{\mathbb{T}_1}$ and $P_2 = \{s_0, s_1, ..., s_n\} \subset [c, d]_{\mathbb{T}_2}$ with $\{\xi_1, \xi_2, ..., \xi_n\}$ denote an arbitrary selection of points in $[a, b]_{\mathbb{T}_1}$. Similarly, $\{\zeta_1, \zeta_2, ..., \zeta_n\}$ denote an arbitrary selection of points in $[c, d]_{\mathbb{T}_2}$.

The following definitions and lemma are of interest.

Definition 2.2. *Let $\mathbb{T}_1, \mathbb{T}_2$ be two given time scales and let $\mathbb{T}_1 \times \mathbb{T}_2 = \{(x, y) : x \in \mathbb{T}_1, y \in \mathbb{T}_2\}$ which is a complete metric space with the metric (distance) D. For I_1, I_2 in the closed bounded region in $\mathcal{R}$, we define $D(I_1, I_2) = D((a, c), (b, d)) =$*

$((b-a)^2+(d-c)^2)^{\frac{1}{2}}$ *as the distance between* I_1 *and* I_2. *Then*

$$\begin{aligned} D(I_1, I_2) &= \max\left(\left|\underline{I_1} - \underline{I_2}\right|, \left|\overline{I_1} - \overline{I_2}\right|\right), \\ &\quad \textit{where it is assumed that } \underline{I_1} \leqslant \underline{I_2} \textit{ and } \overline{I_1} \leqslant \overline{I_2}; \\ &= \max\left(\left|(\underline{a}, \underline{c}) - (\underline{b}, \underline{d})\right|, \left|(\overline{a}, \overline{c}) - (\overline{b}, \overline{d})\right|\right) \\ &= \max((\underline{b} - \underline{a})^2 + (\underline{d} - \underline{c})^2)^{\frac{1}{2}}, ((\overline{b} - \overline{a})^2 + (\overline{d} - \overline{c})^2)^{\frac{1}{2}}. \end{aligned}$$

Let $(I_{\mathbb{R}}, D)$ be a complete metric space. The Hausdorff distance between I_1 and I_2 is defined by

$$D(I_1, I_2) = \max\{|\underline{I_1} - \underline{I_2}|, |\overline{I_1} - \overline{I_2}|\}.$$

DEFINITION 2.3. [**1**] *Let* $F : [a,b)_{\mathbb{T}_1} \times [c,d)_{\mathbb{T}_2} \to I_{\mathbb{R}}$ *be an interval-valued function on* $\mathcal{R}$ *such that* $F = [\underline{F}, \overline{F}]$ *and let* g_1, g_2 *be non-decreasing functions defined on* $[a,b)_{\mathbb{T}_1}$ *and* $[c,d)_{\mathbb{T}_2}$ *respectively with partitions* $P_1 = \{t_0, t_1, ..., t_n\} \subset [a,b]_{\mathbb{T}_1}$ *with tag points* $\xi_i \in [t_{i-1}, t_i]_{\mathbb{T}_1}$ *for* $i = 1, 2, ..., n$ *and* $P_2 = \{s_0, s_1, ..., s_k\} \subset [c,d]_{\mathbb{T}_2}$ *with tag points* $\zeta_j \in [s_{j-1}, s_j]_{\mathbb{T}_2}$ *for* $j = 1, 2, ..., k$. *Then*

$$S(P_1, P_2, F, g_1, g_2) = \sum_{i=1}^{n} \sum_{j=1}^{k} F(\xi_i, \zeta_j)(g_1(t_i) - g_1(t_{i-1}))(g_2(s_j) - g_2(s_{j-1}))$$

is defined as Henstock-Kurweil-Stieltjes-$\diamond$*-double sum of* F *with respect to functions* g_1 *and* g_2, *and partitions* P_1 *and* P_2. *We also adopt the notation* $S(F, P, g)$ *where* $P = P_1 \times P_2$ *and* $g = g_1 \times g_2 : [a,b)_{\mathbb{T}_1} \times [c,d)_{\mathbb{T}_2} \longrightarrow \mathbb{R}^2$; $(t,s) \mapsto (g_1(t), g_2(s))$.

Now, the Henstock-Kurweil-Stieltjes-$\diamond$*-double sum of* F *with respect to increasing functions* g_1 *and* g_2 *can be taken as*

$$S(P, F, g) = \sum_{i=1}^{n} \sum_{j=1}^{k} F(\xi_i, \zeta_j) \diamond g_{1_i} \diamond g_{2_j}.$$

Throughout this paper, we denote the set of all interval valued numbers on a rectangle $\mathcal{R}$ by $E_{\mathcal{R}}$ and $I_{\mathbb{R}}$ by interval numbers on real line. We also denote the class of interval-valued Henstock-Kurweil-Stieltjes-$\diamond$-double integrable functions containing F by $F \in IVHKS$.

DEFINITION 2.4. [**1**] *Let* $F : [a,b)_{\mathbb{T}_1} \times [c,d)_{\mathbb{T}_2} \to I_{\mathbb{R}}$ *be an interval-valued function on* $\mathcal{R} = [a,b)_{\mathbb{T}_1} \times [c,d)_{\mathbb{T}_2}$ *such that* $F = [\underline{F}, \overline{F}]$. *We say that* F *is Henstock-Kurzweil-Stieltjes-*$\diamond$*-integrable with respect to non-decreasing functions* $g_1 : [a,b)_{\mathbb{T}_1} \longrightarrow \mathbb{R}$ *and* $g_2 : [c,d)_{\mathbb{T}_2} \longrightarrow \mathbb{R}$ *if there exists an interval* $I_0 \in I_{\mathbb{R}}$ *such that for every* $\varepsilon > 0$, *there are* $\diamond$*-gauges* δ_1 *and* δ_2 *(or* γ_1 *and* γ_2 *) for* $[a,b)_{\mathbb{T}_1}$ *and* $[c,d)_{\mathbb{T}_2}$ *respectively such that*

$$D\left(S(P, F, g), I_0\right) < \varepsilon$$

provided that $P_1 = \{t_0, t_1, ..., t_n\} \subset [a,b]_{\mathbb{T}_1}$ *with tag points* $\xi_i \in [t_{i-1}, t_i]_{\mathbb{T}_1}$ *for* $i = 1, ..., n$ *is a* δ_1*-fine (or* γ_1*) and* $P_2 = \{s_0, s_1, ..., s_k\} \subset [c,d]_{\mathbb{T}_2}$ *with tag points*

$\zeta_j \in [s_{j-1}, s_j]_{\mathbb{T}_2}$, $j = 1, 2, ..., k$ is a δ_2-fine (or γ_2) are δ-fine (or γ) partitions of $[a, b)_{\mathbb{T}_1}$ and $[c, d)_{\mathbb{T}_2}$ respectively.

We say that I_0 is the Henstock-Kurzweil-Stieltjes-$\diamond$-double integral of F with respect to g_1 and g_2 defined on $[a, b)_{\mathbb{T}_1} \times [c, d)_{\mathbb{T}_2}$, and write

$$\int\int_{\mathcal{R}} F(t, s)\diamond g_1(t)\diamond g_2(s) = I_0.$$

It follows from the right dense continuity of F that there two continuous real-valued functions $\underline{F}$ and $\overline{F}$ such that, for $t, s \in \mathbb{R}$,

$$F(t, s) = [\underline{F(t, s)}, \overline{F(t, s)}].$$

Moreover,

$$\begin{aligned} &\int\int_{\mathcal{R}} F(t, s)\diamond g_1(t)\diamond g_2(s) \\ = &\left[\int\int_{\mathcal{R}} \underline{F(t, s)}\diamond g_1(t)\diamond g_2(s), \int\int_{\mathcal{R}} \overline{F(t, s)}\diamond g_1(t)\diamond g_2(s)\right]. \end{aligned}$$

It is quite natural to expect that $F \in IVHKS(a, b]_{\mathbb{T}_1} \times (c, d]_{\mathbb{T}_2}$. That is to say that F is interval-valued Henstock-Kurweil-Stieltjes-$\diamond$-double integrable on $(a, b]_{\mathbb{T}_1} \times (c, d]_{\mathbb{T}_2}$.

Definition 2.5. *Let $\{F_n\}$ be a sequence of interval-valued Henstock-Kurweil-Stieltjes-$\diamond$-double integrable function with respect to monotonic increasing functions g_1, g_2 on $(a, b]_{\mathbb{T}_1} \times (c, d]_{\mathbb{T}_2}$, then F_n converges to an interval-valued function F if*

$$\begin{aligned} \lim_{n\to\infty}\int\int_{\mathcal{R}} F(t, s)\diamond g_1(t)\diamond g_2(s) &= \left[\int\int_{\mathcal{R}} \underline{F(t, s)}\diamond g_1(t)\diamond g_2(s), \int\int_{\mathcal{R}} \overline{F(t, s)}\diamond g_1(t)\diamond g_2(s)\right] \\ &= \int\int_{\mathcal{R}} F(t, s)\diamond_1 g_1(t)\diamond_2 g_2(s). \end{aligned}$$

Definition 2.6. *A sequence $\{F_n\}$ in $\mathcal{R}$ is said to be interval-valued uniformly convergence to F in $\mathcal{R}$ on $(a, b]_{\mathbb{T}_1} \times (c, d]_{\mathbb{T}_2}$ if for each $\varepsilon > 0$, there exists a $n_0 \in \mathbb{N}$ such that $D(F_n(t, s), F(t, s)) < \varepsilon$ for all $n \geqslant n_0$ and for all $t, s \in (a, b]_{\mathbb{T}_1} \times (c, d]_{\mathbb{T}_2}$.*

3. The main results

The following definition and theorems are useful in our result.

Definition 3.1. *A sequence $\{F_n(t, s)\}, F_n \in E_{\mathcal{R}}$ is called an interval-valued increasing (respectively, interval-valued deceasing in $[a, b)_{\mathbb{T}_1} \times [c, d)_{\mathbb{T}_2}$ if $F_n(t, s) \leqslant F_{n+1}(t, s)$ (respectively, $F_{n+1}(t, s) \leqslant F_n(t, s)$) for all $t, s \in [a, b)_{\mathbb{T}_1} \times [c, d)_{\mathbb{T}_2}$ and $n \in \mathbb{N}$. A sequence $\{F_n\}$ is called an interval-valued monotone on $[a, b)_{\mathbb{T}_1} \times [c, d)_{\mathbb{T}_2}$ if it is either interval-valued increasing or interval-valued decreasing in $[a, b)_{\mathbb{T}_1} \times [c, d)_{\mathbb{T}_2}$.*

Theorem 3.1. [**5, 16**] *Let $D : E_{\mathcal{R}} \times E_{\mathcal{R}} \to \mathbb{R}$ be a mapping defined by $D(\alpha, \beta) = \sup\{\rho([\alpha]^\lambda, [\beta]^\lambda) : \lambda \in [0, 1]\}$ for all $\alpha, \beta \in E_{\mathcal{R}}$ is a metric on $E_{\mathcal{R}}$. Then,*
(i) $(E_{\mathcal{R}}, D)$ is a complete metric space.
(ii) $D(\alpha + \gamma, \beta + \gamma) = D(\alpha, \beta)$ for all $\alpha, \beta, \gamma \in E_{\mathcal{R}}$.

(iii) $D(\lambda\alpha, \lambda\beta) = |\lambda| D(\alpha, \beta)$ *for all* $\alpha, \beta \in E_{\mathcal{R}}$ *and* $\lambda \in \mathbb{R}$.
(iv) $D(\alpha + \gamma, \beta + \eta) \leqslant D(\alpha, \beta) + D(\gamma, \eta)$ *for all* $\alpha, \beta, \gamma, \eta \in E_{\mathcal{R}}$.
(v) $D(\alpha + \beta, \theta) \leqslant D(\alpha, \theta) + D(\beta, \theta)$ *for all* $\alpha, \beta \in E_{\mathcal{R}}$ *and* θ *is the characteristic function of zero.*
(vi) $D(\alpha + \beta, \gamma) \leqslant D(\alpha, \gamma) + D(\beta, \theta)$ *for all* $\alpha, \beta, \gamma \in E_{\mathcal{R}}$.
(vii) If $\alpha, \beta, \gamma \in E_{\mathcal{R}}, D(\alpha, \beta) \leqslant D(\alpha, \gamma)$ *and* $D(\beta, \gamma) \leqslant D(\alpha, \gamma)$.

THEOREM 3.2. *Every monotonically increasing (respectively, monotonically decreasing) sequence* $\{F_n\}$, $F_n \in E_{\mathcal{R}}$ *with an upper bound (respectively, a lower bound) converges to* $\sup\{F_n : n \in \mathbb{N}\}$ *(respectively,* $\inf\{F_n : n \in \mathbb{N}\}$*) in the metric space* $(E_{\mathcal{R}}, D)$.

The following lemma is useful.

LEMMA 3.1. *(Saks-Double Henstock Lemma). Let* $F : [a, b)_{\mathbb{T}_1} \times [c, d)_{\mathbb{T}_2} \to I_{\mathbb{R}}$ *be an interval-valued function with respect to increasing functions* g_1, g_2 *and that the integral* $\int\int_{\mathcal{R}} F(t, s)\Diamond_{1,2}(g_1 \times g_2)$ *exists. For every* $\varepsilon > 0$, *there is a* $\Diamond$*-gauge* δ *(or* γ*) such that*

$$D\left(S(P, F, g_1, g_2), \int\int_{\mathcal{R}} F(t, s)\Diamond_1 g_1(t)\Diamond_2 g_2(s)\right) < \varepsilon$$

provided that $P_1 = \{t_0, t_1, ..., t_n\} \subset [a, b]_{\mathbb{T}_1}$ *with tag points* $\xi_i \in [t_{i-1}, t_i]_{\mathbb{T}_1}$ *for* $i = 1, ..., n$ *and* $P_2 = \{s_0, s_1, ..., s_k\}$ *of* $[c, d]_{\mathbb{T}_2}$ *with tag points* $\zeta_j \in [s_{j-1}, s_j]_{\mathbb{T}_2}$, $j = 1, 2, ..., k$ *are* δ*-fine (or* γ*) partitions of* $[a, b)_{\mathbb{T}_1} \times [c, d)_{\mathbb{T}_2}$, *then*

$$D\left(\sum_{i=1}^{n}\sum_{j=1}^{k}\left(F(\xi_i, \zeta_i)(g_1(t_i) - g_1(t_{i-1}))(g_2(s_j) - g_2(s_{j-1})), \int_{t_{i-1}}^{t_i}\int_{s_{j-1}}^{s_j} F(t, s)\right)\right) \leqslant \varepsilon.$$

Now our monotone convergence result for Henstock-Kurzweil-Stieltjes-$\Diamond$-double integral for interval-valued functions is as follows:

THEOREM 3.3. *Let* $\{F_m(t, s)\}$ *be an interval-valued monotone sequence of Interval-valued Henstock-Kurzweil-Stieltjes-*$\Diamond$*-double integrable functions in* $E_{\mathcal{R}}$, $\{\int\int_{\mathcal{R}} F_m(t, s)\}$ *be interval-valued bounded and* $F(t, s) \in E_{\mathcal{R}}$ *be such that for each* $t, s \in [a, b)_{\mathbb{T}_1} \times [c, d)_{\mathbb{T}_2}, \{F_m(t, s)\}$ *converges to* $F(t, s)$ *in the metric space* $(E_{\mathcal{R}}, D)$. *Then*

(i) F *is Henstock-Kurzweil-Stieltjes-*$\Diamond$*-double integrable on* $[a, b)_{\mathbb{T}_1} \times [c, d)_{\mathbb{T}_2}$ *and*
(ii)

$$\int\int_{\mathcal{R}} F(t, s) = \int\int_{\mathcal{R}} [\overline{F(t, s)}, \underline{F(t, s)}] = \lim_{m\to\infty} \int\int_{\mathcal{R}} F_m(t, s).$$

PROOF. Since $\{\int\int_{\mathcal{R}} F_m(t, s)\}$ is right dense continuous, interval-valued bounded and increasing. It follows from the continuity of F that there two continuous real-valued functions $\underline{F}$ and $\overline{F}$ such that, for $t, s \in \mathbb{R}$,

$$F(t, s) = [\underline{F(t, s)}, \overline{F(t, s)}].$$

Then by Theorem 3.1, $\{\int\int_{\mathcal{R}} F_m(t, s)\}$ must be interval-valued converges to $I_0 = \sup\{\int\int_{\mathcal{R}} F_m(t, s)\}$. Let $\varepsilon > 0$ be given. Then we choose an $r \in \mathbb{N}$ such that

$$\frac{1}{2^{r-2}} < \frac{\varepsilon}{3}$$

and

$$D\left(\int\int_{\mathcal{R}} F_r(t,s), I_0\right) < \frac{\varepsilon}{3}.$$

Again since $\{F_m\}$ is a sequence of interval-valued Henstock-Kurzweil-Stieltjes-$\diamond$-double integrable functions on $[a,b)_{\mathbb{T}_1} \times [c,d)_{\mathbb{T}_2}$, for each $m \in \mathbb{N}$ there exists a $\delta_m \in E_{\mathcal{R}}$ such that every δ_m-fine division (ξ^m, ζ^m) of $[a,b)_{\mathbb{T}_1} \times [c,d)_{\mathbb{T}_2}$ satisfies

$$D\left(\sum_{i=1}^{n_m}\sum_{j=1}^{k_m} F_m(\xi_i^m, \zeta_j^m)(g_1(t_i) - g_1(t_{i-1}))(g_2(s_j) - g_2(s_{j-1}), \int\int_{\mathcal{R}} F_m(t,s)\right)$$
$$< \quad \frac{1}{2^m}.$$

Again by the condition, for each $t, s \in [a,b)_{\mathbb{T}_1} \times [c,d)_{\mathbb{T}_2}$, we can select a $m_{t,s} (\geqslant r) \in \mathbb{N}$ such that

$$D(F_{m_{t,s}}(t,s), F(t,s)) < \frac{\varepsilon}{3[(b-a)(d-c)]}.$$

Consider the function $\delta = \delta_{m_{t,s}}$ and let $(\xi, \zeta) = \{(\xi_i, \zeta_j) : i = 1, 2, ..., n, \quad j = 1, 2, ..., k\}$ be any δ-fine division of $[a,b)_{\mathbb{T}_1} \times [c,d)_{\mathbb{T}_2}$. Then

$$\begin{aligned}
& D\left(\sum_{i=1}^{n}\sum_{j=1}^{k} F(g_1(t_i) - g_1(t_{i-1}))(g_2(s_j) - g_2(s_{j-1}), I_0\right) \\
\leqslant \; & D(\sum_{i=1}^{n}\sum_{j=1}^{k} F(\xi_i, \zeta_j)(g_1(t_i) - g_1(t_{i-1}))(g_2(s_j) - g_2(s_{j-1}), \\
& \sum_{i=1}^{n}\sum_{j=1}^{k} F_{m_{\xi_i,\zeta_j}}(\xi_i, \zeta_j)(g_1(t_i) - g_1(t_{i-1}))(g_2(s_j) - g_2(s_{j-1})) \\
& + D(\sum_{i=1}^{n}\sum_{j=1}^{k} F_{m_{\xi_i,\zeta_j}}(\xi_i, \zeta_j)(g_1(t_i) - g_1(t_{i-1}))(g_2(s_j) - g_2(s_{j-1}), \\
& \sum_{i=1}^{n}\sum_{j=1}^{k} \int_{t_{i-1}}^{t_i}\int_{s_{j-1}}^{s_j} F_{m_{\xi_i,\zeta_j}}(t,s)) \\
& + D\left(\sum_{i=1}^{n}\sum_{j=1}^{k} \int_{t_{i-1}}^{t_i}\int_{s_{j-1}}^{s_j} F_{m_{\xi_i,\zeta_j}}(t,s)\diamond_1 g_1(t)\diamond_2 g_2(s), I_0\right) < \varepsilon
\end{aligned}$$

provided that $P_1 = \{t_0, t_1, ..., t_n\}$ of $[a,b]_{\mathbb{T}_1}$ with tag points $\xi_i \in [t_{i-1}, t_i]_{\mathbb{T}_1}$ and $P_2 = \{s_0, s_1, ..., s_k\}$ of $[c,d]_{\mathbb{T}_2}$ with tag points $\zeta_j \in [s_{j-1}, s_j]_{\mathbb{T}_2}$, $j = 1, 2, ..., k$ are δ-fine (or γ) partitions of $[a,b)_{\mathbb{T}_1} \times [c,d)_{\mathbb{T}_2}$ holds.

Now we estimate the three values in the right-handed sum of the last inequality.

(i) Let us estimate

$$D(\sum_{i=1}^{n}\sum_{j=1}^{k} F(\xi_i,\zeta_j)(g_1(t_i)-g_1(t_{i-1}))(g_2(s_j)-g_2(s_{j-1}),$$
$$\sum_{i=1}^{n}\sum_{j=1}^{k} F_{m_{\xi_i,\zeta_j}}(\xi_i,\zeta_j)(g_1(t_i)-g_1(t_{i-1}))(g_2(s_j)-g_2(s_{j-1})):$$

By Theorem 3.1,

$$\begin{aligned}
& D(\sum_{i=1}^{n}\sum_{j=1}^{k} F(\xi_i,\zeta_j)(g_1(t_i)-g_1(t_{i-1}))(g_2(s_j)-g_2(s_{j-1}),\\
& \sum_{i=1}^{n}\sum_{j=1}^{k} F_{m_{\xi_i,\zeta_j}}(\xi_i,\zeta_j)(g_1(t_i)-g_1(t_{i-1}))(g_2(s_j)-g_2(s_{j-1}))\\
\leqslant\ & \sum_{i=1}^{n}\sum_{j=1}^{k} D(F(\xi_i,\zeta_j),F_{m_{\xi_i,\zeta_j}}(\xi_i,\zeta_j))(g_1(t_i)-g_1(t_{i-1}))(g_2(s_j)-g_2(s_{j-1})\\
<\ & \frac{\varepsilon}{3[(b-a)(d-c)]}[(b-a)(d-c)]=\frac{\varepsilon}{3}.
\end{aligned}$$

(ii) To estimate

$$D(\sum_{i=1}^{n}\sum_{j=1}^{k} F_{m_{\xi_i,\zeta_j}}(\xi_i,\zeta_j)(g_1(t_i)-g_1(t_{i-1}))(g_2(s_j)-g_2(s_{j-1}),$$
$$\sum_{i=1}^{n}\sum_{j=1}^{k}\int_{t_{i-1}}^{t_i}\int_{s_{j-1}}^{s_j} F_{m_{\xi_i,\zeta_j}}(t,s)):$$

Suppose $p=\max\{m_{\xi_i,\zeta_j}: i=1,2,...,n,\ j=1,2,...,k\}$. Then

$$\begin{aligned}
& D(\sum_{i=1}^{n}\sum_{j=1}^{k} F_{m_{\xi_i,\zeta_j}}(\xi_i,\zeta_j)(g_1(t_i)-g_1(t_{i-1}))(g_2(s_j)-g_2(s_{j-1}),\\
& \sum_{i=1}^{n}\sum_{j=1}^{k}\int_{t_{i-1}}^{t_i}\int_{s_{j-1}}^{s_j} F_{m_{\xi_i,\zeta_j}}(t,s))\\
\leqslant\ & \sum_{t,s=r}^{p}(\sum_{m_{\xi_i}=t}\ \sum_{m_{\zeta_j}=s} D(F_{m_{\xi_i,\zeta_j}}(\xi_i,\zeta_j)(g_1(t_i)-g_1(t_{i-1}))(g_2(s_j)-g_2(s_{j-1}),\\
& \sum_{i=1}^{n}\sum_{j=1}^{k}\int_{t_{i-1}}^{t_i}\int_{s_{j-1}}^{s_j} F_{m_{\xi_i,\zeta_j}}(t,s)).
\end{aligned}$$

Now applying Lemma 3.1,

$$\sum_{m_{\xi_i}=t} \sum_{m_{\zeta_j}=s} D\left(F_{m_{\xi_i,\zeta_j}}(\xi_i,\zeta_j)(g_1(t_i)-g_1(t_{i-1}))(g_2(s_j)-g_2(s_{j-1})\right),$$

$$\left(\sum_{i=1}^{n}\sum_{j=1}^{k}\int_{t_{i-1}}^{t_i}\int_{s_{j-1}}^{s_j} F_{m_{\xi_i,\zeta_j}}(t,s)\right) \leqslant \frac{1}{2^{t-1}}\frac{1}{2^{s-1}}$$

and hence

$$\begin{aligned}
&D(\sum_{i=1}^{n}\sum_{j=1}^{k} F_{m_{\xi_i,\zeta_j}}(\xi_i,\zeta_j)(g_1(t_i)-g_1(t_{i-1}))(g_2(s_j)-g_2(s_{j-1}),\\
&\sum_{i=1}^{n}\sum_{j=1}^{k}\int_{t_{i-1}}^{t_i}\int_{s_{j-1}}^{s_j} F_{m_{\xi_i,\zeta_j}}(t,s))\\
< \quad &\sum_{t=r}^{p}\sum_{s=r}^{p}\frac{1}{2^{t-1}}\frac{1}{2^{s-1}} < \frac{1}{2^{2r-2}} < \frac{\varepsilon}{3}.
\end{aligned}$$

(iii) Now, the estimation of

$$D\left(\sum_{i=1}^{n}\sum_{j=1}^{k}\int_{t_{i-1}}^{t_i}\int_{s_{j-1}}^{s_j} F_{m_{\xi_i,\zeta_j}}(t,s)\diamond_1 g_1(t)\diamond_2 g_2(s), I_0\right):$$

Here $r \leqslant m_{\xi_i,\zeta_j} \leqslant p$ implies $F_r(t,s) \leqslant F_{m_{\xi_i,\zeta_j}}(t,s) \leqslant F_p(t,s)$ for all $t,s \in [a,b)_{\mathbb{T}_1} \times [c,d)_{\mathbb{T}_2}$ and so

$$\int_{t_{i-1}}^{t_i}\int_{s_{j-1}}^{s_j} F_r(t,s) \leqslant \int_{t_{i-1}}^{t_i}\int_{s_{j-1}}^{s_j} F_{m_{\xi_i,\zeta_j}}(t,s) \leqslant \int_{t_{i-1}}^{t_i}\int_{s_{j-1}}^{s_j} F_p(t,s) \leqslant I_0.$$

Therefore by Theorem 3.1,

$$D\left(\sum_{i=1}^{n}\sum_{j=1}^{k}\int_{t_{i-1}}^{t_i}\int_{s_{j-1}}^{s_j} F_{m_{\xi_i,\zeta_j}}(t,s), I_0\right) \leqslant D\left(\int_a^b\int_c^d F_r(t,s), I_0\right) < \frac{\varepsilon}{3}.$$

Thus

$$D\left(\sum_{i=1}^{n}\sum_{j=1}^{k} F(\xi_i,\zeta_j)(g_1(t_i)-g_1(t_{i-1}))(g_2(s_j)-g_2(s_{j-1}), I_0\right) < \varepsilon.$$

So F is Henstock-Kurzweil-Stieltjes-$\diamond$-double integrable on $[a,b)_{\mathbb{T}_1} \times [c,d)_{\mathbb{T}_2}$ and

$$\int\int_{\mathcal{R}} [\overline{F(t,s)}, \underline{F(t,s)}] = \int_a^b\int_c^d F(t,s) = I_0 = \lim_{m\to\infty}\int_a^b\int_c^d F_m(t,s)$$

which implies that

$$\lim_{m\to\infty}\int\int_{\mathcal{R}} F_m(t,s) = \int\int_{\mathcal{R}} [\overline{F(t,s)}, \underline{F(t,s)}] = \int\int_{\mathcal{R}} F(t,s) = I_0,$$

which ends the proof. □

4. Application

We give as an application to our main result an error estimates of the Henstock-Kurzweil-Stieltjes-$\diamondsuit$-double integral $\int_a^b \int_c^d F(t,s)\diamondsuit_1 g_1(t)\diamondsuit_2 g_2(s)$ using Henstock-Kurzweil-Stieltjes sum $S(F, I_n, J_k, \xi, \zeta)$ for a cubature formula.

Let P_1 and P_2 be two partitions of $[a,b]_{\mathbb{T}_1}$ and $[c,d]_{\mathbb{T}_2}$ such that $P_1 = \{t_0, t_1, ..., t_n\} \subset [a,b]_{\mathbb{T}_1}$ and $P_2 = \{s_0, s_1, ..., s_n\} \subset [c,d]_{\mathbb{T}_2}$. Let $\{\xi_1, \xi_2, ..., \xi_n\}$ denote an arbitrary selection of points from $[a,b]_{\mathbb{T}_1}$ with $\xi_i \in [t_{i-1}, t_i)_{\mathbb{T}_1}, i = 1, 2, ..., n$. Similarly, let $\{\zeta_1, \zeta_2, ..., \zeta_n\}$ denote an arbitrary selection of points from $[c,d]_{\mathbb{T}_2}$ with $\zeta_j \in [s_{j-1}, s_j)_{\mathbb{T}_2}, j = 1, 2, ..., k$ and g_1, g_2 are monotonic increasing functions.

Let $l_i = t_i - t_{i-1}, \quad i = 1, 2, ..., n$ and $q_j = s_j - s_{j-1}, \quad j = 1, 2, ..., k$ then $k_h(\xi_i) = h\left(\xi_i - \dfrac{t_{i-1}+t_i}{2}\right)$ and $k_h(\zeta_j) = h\left(\zeta_j - \dfrac{s_{j-1}+s_j}{2}\right)$.

Let us consider the sum

$$
\begin{aligned}
S(F, I_n, J_k, \xi, \zeta) \quad = \quad & \sum_{i=1}^{n}\sum_{j=1}^{k} q_j \int_{t_{i-1}}^{t_i} F(t,\zeta_j)\diamondsuit_1 g_1(t) \\
& + \sum_{i=1}^{n}\sum_{j=1}^{k} l_i \int_{s_{j-1}}^{s_j} F(\xi_i, s)\diamondsuit_2 g_2(s) - \sum_{i=1}^{n}\sum_{j=1}^{k} l_i q_j F(\xi_i,\zeta_j) \\
& - \sum_{i=1}^{n}\sum_{j=1}^{k} k_h(\xi_i) \int_{s_{j-1}}^{s_j} [F(t_i,s) - F(t_{i-1},s)]\diamondsuit_2 g_2(s) \\
& - \sum_{i=1}^{n}\sum_{j=1}^{k} k_h(\zeta_j) \int_{t_{i-1}}^{t_i} [F(t,s_j) - F(t,s_{j-1})]\diamondsuit_1 g_1(t) \\
& + \sum_{i=1}^{n}\sum_{j=1}^{k} q_j k_h(\xi_i)[F(t_i,\zeta_j) - F(t_{i-1},\zeta_j)] \\
& + \sum_{i=1}^{n}\sum_{j=1}^{k} l_i k_h(\zeta_j)[F(\xi_i,s_j) - F(\xi_i,s_{j-1})] \\
& - \sum_{i=1}^{n}\sum_{j=1}^{k} k_h(\xi_i)k_h(\zeta_j)[F(t_{i-1},s_{j-1}) - F(t_{i-1},s_j)] \\
& - F(t_i,s_{j-1}) + F(t_i,s_j).
\end{aligned}
$$

THEOREM 4.1. *Suppose $F : [a,b]_{\mathbb{T}_1} \times [c,d]_{\mathbb{T}_2} \to \mathbb{R}$ is Henstock-Kurzweil-Stieltjes-$\diamondsuit$-double integrable function such that the partial derivative of order 2 exists for all $(t,s) \in [a,b]_{\mathbb{T}_1} \times [c,d]_{\mathbb{T}_2}$. If $F_{ts} = \dfrac{\partial^2 F}{\partial t \partial s}$ exists on $(a,b)_{\mathbb{T}_1} \times (c,d)_{\mathbb{T}_2}$ and* *bounded, such that*

$$\|F_{ts}\|_\infty = \sup_{t,s\in(t_{i-1},t_i)_{\mathbb{T}_1}\times(s_{j-1},s_j)_{\mathbb{T}_2}} \left|\frac{\partial^2 F(t,s)}{\partial t\partial s}\right| < \infty,$$

then

$$\int_a^b\int_c^d F(t,s)\diamond_1 g_1(t)\diamond_2 g_2(s) = S(F,I_n,J_k,\xi,\zeta) + R(F,I_n,J_k,\xi,\zeta).$$

PROOF. Firstly, we estimate the remainder $R(F,I_n,J_k,\xi,\zeta)$ to get

$$\begin{aligned} &|R(F,I_n,J_k,\xi,\zeta)| \\ \leqslant\ & \sum_{i=1}^{n}\sum_{j=1}^{k}\left[\frac{l_i^2}{4} + \left(\xi_i - \frac{t_{i-1}+t_i}{2}\right)^2 + (h-2)\left(\xi_i - \frac{t_{i-1}+t_i}{2}\right)k_h(\xi_i)\right] \end{aligned}$$

$$\times\sum_{i=1}^{n}\sum_{j=1}^{k}\left[\frac{q_j^2}{4} + \left(\zeta_i - \frac{s_{j-1}+s_j}{2}\right)^2 + (h-2)\left(\zeta_j - \frac{s_{j-1}+s_j}{2}\right)k_h(\zeta_j)\right]\|F_{ts}\|_\infty \tag{4.1}$$

for all $(\xi_i,\zeta_j)\in[t_{i-1},t_i]_{\mathbb{T}_1}\times[s_{j-1},s_j]_{\mathbb{T}_2}$ with $l_i = t_i - t_{i-1}$, $i = 1,2,...,n$ and $q_j = s_j - s_{j-1}$, $j = 1,2,...,k$ with $k_h(\xi_i) = h\left(\xi_i - \frac{t_{i-1}+t_i}{2}\right)$ and $k_h(\zeta_j) = h\left(\zeta_j - \frac{s_{j-1}+s_j}{2}\right)$ with $h\in[0,2]$.

Considering the interval $[t_{i-1},t_i]_{\mathbb{T}_1}\times[s_{j-1},s_j]_{\mathbb{T}_2}$ with $i = 1,2,...,n$; $j = 1,2,...,k$, we obtain

$$\begin{gathered} |\textstyle\int_{t_{i-1}}^{t_i}\int_{s_{j-1}}^{s_j} F(t,s)\diamond_1 g_1(t)\diamond_2 g_2(s) - q_j\int_{t_{i-1}}^{t_i} F(t,\zeta_j)\diamond_1 g_1(t) \\ -l_i\textstyle\int_{s_{j-1}}^{s_i} F(\xi_i,s)\diamond_2 g_2(s) + l_i q_j F(\xi_i,\zeta_j) + k_h(\xi_i)\int_{s_{j-1}}^{s_j}[F(t_i,s) - F(t_{i-1},s)]\diamond_2 g_2(s) \\ +k_h(\zeta_j)\textstyle\int_{t_{i-1}}^{t_i}[F(t,s_j) - F(t,s_{j-1})]\diamond_1 g_1(t) \\ -q_j k_h(\xi_i)[F(t_i,\zeta_j) - F(t_{i-1},\zeta_j)] - l_i k_h(\zeta_j)[F(\xi_i,s_j) - F(\xi_i,s_{j-1})] \\ +k_h(\xi_i)k_h(\zeta_j)[F(t_{i-1},s_{j-1}) - F(t_{i-1},s_j) - F(t_i,s_{j-1}) + F(t_i,s_j)]| \end{gathered}$$

$$\leqslant \|F_{ts}\|_\infty\left[\frac{l_i^2}{4} + \left(\xi_i - \frac{t_{i-1}+t_i}{2}\right)^2 + (h-2)\left(\xi_i - \frac{t_{i-1}+t_i}{2}\right)k_h(\xi_i)\right]$$

$$\times\sum_{i=1}^{n}\sum_{j=1}^{k}\left[\frac{q_j^2}{4} + \left(\zeta_i - \frac{s_{j-1}+s_j}{2}\right)^2 + (h-2)\left(\zeta_j - \frac{s_{j-1}+s_j}{2}\right)k_h(\zeta_j)\right]$$

for all $i = i,2,...,n$; $j = 1,2,...,k$. Summing over i from 1 to n and over j from 1 to k using the triangular inequality, we have the estimation (4.1). □

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